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JACOBI SPECTRAL GALERKIN METHODS FOR THE EIGENVALUE PROBLEM OF A COMPACT INTEGRAL OPERATOR

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Abstract. In this paper, we propose Jacobi spectral Galerkin methods for solving the eigenvalue problem of a compact integral operator with the smooth kernel. We calculate the error bounds of approximated eigenelements with exacteigen elements both in weighted L^2 and infinity norm. We propose discrete Jacobi spectral Galerkin method where the integral operator and the inner product are approximated by using the quadrature formula with Jacobi weight. We evaluate the error bounds of approximated eigenelements with exact eigenelements in both the infinity and weighted norms. Numerical examples are presented to illustrate the theoretical results.

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1. Introduction

The eigenvalue problem of a compact integral operator defined on Banach spaces plays an important role in Engineering, Mathematical physics. Since the problem can not be solved explicitly, there are many numerical methods have been developed by many authors [1, 2, 3, 9, 13]. Projection methods are the important methods to solve this problem numerically. In [8], the eigenvalue problem of a compact integral operator have been solved by using piecewise polynomial methods. In piecewise polynomial based projection methods the number of partition should be increased to get more accurate approximate solution. Hence, one has to solve a large system of equations which take lots of time to compute. Recently, spectral methods are in attention of many researchers. Legendre and Jacobi spectral projection methods for eigenvalue problem have been developed by [10, 11, 12].

Jacobi spectral collocation and Jacobi spectral Galerkin methods have been applied on different types of integral equations[5, 4, 6]. In this paper we will provide a Jacobi spectral Galerkin method method for solving the eigenvalue problem of a compact integral operator with smooth kernel. By using the sufficiently accurate quadrature rule, the Jacobi spectral Galerkin methods give rise to discrete Jacobi spectral Galerkin methods. We will discuss on the convergence analysis of exact eigenelements with approximated eigenelements in both weighted L^2 and infinity norm.

Throughout this paper, we assume c is the generic constant.

2. Jacobi Spectral Galerkin Methods:

In this section, we will discuss on the Jacobi spectral Galerkin methods for solving eigenvalue problem of a compact linear integral operator with smooth kernels.
Define a weighted space

$$L^2_{w^{\alpha,\beta}}[-1,1] = \{v : v \ is \text{ measurable and } \|v\|_{w^{\alpha,\beta}} < \infty\},$$

where

$$\|v\|_{w^{\alpha,\beta}} = \left(\int_{-1}^{1} w^{\alpha,\beta}(x)v^2(x)dx\right)^{\frac{1}{2}}.$$

Consider the following integral operator $\mathcal{K}$ defined on $\mathbb{X} = L^2_{w^{\alpha,\beta}}[-1,1]$ or $L^\infty[-1,1]$ by

$$\mathcal{K}u(x) = \int_{-1}^{1} k(x,t)u(t)\,\mathrm{d}t, \quad x \in [-1,1], \tag{1}$$

where $k(.,.) \in \mathcal{C}([-1,1]\times[-1,1])$. Then $\mathcal{K} : \mathbb{X} \to \mathbb{X}$ is a compact operator.
We are interested to find $u \neq 0 \in \mathbb{X}$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that

$$\mathcal{K}u = \lambda u, \ \|u\| = 1. \tag{2}$$

Assume $\lambda \neq 0$ be the eigenvalue of $\mathcal{K}$ with algebraic multiplicity m and ascent l. Let $\Gamma \subset \rho(\mathcal{K})$ be a simple closed rectifiable curve such that $\sigma(\mathcal{K}) \cap int\,\Gamma = \{\lambda\},\ 0 \notin int\,\Gamma$, where $int\,\Gamma$ denotes the interior of Γ.

