ON THE φ-ORDER AND THE φ-TYPE OF ENTIRE AND MEROMORPHIC FUNCTIONS IN THE UNIT DISC

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Abstract. The paper is devoted to deriving some estimates of sum, product and derivative of analytic or meromorphic functions in the unit disc involving the concepts of the φ-order and the φ-type. Our results generalise and improve many previous results that have used the usual order [7, 16, 17, 21, 23] and the iterated p-order [4, 13, 14].

1. Introduction

In order to study the growth of solutions of the following higher order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1}$$

where $k \geq 2$, $A_0 \not\equiv 0$ and the coefficients $A_j(j = 0, \ldots, k-1)$ are analytic functions (respectively, meromorphic functions) in unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ or in the whole complex plane $\mathbb{C}$, it is very important to determine the order and the type of the coefficients $A_j(j = 0, \ldots, k-1)$. Through this paper, the reader is assumed to be familiar with the standard notations and the fundamental results of Nevanlinna value distribution theory of meromorphic functions in the complex plane and in the unit disc (see [7, 9, 17, 21, 22]) and can refer to ([3, 4, 8, 10, 11, 15]) for more details about the growth of solutions of equation (1).

For all $r \in [0, 1)$, we define $\exp_1 r = \exp r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r)$, $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Inductively, for all r in $(0, 1)$, $\log^+ r = \max\{0, \log r\}$, $\log_1^+ r = \log^+ r$ and $\log_{p+1}^+ r = \log^+(\log_p^+ r)$, $p \in \mathbb{N} \cup \{0\}$. We also make the conventions $\exp_0 r = 1 = \log_0^+ r$, $\exp_{-1} r = \log_1^+ r$ and $\log_{-1}^+ r = \exp_1 r$.

Definition 1. [4] The iterated p-order of an analytic function f in Δ is defined by

$$\tilde{\rho}_p(f) := \limsup_{r \longrightarrow 1^-} \frac{\log_{p+1}^+ M(r,f)}{-\log(1-r)}, \quad p \in \mathbb{N},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$ is the maximum modulus of f. For a mero-morphic function f in Δ, the iterated p-order is defined by

$$\rho_p(f) := \limsup_{r \longrightarrow 1^-} \frac{\log_p^+ T(r,f)}{-\log(1-r)}, \quad p \in \mathbb{N},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

Definition 2. [10] The iterated p-type of an analytic function f in Δ with $0 < \tilde{\rho}_p(f) < +\infty$ is defined by

$$\tilde{\tau}_p(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\tilde{\rho}_p(f)} \log_p^+ M(r,f).$$

For a meromorphic function f in Δ with $(0 < \rho_p(f) < +\infty)$, the iterated p-type is defined by

$$\tau_p(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\rho_p(f)} \log_{p-1}^+ T(r,f).$$

It is clear that $\rho_1(f)$ and $\tau_1(f)$ coincide with the usual order $\rho(f)$ and the usual type $\tau(f)$ respectively. In [14], Lahiri proved that the iterated p-order of the deriva-tive of an entire function or a meromorphic function in the complex plane is the same as that of the function, i.e., $\rho_p(f') = \rho_p(f), p \geq 1$. Later, Cao-Yi [4] gave the version of this result in the unit disc and obtained other properties that are well-known in the complex plane [7, 17, 23].

Theorem A. [4] Let f_1 and f_2 be two meromorphic functions in Δ. Then, for $p \in \mathbb{N}$ we have

(i) $\rho_p(f_1) = \rho_p(\frac{1}{f_1})$, where $f_1 \not\equiv 0$, $\rho_p(af_1) = \rho_p(f_1)$, where $a \in \mathbb{C}^*$.

(ii) $\rho_p(f_1') = \rho_p(f_1)$.

(iii) $\max\{\rho_p(f_1 + f_2), \rho_p(f_1 f_2)\} \leq \max\{\rho_p(f_1), \rho_p(f_2)\}$.

(iv) If $\rho_p(f_1) < \rho_p(f_2)$, then $\rho_p(f_1 + f_2) = \rho_p(f_1 f_2) = \rho_p(f_2)$.

Theorem B. [4] Let f_1 and f_2 be two analytic functions in Δ. Then, for $p \in \mathbb{N}$ we have

(i) $\tilde{\rho}_p(af_1) = \tilde{\rho}_p(f_1)(a \in \mathbb{C}^*)$.

(ii) $\tilde{\rho}_p(f_1') = \tilde{\rho}_p(f_1)$.

(iii) $\max\{\tilde{\rho}_p(f_1 + f_2), \tilde{\rho}_p(f_1 f_2)\} \leq \max\{\tilde{\rho}_p(f_1), \tilde{\rho}_p(f_2)\}$.

(iv) If $\tilde{\rho}_p(f_1) < \tilde{\rho}_p(f_2)$, then $\tilde{\rho}_p(f_1 + f_2) = \tilde{\rho}_p(f_2)$.

Latreuch-Belaïdi [16] obtained under some conditions similar results for the usual type $(p = 1)$.

