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ON STAR COLORING OF DEGREE SPLITTING OF TENSOR PRODUCT OF GRAPHS

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Abstract. A star coloring of a graph G is a proper vertex coloring which states that every path on four vertices in G is in excess of two dissimilar colors. The star chromatic number $\chi_s(G)$ of G is the fewest number of colors that require to star color G. Let $G = (V, E)$ graph with V_i denote the set of all vertices of degree i, the degree splitting graph $DS(G)$ is obtained from G by adding new vertices w_i for each V_i with $|V_i| \geq 2$, and joining w_i with every vertex in V_i. In this note, we obtain the star chromatic number of degree splitting of tensor product of path with complete graph, wheel graph, cycle graph, complete bipartite graph and path graph.

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1. Introduction

Throughout this paper, the graphs are considered to be finite, simple, connected and undirected [2, 6].

The idea of star chromatic number was introduced by Grünbaum in 1973 [5]. A star coloring of a graph G is a proper vertex coloring which states that every path on four vertices in G is in excess of two dissimilar colors. The star chromatic number $\chi_s(G)$ of G is the fewest number of colors that require to star color G.

The exact value of the star chromatic number for trees, cycles, complete bipartite graphs, outer planar graphs and 2-dimensional grids was showed by Guillaume Fertin et al. [4] and also they gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids ($d \geq 3$), d-dimensional tori ($d \geq 2$), graphs with bounded treewidth and cubic graphs.

Albertson et al. [1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite. Coleman and More [3] proved that finding an optimal star coloring is NP-hard and remain so even for bipartite graphs.

2. Preliminaries

Definition 1. *A graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph with n vertices is denoted by K_n.*

Definition 2. *A trail is called a path P_m if all its vertices are distinct.*

Definition 3. *A closed trail whose origin and internal vertices are distinct is called a cycle C_n.*

Definition 4. *A wheel W_n is defined as $K_1 + C_{n-1}$, $n \geq 4$.*

Definition 5. *A bipartite graph G is a graph whose vertex $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G has one end in V_1 and the other end in V_2; (V_1, V_2) is called a bipartite of G. Further, if every vertex of V_1 is joined to all the vertices of V_2, then G is called a complete bipartite graph. The complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.*

Definition 6. *[7] Given a graph $G = (V, E)$ with $V(G) = S_1 \cup S_2 \cup S_3 \cup \ldots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. Thus to construct the degree splitting graph of G, add new vertices $w_1, w_2, \ldots w_t$ and join w_i to each vertex of S_i for $1 \leq i \leq t$. The degree splitting graph of G is denoted by $DS(G)$.*

Definition 7. *[8] The tensor product of two graphs G_1 and G_2 denoted by $G_1 \times G_2$ has the vertex set $V(G_1 \times G_2)$ and the edge set*

$$E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}.$$

3. Main Results

In this section, we construe the star coloring of degree splitting of tensor product of path with complete graph, wheel graph, cycle graph, complete bipartite graph and path graph.

3.1. Star coloring of degree splitting of tensor product of path with complete graph

Theorem 1. *Let P_m be a path graph with $m \geq 4$ and K_n be a complete graph with $n \geq 3$, then*

$$\chi_s(DS(P_m \times K_n)) = 2n + 2.$$

Proof. Let P_m be a path graph and K_n be a complete graph. Let the tensor product of path with complete graph be denoted by $P_m \times K_n$. Then the vertex set of $|V(P_m \times K_n)| = mn$. We have,

$$V(P_m \times K_n) = \begin{Bmatrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_n, v_1) & (u_n, v_2) & \dots & (u_n, v_n) \end{Bmatrix} = S_1 \cup S_2$$

where

$$S_1 = \{(u_i, v_j) : i = 1, m \text{ and } 1 \le j \le n\}.$$

and

$$S_2 = \{(u_i, v_j) : 2 \le i \le m-1 \text{ and } 1 \le j \le n\}.$$

To obtain $DS(P_m \times K_n)$ from $P_m \times K_n$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2, respectively. Thus we get

$$V(DS(P_m \times K_n) = V(P_m \times K_n) \cup \{w_1, w_2\}.$$

Now, we assign the star coloring as follows:
For $1 \le i \le m$ and for every $1 \le j \le n$