Let us demonstrate the numerical implementation of the Jacobi spectral Galerkin approach to solve (2). Denote $\mathbb{X}_n$ a space consisting of polynomials defined on $[-1,1]$ with degree not more than n. Let $\phi_j(x),\ j = 0,1,\cdots,n$ be the j-th Jacobi polynomial corresponding to the weight function $w^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ with $\alpha,\ \beta > -1$. As a result

$$\mathbb{X}_n = \text{span}\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}.$$

Our aim is to find $u_n \in \mathbb{X}_n$ and λ_n such that

$$\langle \mathcal{K}u_n, v_n\rangle_{w^{\alpha,\beta}} = \lambda_n \langle u_n, v_n\rangle_{w^{\alpha,\beta}},\ \ \forall v_n \in \mathbb{X}_n, \tag{3}$$

where $\langle u, v\rangle_{w^{\alpha,\beta}} = \int_{-1}^{1} w^{\alpha,\beta}(x)u(x)v(x)dx$ is the continuous inner product. Set $u_n(x) = \sum_{j=0}^{n} a_j\phi_j(x)$. Substituting it into equation (3) and taking $v_n = \phi_i(x)$, we obtain

$$\sum_{j=0}^{n} a_j \langle \mathcal{K}\phi_j, \phi_i\rangle_{w^{\alpha,\beta}} = \lambda_n \sum_{j=0}^{n} a_j \langle \phi_j, \phi_i\rangle_{w^{\alpha,\beta}}. \tag{4}$$

Now, we will give some useful lemmas and notations, which play a significant role in the convergence analysis later. First we define the orthogonal projection operator $\mathcal{P}_n^{\alpha,\beta} : \mathbb{X} \to \mathbb{X}_n$ which satisfies

$$\langle \mathcal{P}_n^{\alpha,\beta}u, v_n\rangle_{w^{\alpha,\beta}} = \langle u, v_n\rangle_{w^{\alpha,\beta}},\ \ \forall u \in \mathbb{X},\ \ v_n \in \mathbb{X}_n. \tag{5}$$

Define

$$H^m_{w^{\alpha,\beta}}[-1,1] = \{v : D^k v \in L^2_{w^{\alpha,\beta}}[-1,1],\ 0 \le k \le m\},$$

equipped with the norm

$$\|v\|_{H^m_{w^{\alpha,\beta}}[-1,1]} = \left(\sum_{k=0}^{m} \|D^k v\|^2_{w^{\alpha,\beta}}\right)^{\frac{1}{2}}$$

with $D^k v = \frac{d^k v}{dx^k}$. When $w^{\alpha,\beta}(x) = 1,\ L^2_{w^{\alpha,\beta}}[-1,1],\ H^m_{w^{\alpha,\beta}}[-1,1]$ and $\|.\|_{w^{\alpha,\beta}}$ are denoted simply by $L^2,\ H^m[-1,1]$ and $\|.\|$, respectively.

In bounding the above approximation error, only some of the L^2 norms appearing in the right hand side of above norm enter into play. Thus it is convenient to introduce the seminorm

$$|v|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]} = \left(\sum_{k=\min(m,n+1)}^{m} \|D^k v\|^2_{w^{\alpha,\beta}}\right)^{\frac{1}{2}}.$$

Now the spectral method for the eigenvalue problem (2) is defined as follows: find $u_n \in \mathbb{X}_n$ and $\lambda_n \in \mathbb{C}\backslash\{0\}$, such that

$$\mathcal{P}_n^{\alpha,\beta}\mathcal{K}u_n = \lambda_n u_n,\ \ \ \|u_n\| = 1. \tag{6}$$

The iterated eigenvector is defined by $\widehat{u}_n = \frac{1}{\lambda_n}\mathcal{K}u_n$.

We quote the following properties of $\mathcal{P}_n^{\alpha,\beta}$ from [4, 5, 6, 7].