Theorem C. *[16] If f_1 and f_2 are two analytic functions in Δ satisfying $0 < \tilde{\rho}(f_1) = \tilde{\rho}(f_2) = \rho < +\infty$ and $\tilde{\tau}(f_1) \neq \tilde{\tau}(f_2)$, then $\tilde{\rho}(f_1 + f_2) = \rho$ and $\tilde{\tau}(f_1 + f_2) = \max\{\tilde{\tau}(f_1), \tilde{\tau}(f_2)\}$.*

As it is shown by Chyzhykov-Semochko in [6, Example 1.4], the iterated p-order does not cover an arbitrary growth of entire solutions of equation (1) and the concept of the φ-order (cf. [20]) was investigated in the same paper. This concept is quickly adopted by Semochko [18] and by Belaïdi [1, 2], where the φ-type's concept is given.

Definition 3. *[18] Let φ be an increasing unbounded function in the unit disc Δ. The φ-orders of an analytic function f in Δ are defined by*

$$\tilde{\rho}_\varphi^0(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(M(r,f))}{-\log(1-r)}, \quad \tilde{\rho}_\varphi^1(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(\log M(r,f))}{-\log(1-r)}.$$

If f is meromorphic in Δ, then the φ-orders are defined by

$$\rho_\varphi^0(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(e^{T(r,f)})}{-\log(1-r)}, \quad \rho_\varphi^1(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(T(r,f))}{-\log(1-r)}.$$

We can see that if $\varphi(r) = \log_p^+ r$, $p \in \mathbb{N}$ and f is an analytic function in Δ, then $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_p(f)$. By Φ we define the class of positive unbounded increasing functions in the unit disc Δ such that $\varphi(e^t)$ is slowly growing, i.e., $\forall c > 0: \lim\limits_{t \to +\infty} \dfrac{\varphi(e^{ct})}{\varphi(e^t)} = 1$. As examples, $\varphi(r) = \log_p^+ r$, $(p \geq 2)$ belongs to the class Φ and $\varphi(r) = \log^+ r \notin \Phi$.

Now, we introduce by analogous manner the definitions of the φ-types in the unit disc Δ related to the φ-order.

Definition 4. *Let φ be an increasing unbounded function in the unit disc Δ. We define the φ-types of an analytic function f in Δ with $0 < \tilde{\rho}_\varphi^i(f) < +\infty$ $(i = 0, 1)$ by*

$$\tilde{\tau}_\varphi^0(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\tilde{\rho}_\varphi^0(f)} \exp\{\varphi(M(r,f))\},$$

$$\tilde{\tau}_\varphi^1(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\tilde{\rho}_\varphi^1(f)} \exp\{\varphi(\log M(r,f))\}.$$

*If f is meromorphic in Δ, then we define the φ-types with $0 < \rho_\varphi^i(f) < +\infty$ $(i = 0, 1)$
by*

$$\tau_\varphi^0(f) = \limsup_{r \longrightarrow 1^-} (1 - r)^{\rho_\varphi^0(f)} \exp\{\varphi(e^{T(r,f)})\},$$

$$\tau_\varphi^1(f) = \limsup_{r \longrightarrow 1^-} (1 - r)^{\rho_\varphi^1(f)} \exp\{\varphi(T(r, f))\}.$$

2. Main results

The main purpose of this paper is to investigate under suitable conditions the φ-order
and the φ-type of $f_1 + f_2$, $f_1 f_2$ and f_1', where f_1, f_2 are analytic or meromorphic
functions in Δ.

Theorem 1. *Let $\varphi \in \Phi$ and f_1, f_2 be two meromorphic functions in Δ. Then*

(i) $\rho_\varphi^j(f_1 + f_2) \leq \max\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\}$, $(j = 0, 1)$.

(ii) $\rho_\varphi^j(f_1 f_2) \leq \max\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\}$, $(j = 0, 1)$.

(iii) $\rho_\varphi^j(af_1) = \rho_\varphi^j(f_1)$ and $\tau_\varphi^j(af_1) = \tau_\varphi^j(f_1)$, $(a \in \mathbb{C}^*; j = 0, 1)$.

(iv) $\rho_\varphi^j\left(\frac{1}{f_1}\right) = \rho_\varphi^j(f_1)$ and $\tau_\varphi^j\left(\frac{1}{f_1}\right) = \tau_\varphi^j(f_1)$, $(f_1 \not\equiv 0; j = 0, 1)$.

Theorem 2. *Let $\varphi \in \Phi$ and f_1, f_2 be two meromorphic functions in Δ. If $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$, $(j = 0, 1)$, then $\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2)$, $(j = 0, 1)$.*

Theorem 3. *Let $\varphi \in \Phi$ and f_1, f_2 be two analytic functions in Δ. Then*

(i) $\tilde{\rho}_\varphi^j(af_1) = \tilde{\rho}_\varphi^j(f_1)$ and $\tilde{\tau}_\varphi^j(af_1) = \tilde{\tau}_\varphi^j(f_1)$, $(a \in \mathbb{C}^*; j = 0, 1)$.

(ii) $\max\{\tilde{\rho}_\varphi^j(f_1 + f_2), \tilde{\rho}_\varphi^j(f_1 f_2)\} \leq \max\{\tilde{\rho}_\varphi^j(f_1), \tilde{\rho}_\varphi^j(f_2)\}$, $(j = 0, 1)$.

(iii) If $\tilde{\rho}_\varphi^j(f_1) < \tilde{\rho}_\varphi^j(f_2)$, then $\tilde{\rho}_\varphi^j(f_1 + f_2) = \tilde{\rho}_\varphi^j(f_2)$, $(j = 0, 1)$.