When $i \equiv 1 \pmod 3$

$$c(u_i, v_j) = 1$$

When $i \equiv 2 \pmod 3$

$$c(u_i, v_j) = j + 1$$

When $i \equiv 0 \pmod 3$

$$c(u_i, v_j) = n + j + 1$$

and also assign

$$c(w_1) = c(w_2) = 2n + 2.$$

Thus the star coloring for degree splitting of tensor product of path with complete graph is $2n + 2$.

3.2. Star coloring of degree splitting of tensor product of path with wheel graph

Theorem 2. *Let P_m be a path graph with $m \geq 4$ and W_n be a complete graph with $n \geq 4$, then*

$$\chi_s(DS(P_m \times W_n)) = 2n + 2.$$

Proof. Let P_m be a path graph and W_n be a wheel graph. Let the tensor product of path with wheel graph be denoted by $P_m \times W_n$. Then the vertex set of $|V(P_m \times W_n)| = mn$.
We have,

$$V(P_m \times W_n) = \begin{Bmatrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_n, v_1) & (u_n, v_2) & \dots & (u_n, v_n) \end{Bmatrix} = S_1 \cup S_2 \cup S_3 \cup S_4$$

where

$$S_1 = \{(u_i, v_j) : i = 1, m \textit{ and } 1 \leq j \leq n-1\}.$$

$$S_2 = \{(u_i, v_j) : 2 \leq i \leq m-1 \textit{ and } 1 \leq j \leq n-1\},$$

$$S_3 = \{(u_i, v_j) : 2 \leq i \leq m-1 \textit{ and } j = n\}$$

and

$$S_4 = \{(u_i, v_j) : i = 1, m \textit{ and } j = n\}.$$

To obtain $DS(P_m \times W_n)$ from $P_m \times W_n$, we add four vertices w_1, w_2, w_3 and w_4 corresponding to S_1, S_2, S_3 and S_4, respectively. Thus we get

$$V(DS(P_m \times W_n) = V(P_m \times W_n) \cup \{w_1, w_2, w_3, w_4\}.$$

Now, we assign the star coloring as follows:
For $1 \leq i \leq m$ and for every $1 \leq j \leq n$
When $i \equiv 1 \pmod 3$

$$c(u_i, v_j) = 1$$

When $i \equiv 2 \pmod 3$

$$c(u_i, v_j) = j + 1$$

When $i \equiv 0 \pmod 3$

$$c(u_i, v_j) = n + j + 1$$

and also assign

$$c(w_1) = c(w_2) = c(w_3) = c(w_4) = 2n + 2.$$

Thus the star coloring for degree splitting of tensor product of path with wheel graph is $2n + 2$.

3.3. Star coloring of degree splitting of tensor product of path with cycle graph

Theorem 3. *Let P_m be a path graph with $m \geq 4$ and C_n be a cycle graph with $n \geq 3$, then*

$$\chi_s(DS(P_m \times C_n)) = 7.$$

Proof. Let P_m be a path graph and C_n be a cycle graph. Let the tensor product of path with cycle graph be denoted by $P_m \times C_n$. Then the vertex set of $|V(P_m \times C_n)| = mn$.
We have,

$$V(P_m \times C_n) = \begin{Bmatrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_n, v_1) & (u_n, v_2) & \dots & (u_n, v_n) \end{Bmatrix} = S_1 \cup S_2$$

where

$$S_1 = \{(u_i, v_j) : i = 1, m \text{ and } 1 \leq j \leq n\}$$

and

$$S_2 = \{(u_i, v_j) : 2 \leq i \leq m - 1; 1 \leq j \leq n\},$$

To obtain $DS(P_m \times C_n)$ from $P_m \times C_n$, we add two vertices w_1, and w_2, corresponding to S_1 and S_2, respectively. Thus we get