Lemma 1. *Suppose that* $v \in H^m_{w^{\alpha,\beta}}[-1,1]$ *and* $m \geq 1$. *Then the following holds.*

(i) $\|\mathcal{P}_n^{\alpha,\beta}\|_\infty = \max_{\|u\|_\infty=1} \|\mathcal{P}_n^{\alpha,\beta} u\|_\infty = \begin{cases} \mathcal{O}(\log n), & -1 < \alpha, \beta < -\frac{1}{2}; \\ \mathcal{O}(n^{\frac{1}{2}+\max\{\alpha,\beta\}}), & \textit{otherwise.} \end{cases}$

(ii)*For any* $u \in \mathcal{C}[-1,1]$, *there exists a positive constant* ξ *independent of* n *such that*

$$\|\mathcal{P}_n^{\alpha,\beta} u\|_{w^{\alpha,\beta}} \leq \xi \|u\|_\infty.$$

(iii) *For* $\alpha, \beta > -1$, *and* $1 \leq l \leq m$,

$$\|v - \mathcal{P}_n^{\alpha,\beta} v\|_{w^{\alpha,\beta}} \leq c\, n^{-m} |v|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]},$$
$$\|v - \mathcal{P}_n^{\alpha,\beta} v\|_{H^l_{w^{\alpha,\beta}}} \leq c\, n^{2l-\frac{1}{2}-m} |v|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]}.$$

(iv) *For* $-1 < \alpha, \beta \leq 0$,,

$$\|v - \mathcal{P}_n^{\alpha,\beta} v\|_{L^\infty} \leq c\, n^{\frac{3}{4}-m} |v|_{H^m_{w^{\alpha,\beta}}[-1,1]}.$$

Next we discuss the convergence of approximated eigenvalues and eigenvectors to the exact eigenvalues and eigenvectors of the integral operator $\mathcal{K}$. To do this, we set the following notations.
Set $k(x,t) = k_x(t)$ for $x, t \in [-1,1]$.

$$|\mathcal{K}u|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]} = \Big\| \int_{-1}^1 \frac{\partial^m}{\partial x^m} k_x(t) u(t) \mathrm{d}t \Big\|_{w^{\alpha,\beta}} \leq c \Big\| \int_{-1}^1 |u(t)| \mathrm{d}t \Big\|_{w^{\alpha,\beta}} \leq c\|u\|_{w^{\alpha,\beta}}, \quad (7)$$

in which, we have implemented the fact that

$$\Big\| \int_{-1}^1 |u(t)| \mathrm{d}t \Big\|^2_{w^{\alpha,\beta}} \leq c\|u\|^2_{w^{\alpha,\beta}}.$$

Lemma 2. ([1]) *Let* $\mathbb{X}$ *be a Banach space and* $S \subset \mathbb{X}$ *is a relatively compact set. Assume that* $\mathcal{T}$ *and* $\mathcal{T}_n$ *are bounded linear operators from* $\mathbb{X}$ *to* $\mathbb{X}$ *satisfying* $\|\mathcal{T}_n\| \leq c$, *for all* $n \in \mathbb{N}$, *and for each* $x \in S$,

$$\|\mathcal{T}_n - \mathcal{T}\| \to 0 \ \textit{as} \ \ n \to \infty,$$

where c *is a constant independent of* n. *Then* $\|\mathcal{T}_n - \mathcal{T}\| \to 0$ *uniformly for all* $x \in S$.

We first show that the spectral projection operator $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}$ converges to $\mathcal{K}$ in both weighted L^2 norm and infinity norm.

Theorem 3. $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}$ *is norm-convergent to* $\mathcal{K}$ *in weighted* $L^2_{\omega^{\alpha,\beta}}$ *norm and infinity norm.*

Proof. Using Lemma-1 and equation (7), we obtain

$$\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u\|_{w^{\alpha,\beta}} \leq c\,n^{-m}|\mathcal{K}u|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]} \leq c\,n^{-m}\|u\|_{w^{\alpha,\beta}},$$
$$\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u\|_{L^\infty} \leq c\,n^{\frac{3}{4}-m}|\mathcal{K}u|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]} \leq c\,n^{\frac{3}{4}-m}\|u\|_{w^{\alpha,\beta}}.$$

From these estimates, it follows that $\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}\|_{w^{\alpha,\beta}} \to 0$ and $\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}\|_{L^\infty} \to 0$ for sufficiently large n.