Theorem 4. *Let $\varphi \in \Phi$ and f_1, f_2 be two meromorphic functions in Δ. Then*

(i) *If $0 < \rho_\varphi^j(f_1) < \rho_\varphi^j(f_2) < +\infty$ and $\tau_\varphi^j(f_1) < \tau_\varphi^j(f_2), (j = 0, 1)$, then*

$$\tau_\varphi^j(f_1 + f_2) = \tau_\varphi^j(f_1 f_2) = \tau_\varphi^j(f_2), j = 0, 1. \tag{2}$$

(ii) *If $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2) < +\infty, (j = 0, 1)$, then*

$$\tau_\varphi^j(f_1 + f_2) \leq \max\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\}.$$

Moreover, if $\tau_\varphi^j(f_1) \neq \tau_\varphi^j(f_2)$, $(j = 0, 1)$, then

$$\tau_\varphi^j(f_1 + f_2) = \max\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\}. \tag{3}$$

(iii) If $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1\, f_2) < +\infty, (j = 0, 1)$, then

$$\tau_\varphi^j(f_1\, f_2) \leq \max\{\tau_\varphi^j(f), \tau_\varphi^j(f_2)\}.$$

Moreover, if $\tau_\varphi^j(f_1) \neq \tau_\varphi^j(f_2), (j = 0, 1)$, then

$$\tau_\varphi^j(f_1\, f_2) = \max\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\}. \tag{4}$$

Theorem 5. *Let* $\varphi \in \Phi$ *and* f_1, f_2 *be two meromorphic functions in* Δ. *If* $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho < +\infty$ *and* $\tau_\varphi^j(f_1) < \tau_\varphi^j(f_2), (j = 0, 1)$, *then*

$$\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1\, f_2) = \rho, \tag{5}$$

$$\tau_\varphi^j(f_1 + f_2) = \tau_\varphi^j(f_1\, f_2) = \tau_\varphi^j(f_2). \tag{6}$$

Theorem 6. *Let* $\varphi \in \Phi$ *and* f_1, f_2 *be two analytic functions in* Δ. *Then*

(i) If $0 < \tilde{\rho}_\varphi^j(f_1) < \tilde{\rho}_\varphi^j(f_2) < +\infty$ *and* $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2), (j = 0, 1)$, then

$$\tilde{\tau}_\varphi^j(f_1 + f_2) = \tilde{\tau}_\varphi^j(f_2). \tag{7}$$

(ii) If $0 < \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2) = \tilde{\rho}_\varphi^j(f_1 + f_2) < +\infty, (j = 0, 1)$, then

$$\tilde{\tau}_\varphi^j(f_1 + f_2) \leq \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}.$$

Moreover, if $\tilde{\tau}_\varphi^j(f_1) \neq \tilde{\tau}_\varphi^j(f_2)$, then $\tilde{\tau}_\varphi^j(f_1 + f_2) = \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}$.

(iii) If $0 < \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2) = \tilde{\rho}_\varphi^j(f_1\, f_2) < +\infty, (j = 0, 1)$, then

$$\tilde{\tau}_\varphi^j(f_1\, f_2) \leq \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}.$$

Theorem 7. *Let* $\varphi \in \Phi$ *and* f_1, f_2 *be two analytic functions in* Δ. *If* $0 < \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2) < +\infty$ *and* $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2), (j = 0, 1)$, *then*

$$\tilde{\rho}_\varphi^j(f_1 + f_2) = \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2), \tag{8}$$

$$\tilde{\tau}_\varphi^j(f_1 + f_2) = \tilde{\tau}_\varphi^j(f_2). \tag{9}$$

Theorem 8. *If* f *is a meromorphic function in* Δ *and* $\varphi \in \Phi$, *then* $\rho_\varphi^j(f') = \rho_\varphi^j(f)$, $(j = 0, 1)$.

Theorem 9. *If* f *is an analytic function in* Δ *and* $\varphi \in \Phi$, *then* $\tilde{\rho}_\varphi^j(f') = \tilde{\rho}_\varphi^j(f)$, $(j = 0, 1)$.

Remark 1. *For some related results in the complex plane, see [12].*

3. Basic properties and lemmas

In this section, we recall some basic properties of functions from the class Φ and give only lemmas that we need to prove our results.

Proposition 1. *[6] If $\varphi \in \Phi$, then*

$$\forall m > 0, \forall k \geq 0 : \quad \frac{\varphi^{-1}(\log x^m)}{x^k} \longrightarrow +\infty, \quad x \longrightarrow +\infty, \tag{10}$$

$$\forall \delta > 0 : \quad \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} \longrightarrow +\infty, \quad x \longrightarrow +\infty. \tag{11}$$

Remark 2. *[6] One can show that* (11) *implies that*

$$\forall c > 0, \varphi(ct) \leq \varphi(t^c) \leq (1 + o(1))\varphi(t), \quad t \longrightarrow +\infty. \tag{12}$$

Proposition 2. *[18] Let $\varphi \in \Phi$ and f be an analytic function in the unit disc Δ. Then $\rho_\varphi^1(f) = \tilde{\rho}_\varphi^1(f)$.*