$$V(DS(P_m \times C_n) = V(P_m \times C_n) \cup \{w_1, w_2\}.$$

Now, we assign the star coloring as follows:
For $1 \leq i \leq m$
Case (i): When $n \equiv 0 \pmod 3$ and When $n \equiv 1 \pmod 3$
For $i \equiv 1 \pmod 3$

$$c(u_i, v_j) = 1$$

For $i \equiv 2 \pmod 3$

$$c(u_i, v_j) = \begin{cases} 2, & if \quad j \equiv 1 \pmod 3 \\ 4, & if \quad j \equiv 2 \pmod 3 \\ 6, & if \quad j \equiv 0 \pmod 3 \end{cases}$$

For $i \equiv 0 \pmod 3$

$$c(u_i, v_j) = \begin{cases} 3, & if \quad j \equiv 1 \pmod 3 \\ 4, & if \quad j \equiv 2 \pmod 3 \\ 5, & if \quad j \equiv 0 \pmod 3 \end{cases}$$

Case (ii): When $n \equiv 2 \pmod 3$
For every $1 \leq j \leq n-1$
For $i \equiv 1 \pmod 3$

$$c(u_i, v_j) = 1$$

For $i \equiv 2 \pmod 3$

$$c(u_i, v_j) = \begin{cases} 2, & if \quad j \equiv 1 \pmod 3 \\ 4, & if \quad j \equiv 2 \pmod 3 \\ 6, & if \quad j \equiv 0 \pmod 3 \end{cases}$$

For $i \equiv 0 \pmod 3$

$$c(u_i, v_j) = \begin{cases} 3, & if \quad j \equiv 1 \pmod 3 \\ 4, & if \quad j \equiv 2 \pmod 3 \\ 5, & if \quad j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_i, v_n) = \begin{cases} 1, & if \quad j \equiv 1 \pmod 3 \\ 6, & if \quad j \equiv 2 \pmod 3 \\ 5, & if \quad j \equiv 0 \pmod 3 \end{cases}$$

and also assign

$$c(w_1) = c(w_2) = 7.$$

Thus the star coloring for degree splitting of tensor product of path with cycle graph is 7 if $n = 3k$. This completes the proof of the theorem.

3.4. Star coloring of degree splitting of tensor product of path with complete bipartite graph

Theorem 4. *Let P_m be a path graph with $m \geq 4$ and $K_{n1,n2}$ be a complete bipartite graph with $n1 \geq 2$ and $n2 \geq 3$, then*

$$\chi_s(DS(P_m \times K_{n1,n2})) = m + n1(or\ n2) + 2,\ if either\ m = n1\ or\ m = n2.$$

Proof. Let P_m be a path graph and $K_{n1,n2}$ be a complete bipartite graph. Let the tensor product of path with complete bipartite graph be denoted by $P_m \times K_{n1,n2}$. Then the vertex set of $|V(P_m \times K_{n1,n2})| = m(n1 + n2)$.
We have,

$$V(P_m \times K_{n1,n2}) = \begin{Bmatrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_n, v_1) & (u_n, v_2) & \dots & (u_n, v_n) \end{Bmatrix} = S_1 \cup S_2 \cup S_3 \cup S_4$$

where

$$S_1 = \{(u_i, v_j) : i = 1, m\ and\ 1 \leq j \leq n1\}.$$

$$S_2 = \{(u_i, v_j) : 2 \leq i \leq m - 1; 1 \leq j \leq n1\},$$

$$S_3 = \{(u_i, v_j) : 2 \leq i \leq m - 1; n1 + 1 \leq j \leq n1 + n2\}$$

and

$$S_4 = \{(u_i, v_j) : i = 1, m; n1 + 1 \leq j \leq n1 + n2\}.$$

To obtain $DS(P_m \times K_{n1,n2})$ from $P_m \times K_{n1,n2}$, we add four vertices w_1, w_2, w_3 and w_4 corresponding to S_1, S_2, S_3 and S_4, respectively. Thus we get

$$V(DS(P_m \times K_{n1,n2}) = V(P_m \times K_{n1,n2}) \cup \{w_1, w_2, w_3, w_4\}.$$