Theorem 4. *$\mathcal{K}\mathcal{P}_n^{\alpha,\beta}$ is norm-convergent to $\mathcal{K}$ in weighted $L^2_{\omega^{\alpha,\beta}}$ norm.*

Proof. Using Cauchy-Schwarz inequality, it can be easily shown that

$$\|\mathcal{K}u\|_\infty \leq \|u\|_{w^{\alpha,\beta}}\|k\|_\infty\|1\|^{1/2}_{w^{-\alpha,-\beta}}.$$

Replacing u by $(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})u$ in above equation, we obtain

$$\begin{aligned}\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})u\|_{w^{\alpha,\beta}} &\leq \|1\|_{w^{\alpha,\beta}}\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})u\|_\infty \\ &\leq c\,\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})u\|_{w^{\alpha,\beta}} \leq c\,n^{-r}|u|_{H^{r;n}_{w^{\alpha,\beta}}[-1,1]}.\end{aligned}$$

This shows that $\mathcal{K}\mathcal{P}_n^{\alpha,\beta}$ is point-wise converges to $\mathcal{K}$ on $\mathcal{C}[-1,1]$. Let $B=\{u\in\mathbb{X}: \|u\|\leq 1\}$ be a closed unit ball in $\mathcal{C}[-1,1]$. Since $\mathcal{K}$ is compact operator, the set $S=\{\mathcal{K}u : u\in B\}$ is a relatively compact set in $\mathcal{C}[-1,1]$. Now using Lemma-2, we have

$$\begin{aligned}\|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}\|_{w^{\alpha,\beta}} &= \sup\{\|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}u\|_{w^{\alpha,\beta}} : u\in B\} \\ &= \sup\{\|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})u\|_{w^{\alpha,\beta}} : u\in S\} \to 0 \ as\ n\to\infty.\end{aligned}$$

Again for a relatively compact set $S_2=\{\mathcal{K}\mathcal{P}_n^{\alpha,\beta}u : u\in B\}$, we conclude that

$$\begin{aligned}\|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta}-\mathcal{K})\mathcal{K}\mathcal{P}_n^{\alpha,\beta}\|_{w^{\alpha,\beta}} &= \sup\{\|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}\mathcal{P}_n^{\alpha,\beta}u\|_{w^{\alpha,\beta}} : u\in B\} \\ &= \sup\{\|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})u\|_{w^{\alpha,\beta}} : u\in S_2\} \to 0 \ as\ n\to\infty.\end{aligned}$$

This gives $\mathcal{K}\mathcal{P}_n^{\alpha,\beta}$ is ν-convergent to $\mathcal{K}$. This completes the proof.

Since $\mathcal{K}\mathcal{P}_n^{\alpha,\beta}$ is norm-convergent to $\mathcal{K}$, in $L^2_{\omega^{\alpha,\beta}}$, the spectrum of $\mathcal{K}\mathcal{P}_n^{\alpha,\beta}$ inside Γ consists of m eigenvalues, say $\lambda_{n,1},\lambda_{n,2},\cdots,\lambda_{n,m}$ counted accordingly to their algebraic multiplicities inside Γ. Let

$$\widehat{\lambda}_n = \frac{\lambda_{n,1}+\lambda_{n,2}+\cdots+\lambda_{n,m}}{m}$$

denote their arithmetic mean and we approximate λ by $\widehat{\lambda}_n$. Let $\mathcal{P}^S$ and $\mathcal{P}_n^S$ be the spectral projections of $\mathcal{K}$ and $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}$, respectively, associated with their corresponding spectral inside Γ.