Remark 3. *For an analytic function f in the unit disc Δ and $\varphi \in \Phi$, the equality $\rho_\varphi^0(f) = \tilde{\rho}_\varphi^0(f)$ is not always verified. As a counter-example, we consider the function $\varphi(.) = \log_2(.) = \log\log(.)$ which belongs to the class Φ and satisfies $\rho_{\log_2}^0(f) = \rho_1(f)$ and $\tilde{\rho}_{\log_2}^0(f) = \tilde{\rho}_1(f)$. Since $\rho_1(f) \leq \tilde{\rho}_1(f) \leq \rho_1(f) + 1$ (see [21] or [15, Proposition 2.2.2]), then $\rho_{\log_2}^0(f) \leq \tilde{\rho}_{\log_2}^0(f) \leq \rho_{\log_2}^0(f) + 1$.*

Lemma 10. *[11] Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $F \subset [0,1)$ with $\int_F \frac{dr}{1-r} < +\infty$. Then, there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

Lemma 11. *[11, 21] Let f be a meromorphic function in the unit disc Δ and $k \in \mathbb{N}$. Then we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right)$$

possibly outside of an exceptional set $F \subset [0,1)$ with $\int_F \frac{dr}{1-r} < +\infty$. If $\rho_1(f) < +\infty$, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\frac{1}{1-r}\right).$$

4. Proofs of the main results

Proof of Theorem 1. (i) The definition of ρ_φ^1-order implies that for any given $\varepsilon > 0$ and for r sufficiently large in $[0, 1)$, we have

$$T(r, f_i) \leq \varphi^{-1}\left((\rho_\varphi^1(f_i) + \varepsilon)\log\frac{1}{1-r}\right), \ (i = 1, 2). \tag{13}$$

Hence

$$T(r, f_1 + f_2) \leq T(r, f_1) + T(r, f_2) + O(1)$$

$$= O\left(\varphi^{-1}\left((\max\{\rho_\varphi^1(f_1), \rho_\varphi^1(f_2)\} + \varepsilon)\log\frac{1}{1-r}\right)\right).$$

By the monotonicity of φ and (12), we obtain

$$\frac{\varphi(T(r, f_1 + f_2))}{\log\frac{1}{1-r}} \leq \max\{\rho_\varphi^1(f_1), \rho_\varphi^1(f_2)\} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then we get $\rho_\varphi^1(f_1 + f_2) \leq \max\{\rho_\varphi^1(f_1), \rho_\varphi^1(f_2)\}$. $(ii) - (iv)$ follow immediately from the properties

$$T(r, f_1\, f_2) \leq T(r, f_1) + T(r, f_2),$$

$$T(r, af_1) = T(r, f_1) + O(1), (a \in \mathbb{C}^*),$$

$$T\left(r, \frac{1}{f_1}\right) = T(r, f_1) + O(1)$$

and the definitions of φ-order and φ-type. Similar proofs for $j = 0$.

Proof of Theorem 2. By Theorem 1, we have $\rho_\varphi^j(f_1 + f_2) \leq \rho_\varphi^j(f_2)$ and then

$$\rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2 - f_1) \leq \max\{\rho_\varphi^j(f_1 + f_2), \rho_\varphi^j(f_1)\}.$$

Suppose that $\rho_\varphi^j(f_1) > \rho_\varphi^j(f_1 + f_2)$, then

$$\rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2 - f_1) \leq \max\{\rho_\varphi^j(f_1 + f_2), \rho_\varphi^j(f_1)\} = \rho_\varphi^j(f_1)$$

this contradicts the hypothesis $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$. Then $\rho_\varphi^j(f_2) \leq \rho_\varphi^j(f_1 + f_2)$ and therefore $\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_2), (j = 0, 1)$. Similarly, again by Theorem 1 and the fact that

$$\rho_\varphi^j(f_2) = \rho_\varphi^j\left(f_1\, f_2\, \frac{1}{f_1}\right) \leq \max\{\rho_\varphi^j(f_1\, f_2), \rho_\varphi^j(f_1)\} = \rho_\varphi^j(f_1),$$

we obtain $\rho_\varphi^j(f_1\, f_2) = \rho_\varphi^j(f_2),\, (j = 0, 1)$.

Proof of Theorem 3. (i) It is obvious from the definitions of $\tilde{\rho}_\varphi^j(f_1)$ and $\tilde{\tau}_\varphi^j(f_1)$, ($j = 0, 1$). ($ii$) holds by using the known inequalities $|f_1 + f_2| \leq 2 \max\{|f_1|, |f_2|\}$ and $|f_1\, f_2| \leq [\max\{|f_1|, |f_2|\}]^2$ and applying (12). We can obtain (iii) by analogous discussion as in the proof of the first part of Theorem 2.