Now, we assign the star coloring as follows:
For $1 \leq i \leq m$
Case (i): If $n1 \leq n2$
For $i \equiv 1 \pmod 4$ and $i \equiv 2 \pmod 4$

$$c(u_i, v_j) = \begin{cases} j + 1, & if \quad 1 \leq j \leq n1 \\ 1, & if \quad n1 + 1 \leq j \leq n1 + n2 \end{cases}$$

For $i \equiv 3 \pmod 4$ and $i \equiv 0 \pmod 4$

$$c(u_i, v_j) = \begin{cases} n1 + j + 1, & if \quad 1 \leq j \leq n1 \\ 1, & if \quad n1 + 1 \leq j \leq n1 + n2 \end{cases}$$

Case (ii): If $n1 > n2$
For $i \equiv 1 \pmod 4$ and $i \equiv 2 \pmod 4$

$$c(u_i, v_j) = \begin{cases} 1, & if \quad 1 \leq j \leq n1 \\ 1 - n1 + j, & if \quad n1 + 1 \leq j \leq n1 + n2 \end{cases}$$

For $i \equiv 3 \pmod 4$ and $i \equiv 0 \pmod 4$

$$c(u_i, v_j) = \begin{cases} 1, & if \quad 1 \leq j \leq n1 \\ n2 - n1 + 1 + j, & if \quad n1 + 1 \leq j \leq n1 + n2 \end{cases}$$

and also assign

$$c(w_1) = c(w_2) = c(w_3) = c(w_4) = m + n1(\ or\ n2) + 2.$$

This completes the proof of the theorem.

3.5. Star coloring of degree splitting of tensor product of path with path graph

Theorem 5. *Let P_m and P_n be a path graph of order $m \geq 4$ and $n \geq 4$, respectively. Then*

$$\chi_s(DS(P_m \times P_n)) = 6.$$

Proof. Let P_m be a path graph and P_n be a path graph. Let the tensor product of path with path graph be denoted by $P_m \times P_n$. Then the vertex set of $|V(P_m \times P_n)| = mn$.
We have,

$$V(P_m \times P_n) = \begin{Bmatrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_m, v_1) & (u_m, v_2) & \dots & (u_m, v_n) \end{Bmatrix} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{(u_i, v_j) : i = 1, m \ and \ j = 1, n\},$$

$$S_2 = \{(u_i, v_j) : i = 1, m \ and \ 2 \leq j \leq n - 1\} \cup \{(u_i, v_j) : j = 1, n \ and \ 2 \leq i \leq m - 1\}$$

and

$$S_3 = \{(u_i, v_j) : 2 \leq i \leq m - 1;\ 2 \leq j \leq n - 1\}.$$

To obtain $DS(P_m \times P_n)$ from $P_m \times P_n$, we add three vertices w_1, w_2 and w_3 corresponding to S_1, S_2 and S_3, respectively. Thus we get

$$V(DS(P_m \times P_n)) = V(P_m \times P_n) \cup \{w_1, w_2, w_3\}.$$

Now, we assign the star coloring as follows:
For $i \equiv 1 \pmod 4$ and $i \equiv 3 \pmod 4$

$$c(u_i, v_j) = 1, \quad \forall\ j$$

For $i \equiv 2 \pmod 4$

$$c(u_i, v_j) = \begin{cases} 2, & if\ j \equiv 1\ and\ 2 \pmod 4 \\ 3, & if\ j \equiv 3\ and\ 4 \pmod 4 \end{cases}$$

For $i \equiv 0 \pmod 4$

$$c(u_i, v_j) = \begin{cases} 4, & if\ j \equiv 1\ and\ 2 \pmod 4 \\ 5, & if\ j \equiv 3\ and\ 4 \pmod 4 \end{cases}$$

and also assign

$$c(w_1) = c(w_2) = c(w_3) = 6.$$

Thus the star coloring of degree splitting of tensor product of path with path graph is 6, when $m \geq 4$ and $n \geq 4$. This completes the proof of the theorem.

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