To discuss the closeness of eigenfunctions of the integral operator $\mathcal{K}$ and those of the approximate operators $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}$, we recall the concept of the gap between the spectral subspaces. For nonzero subspaces $\mathbb{Y}_1$ and $\mathbb{Y}_2$ of $\mathbb{X}$, and for $q = w^{\alpha,\beta}$ or ∞, let

$$\delta_q(\mathbb{Y}_1, \mathbb{Y}_2) = \sup\{\text{dist}_q(y, \mathbb{Y}_2) : y \in \mathbb{Y}_1, \|y\|_q = 1\},$$

then

$$\widehat{\delta}_q(\mathbb{Y}_1, \mathbb{Y}_2) = \max\{\delta_q(\mathbb{Y}_1, \mathbb{Y}_2), \delta_q(\mathbb{Y}_2, \mathbb{Y}_1)\},$$

denotes the gap between the spectral subspaces in weighted norm and infinity norm.

We quote the following theorems, which helps us to calculate the error bounds of eigenvalue, eigenvector and iterated eigenvectors.

Theorem 5. ([9, 10]) *Then for sufficiently large n and for $i = 1, 2, \cdots, m$, there exists a constant c independent of n such that*

$$|\lambda - \widehat{\lambda}_n| \leq c\, \|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}},$$
$$|\lambda - \lambda_{n,i}|^l \leq c\, \|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}}.$$

Theorem 6. ([9, 10]) *Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\mathcal{P}_n^S)$ be the ranges of the spectral projections $\mathcal{P}^S$ and $\mathcal{P}_n^S$, respectively. Then for sufficiently large n, there exists a constant c for $q = w^{\alpha,\beta}$ or ∞, independent of n such that*

$$\widehat{\delta}_q(\mathcal{R}(\mathcal{P}^S), \mathcal{R}(\mathcal{P}_n^S)) \leq c\, \|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\mathcal{K}\|_q,$$
$$\delta_q(\mathcal{K}\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq c\, \|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_q.$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$ and $\tilde{u}_n = \mathcal{K}u_n$, we have

$$\|u_n - \mathcal{P}^S u_n\|_q \leq c\, \|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\mathcal{K}\|_q,$$
$$\|\tilde{u}_n - \mathcal{P}^S \tilde{u}_n\|_q \leq c\, \|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_q.$$

3. Convergence Rates for Jacobi spectral Galerkin Method:

In this section, we will discuss on the convergence rates of eigen elements by using Jacobi spectral Galerkin method. To do this, we need the following results.

Theorem 7. *Let $\mathcal{K}$ be a compact integral operator with kernel $k(x,t) \in H^m_{\omega^{\alpha,\beta}}[-1,1]\times[-1,1]$ and $\mathcal{P}_n^{\alpha,\beta}$ be the projection defined by* (5)*. Then the followings hold.*

$$\begin{array}{ll} \|\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}), & \|\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}\|_{\infty} = \mathcal{O}(n^{\frac{3}{4}-m}), \\ \|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-2m}), & \|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}\|_{\infty} = \mathcal{O}(n^{-2m}). \end{array}$$

Proof. The first two estimates can be proved by using Theorem-3 directly. Now we will prove the last two estimates. Using Lemma-1 and estimate (7), we obtain