Proof of Theorem 4. We only give the proofs for $j = 1$, the proofs for $j = 0$ are analogous. (i) The definition of τ_φ^1-type implies that there exists a sequence $\{r_n, n \geq 1\}$ tending to 1^- satisfying $1 - (1 - \frac{1}{n})(1 - r_n) < r_{n+1}$ such that for any given $\varepsilon > 0$ we have

$$T(r_n, f_2) \geq \varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_2) - \varepsilon}{(1 - r_n)^{\rho_\varphi^1(f_2)}}\right)\right) \tag{14}$$

and for $r \longrightarrow 1^-$ there holds

$$T(r, f_i) \leq \varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_i) + \varepsilon}{(1 - r)^{\rho_\varphi^1(f_i)}}\right)\right), \, (i = 1, 2). \tag{15}$$

By Proposition 1 and the fact that

$$T(r, f_1 + f_2) \geq T(r, f_2) - T(r, f_1) - \log 2, \tag{16}$$

we obtain

$$T(r_n, f_1 + f_2) \geq \varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_2) - \varepsilon}{(1 - r_n)^{\rho_\varphi^1(f_2)}}\right)\right) - \varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_1) + \varepsilon}{(1 - r_n)^{\rho_\varphi^1(f_1)}}\right)\right) - \log 2$$

$$\geq \varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_2) - 2\varepsilon}{(1 - r_n)^{\rho_\varphi^1(f_2)}}\right)\right) \tag{17}$$

provided ε such that $0 < 2\varepsilon < \tau_\varphi^1(f_2) - \tau_\varphi^1(f_1)$. It follows from Theorem 2 that $\rho_\varphi^1(f_1 + f_2) = \rho_\varphi^1(f_2)$. Then, by (17) and the monotonicity of φ, we have

$$(1 - r_n)^{\rho_\varphi^1(f_1 + f_2)} \exp\{\varphi(T(r_n, f_1 + f_2))\} \geq \tau_\varphi^1(f_2) - 2\varepsilon.$$

By arbitrariness of $\varepsilon\,(0 < 2\varepsilon < \tau_\varphi^1(f_2) - \tau_\varphi^1(f_1))$, we obtain

$$\tau_\varphi^1(f_1 + f_2) \geq \tau_\varphi^1(f_2). \tag{18}$$

On the other hand, since $\rho_\varphi^1(f_1 + f_2) = \rho_\varphi^1(f_2) > \rho_\varphi^1(f_1) = \rho_\varphi^1(-f_1)$, then it follows from (18) that $\tau_\varphi^1(f_2) = \tau_\varphi^1(f_1 + f_2 - f_1) \geq \tau_\varphi^1(f_1 + f_2)$ and therefore $\tau_\varphi^1(f_1 + f_2) = \tau_\varphi^1(f_2)$.

By a similar discussion as in the above proof such that

$$T(r, f_1 f_2) \geq T(r, f_1) - T(r, f_2) + O(1), \tag{19}$$

we obtain

$$\tau_\varphi^1(f_1 f_2) \geq \tau_\varphi^1(f_2). \tag{20}$$

Since $\rho_\varphi^1(f_1 f_2) = \rho_\varphi^1(f_2) > \rho_\varphi^1(f_1) = \rho_\varphi^1\left(\frac{1}{f_1}\right)$, then by (20) we have $\tau_\varphi^1(f_2) = \tau_\varphi^1\left(f_1 f_2 \frac{1}{f_1}\right) \geq \tau_\varphi^1(f_1 f_2)$ and therefore $\tau_\varphi^1(f_1 f_2) = \tau_\varphi^1(f_2)$.

(ii) It follows from the assumption $0 < \rho_\varphi^1(f_1) = \rho_\varphi^1(f_2) = \rho_\varphi^1(f_1 + f_2) < +\infty$, (15) and Proposition 1 that

$$T(r, f_1 + f_2) \leq T(r, f_1) + T(r, f_2) + O(1)$$

$$\leq \varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_1) + \varepsilon}{(1-r)^{\rho_\varphi^1(f_1)}}\right)\right) + \varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_2) + \varepsilon}{(1-r)^{\rho_\varphi^1(f_2)}}\right)\right) + O(1)$$

$$\leq \varphi^{-1}\left(\log\left(\frac{\max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\} + 3\varepsilon}{(1-r)^{\rho_\varphi^1(f_1 + f_2)}}\right)\right).$$

By the monotonicity of φ and arbitrariness of $\varepsilon > 0$ we obtain

$$\tau_\varphi^1(f_1 + f_2) \leq \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}. \tag{21}$$

Moreover, we may suppose without loss of generality that $\tau_\varphi^1(f_1) < \tau_\varphi^1(f_2)$. Then, by (21) and since $\rho_\varphi^1(f_1 + f_2) = \rho_\varphi^1(f_1) = \rho_\varphi^1(-f_1)$, we have

$$\tau_\varphi^1(f_2) = \tau_\varphi^1(f_1 + f_2 - f_1) \leq \max\{\tau_\varphi^1(f_1 + f_2), \tau_\varphi^1(f_1)\} = \tau_\varphi^1(f_1 + f_2)$$

and therefore $\tau_\varphi^1(f_1 + f_2) = \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}$.

(iii) Since $T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2)$, we obtain by a similar discussion as in (21) that

$$\tau_\varphi^1(f_1 f_2) \leq \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}. \tag{22}$$

Moreover, if we suppose that $\tau_\varphi^1(f_1) < \tau_\varphi^1(f_2)$, then by (22) and since $\rho_\varphi^1(f_1 f_2) = \rho_\varphi^1(f_1) = \rho_\varphi^1\left(\frac{1}{f_1}\right)$, we obtain

$$\tau_\varphi^1(f_2) = \tau_\varphi^1\left(f_1 f_2 \frac{1}{f_1}\right) \leq \max\{\tau_\varphi^1(f_1 f_2), \tau_\varphi^1(f_1)\} = \tau_\varphi^1(f_1 f_2).$$

Hence, $\tau_\varphi^1(f_1\,f_2) = \max\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\}$.