$$\begin{array}{rcl} \|\mathcal{K}(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u\|_{\infty} & = & \sup\limits_{x\in[-1,1]} \Big| \int_{-1}^{1} k(x,t)(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u(t)\mathrm{d}t \Big| \\ & = & \sup\limits_{x\in[-1,1]} \Big| \int_{-1}^{1} w^{\alpha,\beta}(t)w^{-\alpha,-\beta}(t)k(x,t)(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u(t)\mathrm{d}t \Big| \\ & \leq & c\,|\langle k(x,t), (\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u(t)\rangle_{\omega^{\alpha,\beta}}| \\ & = & c\,|\langle (\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})k_x(.), (\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u(.)\rangle_{\omega^{\alpha,\beta}}| \\ & \leq & c\,\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})k_x(.)\|_{\omega^{\alpha,\beta}}\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u\|_{\omega^{\alpha,\beta}} \\ & \leq & c\,n^{-m}|k_x(.)|_{H^{m,n}_{\omega^{\alpha,\beta}}[-1,1]}n^{-m}|\mathcal{K}u|_{H^{m,n}_{\omega^{\alpha,\beta}}[-1,1]} \\ & \leq & c\,n^{-2m}|k_x(.)|_{H^{m,n}_{\omega^{\alpha,\beta}}[-1,1]}\|u\|_{\omega^{\alpha,\beta}}. \end{array}$$

Hence,

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}\|_{\infty} \leq c\,n^{-2m}|k_x(.)|_{H^{m,N}_{\omega^{\alpha,\beta}}[-1,1]}.$$

Again using the above estimate, we obtain

$$\begin{array}{rcl} \|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}u\|_{w^{\alpha,\beta}} & = & \|1\|_{\omega^{\alpha,\beta}}\|\mathcal{K}(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}u\|_{\infty} \\ & \leq & c\,n^{-2m}|k_x|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]}\|u\|_{w^{\alpha,\beta}}. \end{array}$$

This completes the proof.

In the following theorems, we give the superconvergence rates for eigenvalues and eigenvectors.

Theorem 8. *Then for sufficiently large n and for $i = 1, 2, \cdots, m$, there exists a constant c independent of n such that*

$$\begin{array}{l} |\lambda-\widehat{\lambda}_n| = \mathcal{O}(n^{-2m}), \\ |\lambda-\lambda_{n,i}|^l = \mathcal{O}(n^{-2m}). \end{array}$$

Proof. By using Theorem-5 and Theorem-7, for $i = 1, 2, \cdots, m$, it follows that

$$|\lambda - \widehat{\lambda}_n| \le c\,\|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-2m}),$$
$$|\lambda - \lambda_{n,i}|^l \le c\,\|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-2m}).$$

This completes the proof.

Theorem 9. *Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\mathcal{P}_n^S)$ be the ranges of the spectral projections $\mathcal{P}^S$ and $\mathcal{P}_n^S$, respectively. Then for sufficiently large n, there exists a constant c independent of n such that*

$$\widehat{\delta}_{w^{\alpha,\beta}}(\mathcal{R}(\mathcal{P}^S), \mathcal{R}(\mathcal{P}_n^S)) = \mathcal{O}(n^{-m}),$$
$$\delta_{w^{\alpha,\beta}}(\mathcal{R}(\mathcal{P}^S), \mathcal{K}\mathcal{R}(\mathcal{P}_n^S)) = \mathcal{O}(n^{-2m}).$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, we have

$$\|u_n - \mathcal{P}^S u_n\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}),$$
$$\|\mathcal{K}u_n - \mathcal{P}^S\mathcal{K}u_n\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-2m}).$$

Proof. It follows from Theorem-6 and Theorem-8 that

$$\widehat{\delta}_{w^{\alpha,\beta}}(\mathcal{R}(\mathcal{P}^S), \mathcal{R}(\mathcal{P}_n^S)) \le c\,\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}),$$
$$\delta_{w^{\alpha,\beta}}(\mathcal{R}(\mathcal{P}^S), \mathcal{K}\mathcal{R}(\mathcal{P}_n^S)) \le c\|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-2m}).$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, we have

$$\|u_n - \mathcal{P}^S u_n\|_{w^{\alpha,\beta}} \le c\,\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}),$$
$$\|\widehat{u}_n - \mathcal{P}^S\widehat{u}_n\|_{w^{\alpha,\beta}} \le c\,\|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-2m}).$$

This completes the proof.

In the following theorem we give the superconvergence rates for the eigenvectors and iterated eigenvectors in the infinity norm.

Theorem 10. *Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\mathcal{P}_n^S)$ be the ranges of the spectral projections $\mathcal{P}^S$ and $\mathcal{P}_n^S$, respectively. Then for sufficiently large n, there exists a constant c independent of n such that*

$$\widehat{\delta}_\infty(\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) = \mathcal{O}(n^{\frac{3}{4}-m}),$$
$$\delta_\infty(\mathcal{K}\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) = \mathcal{O}(n^{-2m}).$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, we have

$$\|u_n - \mathcal{P}^S u_n\|_\infty = \mathcal{O}(n^{\frac{3}{4}-m}),$$
$$\|\mathcal{K}u_n - \mathcal{P}^S\mathcal{K}u_n\|_\infty = \mathcal{O}(n^{-2m}).$$

Proof. It follows from Theorem-6 and estimate (8) that,

$$\widehat{\delta}_\infty(\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq c\,\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\mathcal{K}\|_\infty = \mathcal{O}(n^{\frac{3}{4}-m}).$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, we have

$$\|u_n - \mathcal{P}^S u_n\|_\infty \leq c\,\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\mathcal{K}\|_\infty = \mathcal{O}(n^{\frac{3}{4}-m}).$$

Now using Theorem-6 and estimate (8), we obtain

$$\delta_\infty(\mathcal{K}\mathcal{R}(\mathcal{P}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq c\,\|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_\infty = \mathcal{O}(n^{-2m}).$$

In particular, for any $u_n \in \mathcal{R}(\mathcal{P}_n^S)$, we have

$$\|\widehat{u}_n - \mathcal{P}^S\widehat{u}_n\|_\infty = \|\mathcal{K}u_n - \mathcal{P}^S\mathcal{K}u_n\|_\infty \leq c\,\|(\mathcal{K}\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_\infty = \mathcal{O}(n^{-2m}).$$

This completes the proof.

4. Discrete Jacobi spectral Galerkin Methods

In this section, we will describe the discrete Jacobi spectral Galerkin methods for the eigenvalue problem of a compact integral operator with smooth kernel. For a given positive integer n, we denote $\{\theta_k^{\alpha,\beta}\}_{k=0}^n$ the points of the Gauss-Jacobi quadrature formula, which are the roots of the Jacobi polynomial $\{J_{n+1}^{\alpha,\beta}(x)\}_{n=0}^\infty$ and $\{w_k^{\alpha,\beta}\}$ are the corresponding weights. By using Gauss-Jacobi quadrature rule with $n+1$ points, the integral operator is approximated as

$$\mathcal{K}_n(x) = \sum_{k=0}^{n} k(x, \theta_k^{\alpha,\beta}) u(\theta_k^{\alpha,\beta}) w_k^{\alpha,\beta}. \tag{8}$$

On the other hand, instead of the continuous inner product, the discrete inner product will be implemented in (3) and (4) which is defined by

$$\langle u, v\rangle_{w^{\alpha,\beta},n} = \sum_{k=0}^{n} u(\theta_k^{\alpha,\beta}) v(\theta_k^{\alpha,\beta}) w_k^{\alpha,\beta}, \quad u, v \in \mathbb{X}_n, \tag{9}$$

where $\{\theta_k^{\alpha,\beta}\}_{k=0}^n$ and $\{w_k^{\alpha,\beta}\}_{k=0}^n$ are the $(n+1)$-degree Jacobi Gauss points and their corresponding Jacobi weights.
The discrete Jacobi spectral Galerkin method is to find

$$\widetilde{u}_n(x) = \sum_{j=0}^{n} \widetilde{a}_j \phi_j(x), \quad \widetilde{u}_n(x) \in \mathbb{X}_n, \tag{10}$$