Proof of Theorem 5. We have from Theorem 1 that

$$\rho_\varphi^j(f_1 + f_2) \leq \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) \quad \text{and} \quad \rho_\varphi^j(f_1\,f_2) \leq \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2).$$

Analogous reasoning as in the proof of Theorem 4 especially (16) and (19) leads to

$$\rho_\varphi^j(f_1 + f_2) \geq \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1) \quad \text{and} \quad \rho_\varphi^j(f_1\,f_2) \geq \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1).$$

Hence, (5) holds. On the other hand, (6) is obvious from (5) and Theorem 4.

Proof of Theorem 6. We will prove the theorem for $j = 0$, the proofs for $j = 1$ are analogous and follow also from Proposition 2 and Theorem 4. (i) The definition of the $\tilde{\tau}_\varphi^0$-type implies that for any $\varepsilon > 0$ there exists a sequence $\{r_n : n \geq 1\}$ tending to 1^- satisfying $1 - (1 - \frac{1}{n})(1 - r_n) < r_{n+1}$ such that for any given $\varepsilon > 0$ we have

$$M(r_n, f_2) \geq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_2) - \varepsilon}{(1 - r_n)^{\tilde{\rho}_\varphi^0(f_2)}}\right)\right) \tag{23}$$

and for all $r \longrightarrow 1^-$

$$M(r, f_i) \leq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_i) + \varepsilon}{(1 - r)^{\tilde{\rho}_\varphi^0(f_i)}}\right)\right), \quad (i = 1, 2). \tag{24}$$

We consider a sequence $\{z_n : n \geq 1\}$ satisfying $|f_2(z_n)| = M(r_n, f_2)$ in each circle $|z| = r_n$. Then, by (23), (24) and Proposition 1 we obtain

$$M(r_n, f_1 + f_2) \geq |f_1(z_n) + f_2(z_n)| \geq |f_2(z_n)| - |f_1(z_n)|$$

$$\geq M(r_n, f_2) - M(r_n, f_1)$$

$$\geq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_2) - \varepsilon}{(1 - r_n)^{\tilde{\rho}_\varphi^0(f_2)}}\right)\right) - \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_1) + \varepsilon}{(1 - r_n)^{\tilde{\rho}_\varphi^0(f_1)}}\right)\right)$$

$$\geq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_2) - 2\varepsilon}{(1 - r_n)^{\tilde{\rho}_\varphi^0(f_2)}}\right)\right) \tag{25}$$

provided ε such that $0 < 2\varepsilon < \tilde{\tau}_\varphi^0(f_2) - \tilde{\tau}_\varphi^0(f_1)$ and $r_n \longrightarrow 1^-$. We have from Theorem 3 that $\tilde{\rho}_\varphi^0(f_1 + f_2) = \tilde{\rho}_\varphi^0(f_2)$. Thus, by the monotonicity of φ, (25) and (12), we obtain

$$(1 - r_n)^{\tilde{\rho}_\varphi^0(f_1 + f_2)} \exp\{\varphi(M(r_n, f_1 + f_2))\} \geq \tilde{\tau}_\varphi^0(f_2) - 2\varepsilon.$$

By arbitrariness of ε $(0 < 2\varepsilon < \tilde{\tau}_\varphi^0(f_2) - \tilde{\tau}_\varphi^0(f))$, we get

$$\tilde{\tau}_\varphi^0(f_1 + f_2) \geq \tilde{\tau}_\varphi^0(f_2). \tag{26}$$

On the other hand, by applying (26) and since $\tilde{\rho}_\varphi^0(f_1 + f_2) = \tilde{\rho}_\varphi^0(f_2) > \tilde{\rho}_\varphi^0(f_1) = \tilde{\rho}_\varphi^0(-f_1)$, then we obtain

$$\tilde{\tau}_\varphi^0(f_2) = \tilde{\tau}_\varphi^0(f_1 + f_2 - f_1) \leq \tilde{\tau}_\varphi^0(f_1 + f_2)$$

and therefore $\tilde{\tau}_\varphi^0(f_1 + f_2) = \tilde{\tau}_\varphi^0(f_2)$.

(ii) It follows from the assumption $0 < \tilde{\rho}_\varphi^0(f_1) = \tilde{\rho}_\varphi^0(f_2) = \tilde{\rho}_\varphi^0(f_1 + f_2) < +\infty$, (24) and Proposition 1 that

$$M(r, f_1 + f_2) \leq M(r, f_1) + M(r, f_2)$$

$$\leq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_1) + \varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1+f_2)}}\right)\right) + \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_2) + \varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1+f_2)}}\right)\right)$$

$$\leq \varphi^{-1}\left(\log\left(\frac{\max\{\tilde{\tau}_\varphi^0(f_1), \tilde{\tau}_\varphi^0(f_2)\} + 2\varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1+f_2)}}\right)\right).$$

The monotonicity of φ and arbitrariness of $\varepsilon > 0$ yield

$$\tilde{\tau}_\varphi^0(f_1 + f_2) \leq \max\{\tilde{\tau}_\varphi^0(f_1), \tilde{\tau}_\varphi^0(f_2)\}. \tag{27}$$

Moreover, we may suppose without loss of generality that $\tilde{\tau}_\varphi^0(f_1) < \tilde{\tau}_\varphi^0(f_2)$. Since $\tilde{\rho}_\varphi^0(f_1 + f_2) = \tilde{\rho}_\varphi^0(f_1) = \tilde{\rho}_\varphi^0(-f_1)$, then by applying (27) we get

$$\tilde{\tau}_\varphi^0(f_2) = \tilde{\tau}_\varphi^0(f_1 + f_2 - f_1) \leq \max\{\tilde{\tau}_\varphi^0(f_1 + f_2), \tilde{\tau}_\varphi^0(f_1)\} = \tilde{\tau}_\varphi^0(f_1 + f_2). \tag{28}$$

We deduce from (27) and (28) that $\tilde{\tau}_\varphi^0(f_1 + f_2) = \max\{\tilde{\tau}_\varphi^0(f_1), \tilde{\tau}_\varphi^0(f_2)\}$.

(iii) By the assumption $0 < \tilde{\rho}_\varphi^0(f_1) = \tilde{\rho}_\varphi^0(f_2) = \tilde{\rho}_\varphi^0(f_1 f_2) < +\infty$ and (24) it follows that

$$M(r, f_1 f_2) \leq M(r, f_1)\, M(r, f_2)$$

$$\leq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_1) + \varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1 f_2)}}\right)\right) \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_\varphi^0(f_2) + \varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1 f_2)}}\right)\right)$$

$$\leq \left[\varphi^{-1}\left(\log\left(\frac{\max\{\tilde{\tau}_\varphi^0(f_1), \tilde{\tau}_\varphi^0(f_2)\} + \varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1 f_2)}}\right)\right)\right]^2.$$

By the monotonicity of φ and (12), we obtain

$$\varphi(M(r, f_1 f_2)) \leq (1 + o(1)) \log\left(\frac{\max\{\tilde{\tau}_\varphi^0(f_1), \tilde{\tau}_\varphi^0(f_2)\} + \varepsilon}{(1 - r)^{\tilde{\rho}_\varphi^0(f_1 f_2)}}\right)$$

$$\leq \log\left(\frac{\max\{\tilde{\tau}_\varphi^0(f_1), \tilde{\tau}_\varphi^0(f_2)\} + 2\varepsilon}{(1 - r)^{\tilde{\rho}_\varphi^0(f_1 f_2)}}\right).$$

Since $\varepsilon > 0$ is arbitrary, we deduce $\tilde{\tau}_\varphi^0(f_1 f_2) \leq \max\{\tilde{\tau}_\varphi^0(f_1), \tilde{\tau}_\varphi^0(f_2)\}$.

Proof of Theorem 7. We have from Theorem 3 that $\tilde{\rho}_\varphi^j(f_1 + f_2) \leq \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2)$. To prove the converse inequality, we assume the contrary $\tilde{\rho}_\varphi^j(f_1 + f_2) < \tilde{\rho}_\varphi^j(f_1)$. Then, by (7) we have $\tilde{\tau}_\varphi^j(f_1) = \tilde{\tau}_\varphi^j(f_1 + f_2 - f_2) = \tilde{\tau}_\varphi^j(f_2)$ which contradicts our hypothesis $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2)$. Hence, $\tilde{\rho}_\varphi^j(f_1 + f_2) \geq \tilde{\rho}_\varphi^j(f_1)$ and therefore (8) holds. It is clear that (9) follows immediately from our assertion (8) and the second part of Theorem 6.

Proof of Theorem 8. Denote $\rho_\varphi^1(f) = \rho$ and $\rho_\varphi^1(f') = \rho'$. By (13), Lemma 11 and (10), for all $r \longrightarrow 1^-$, $r \notin F$, where F is a set satisfying $\int_F \frac{dr}{1-r} < +\infty$, we have

$$T(r, f') = m(r, f') + N(r, f') \leq m(r, \frac{f'}{f}) + m(r, f) + 2N(r, f)$$

$$\leq m(r, \frac{f'}{f}) + 2T(r, f) = O\left(\log \varphi^{-1}\left((\rho + \varepsilon) \log \frac{1}{1 - r}\right) + \log \frac{1}{1 - r}\right)$$

$$+ O\left(\varphi^{-1}\left((\rho + \varepsilon) \log \frac{1}{1 - r}\right)\right) = O\left(\varphi^{-1}\left((\rho + \varepsilon) \log \frac{1}{1 - r}\right)\right).$$

By the monotonicity of φ and (12), we obtain

$$\varphi(T(r, f')) \leq (1 + o(1))\left((\rho + \varepsilon) \log \frac{1}{1 - r}\right) \leq (\rho + 2\varepsilon) \log \frac{1}{1 - r}, \quad r \notin F.$$

By Lemma 10, since $\varepsilon > 0$ is arbitrary, there holds $\rho_\varphi^1(f') \leq \rho_\varphi^1(f) = \rho$.
    Now, we prove $\rho_\varphi^1(f) \leq \rho_\varphi^1(f')$. Indeed, by following analogous demonstration techniques as in [22, Theorem 4.1] or in [5], we can obtain the following estimate

$$T(r, f) < O\left(T(\frac{1 + r}{2}, f') + \log \frac{1}{1 - r}\right), \quad r \longrightarrow 1^-.$$

By (13) and (10) we have that $T(r, f') \leq \varphi^{-1}\left((\rho' + \varepsilon)\log\frac{1}{1-r}\right)$ and then

$$T(r, f) \leq O\left(\varphi^{-1}\left((\rho' + \varepsilon)\log\frac{2}{1-r}\right) + \log\frac{1}{1-r}\right)$$

$$= O\left(\varphi^{-1}\left((\rho' + 2\varepsilon)\log\frac{1}{1-r}\right)\right).$$

By the monotonicity of φ and (12), we obtain

$$\varphi(T(r, f)) \leq (1 + o(1))\left((\rho' + 2\varepsilon)\log\frac{1}{1-r}\right) \leq (\rho' + 3\varepsilon)\log\frac{1}{1-r}.$$

By arbitrariness of $\varepsilon > 0$ we deduce that $\rho_\varphi^1(f) \leq \rho_\varphi^1(f') = \rho'$ and therefore Theorem 8 is proved.

Proof of Theorem 9. It is well-known that for an analytic function f we have

$$f(z) = f(0) + \int_0^z f'(\zeta)d\zeta.$$

Then, for $|z| = r < 1$ we obtain

$$M(r, f) \leq |f(0)| + rM(r, f') \leq |f(0)| + M(r, f').$$

Clearly, it follows from the monotonicity of φ that $\tilde{\rho}_\varphi^0(f) \leq \tilde{\rho}_\varphi^0(f')$. Now, we prove the reverse inequality. Taking $z_0 = re^{i\theta}$ for any $|z| = r \in [0, 1)$ such that $M(r, f') = |f'(z_0)|$ and $R = \frac{1+r}{2}$. By Cauchy's integral formula, we have

$$f'(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta,$$

where $\Gamma = \{\zeta : |\zeta - z_0| = R - r\}$. Set $\zeta - z_0 = (R - r)e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, $d\zeta = (R - r)ie^{i\theta}d\theta$. Since $\max\{|f(\zeta)| : \zeta \in \Gamma\} \leq M(R, f)$, then we obtain

$$M(r, f') = |f'(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(\zeta)|}{|\zeta - z_0|^2}(R - r)d\theta$$

$$\leq \frac{M(R, f)}{R - r} = \frac{2}{1 - r}M(\frac{1+r}{2}, f). \tag{29}$$

We consider now a set $E \subset [0, 1)$ such that

$$E = \left\{r \in [0, 1) : \log\frac{2}{1-r} < \log M(\frac{1+r}{2}, f)\right\}.$$

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Then, by the monotonicity of φ, (29) and (12) we obtain

$$\frac{\varphi(M(r, f'))}{\log\frac{1}{1-r}} \leq \frac{\varphi\left(\exp\left\{\log\frac{2}{1-r} + \log M(\frac{1+r}{2}, f)\right\}\right)}{\log\frac{1}{1-r}}$$

$$\leq \frac{\varphi\left(\exp\left\{2\log M(\frac{1+r}{2}, f)\right\}\right)}{\log\frac{1}{1-r}} = \frac{\varphi\left(\exp\left\{2\log M(R, f)\right\}\right)}{\log\frac{2}{1-R}}$$

$$\leq \frac{(1 + o(1))\varphi(M(R, f))}{\log\frac{1}{1-R}}.$$

Thus,

$$\tilde{\rho}_\varphi^0(f') = \limsup_{r \longrightarrow 1^-} \frac{\varphi(M(r, f'))}{-\log(1-r)} \leq \limsup_{R \longrightarrow 1^-} \frac{(1 + o(1))\varphi(M(R, f))}{\log\frac{1}{1-R}} = \tilde{\rho}_\varphi^0(f)$$

holds on E. It remains to estimate $\dfrac{\varphi(M(r, f'))}{\log\frac{1}{1-r}}$ on E^c such that

$$E^c = \left\{r \in [0; 1) : \log\frac{2}{1-r} \geq \log M(\frac{1+r}{2}, f)\right\}.$$

In fact, by Karamata's theorem (see [19]) we have $\varphi(e^t) = t^{o(1)}$ as $t \longrightarrow +\infty$. Then, for $r \longrightarrow 1^-$ there holds

$$\frac{\varphi(M(r, f'))}{\log\frac{1}{1-r}} \leq \frac{\varphi\left(\exp\left\{\log\frac{2}{1-r} + \log\frac{2}{1-r}\right\}\right)}{\log\frac{1}{1-r}} = \frac{\left(2\log\frac{2}{1-r}\right)^{o(1)}}{\log\frac{1}{1-r}}.$$

Hence,

$$\tilde{\rho}_\varphi^0(f') := \limsup_{r \longrightarrow 1^-} \frac{\varphi(M(r, f'))}{-\log(1-r)} = 0.$$

We deduce that $\tilde{\rho}_\varphi^0(f') \leq \tilde{\rho}_\varphi^0(f)$ and therefore $\tilde{\rho}_\varphi^0(f') = \tilde{\rho}_\varphi^0(f)$.

The case for $j = 1$ follows immediately from Proposition 2 and Theorem 8.

5. Conclusion

In our present paper, we have investigated the concepts of the φ-order and the φ-type and discussed their nice properties. We consider our results very helpful and will certainly contribute in future possible studies of the growth of solutions of equation (1). Moreover, the general concept of (α, β)-order introduced by Sheremeta [20] shows that there are still many things can be done in this subject.

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