such than

$$\langle \mathcal{K}_n \widetilde{u}_n, v_n \rangle_{w^{\alpha,\beta},n} = \tilde{\lambda}_n \langle \widetilde{u}_n, v_n \rangle_{w^{\alpha,\beta},n}, \tag{11}$$

where $\{\widetilde{a}_j\}_{j=0}^n$ are determined by

$$\sum_{j=0}^{n} \widetilde{a}_j \langle \mathcal{K}_n \phi_j, \phi_i \rangle_{w^{\alpha,\beta},n} = \tilde{\lambda}_n \sum_{j=0}^{n} \widetilde{a}_j \langle \phi_j, \phi_i \rangle_{w^{\alpha,\beta},n}, \; i = 1, 2, 3, \cdots, n. \tag{12}$$

Using these notation we can reformulate the equation (2) into the operator form

$$\mathcal{P}_n^{\alpha,\beta} \mathcal{K}_n \widetilde{u}_n = \tilde{\lambda}_n \widetilde{u}_n. \tag{13}$$

Using Cauchy-Schwarz inequality and Lemma-1 with $\|u\|_{w^{\alpha,\beta},n} \leq \|u\|_{w^{\alpha,\beta}}$, we obtain

$$\begin{aligned} \|\mathcal{K}_n u\|_\infty &= \sup_x |\mathcal{K}_n u(x)| \\ &= \sup_x \Big| \sum_{k=0}^{n} k(x, \theta_k^{\alpha,\beta}) u(\theta_k^{\alpha,\beta}) w_k^{\alpha,\beta} \Big| \\ &= \sup_x \Big| \langle k_x, u \rangle_{w^{\alpha,\beta},n} \Big| \\ &\leq \|k_x\|_{w^{\alpha,\beta},n} \|u\|_{w^{\alpha,\beta},n} \\ &\leq \|k_x\|_{w^{\alpha,\beta}} \|u\|_{w^{\alpha,\beta}} \leq c\|u\|_{w^{\alpha,\beta}}. \end{aligned} \tag{14}$$

Lemma 11. *Suppose that $v \in H^m_{w^{\alpha,\beta}}[-1,1]$ with $\alpha, \beta > -1$, $m \geq 1$ and $\phi \in \mathbb{X}_n$. Then the following holds.*

$$|\langle v, \phi \rangle_{w^{\alpha,\beta}} - \langle v, \phi \rangle_{w^{\alpha,\beta},n}| \leq cn^{-m} |v|_{H^m_{w^{\alpha,\beta}}[-1,1]} \|\phi\|_{w^{\alpha,\beta}}. \tag{15}$$

Theorem 12. *Assume that the kernel function $k(.,.) \in H^m_{w^{\alpha,\beta}}[-1,1] \times [-1,1]$ with $\alpha, \beta > -1$ and $u(.) \in \mathcal{C}^m[-1,1]$ for $m \in \mathbb{N}$. If two parameters α and β satisfy the conditions $\alpha, \beta > -1$, then there exists positive constants c such that*

$$\|\mathcal{K}u - \mathcal{K}_n u\|_\infty \leq cn^{-m} |k(x,.)|_{H^m_{w^{\alpha,\beta}}[-1,1]} \|u\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}). \tag{16}$$

Proof. By the definition of $\mathcal{K}$ and $\mathcal{K}_n$, we have

$$\begin{aligned} (\mathcal{K}u)(x) - (\mathcal{K}_n u)(x) &= \int_{-1}^{1} k(x,t) u(t) \mathrm{d}t - \sum_{k=0}^{n} k(x, \theta_k) u(\theta_k) v_k \\ &= \langle k(x,.), u(.) \rangle - \langle k(x,.), u(.) \rangle_n \end{aligned}$$

where $\langle .,. \rangle$ represents the continuous inner product with respect to s and $\langle .,. \rangle_n$ represents the discrete inner product defined by the Gauss-Legendre quadrature formula. By using Lemma-11, we have

$$\begin{aligned} |(\mathcal{K}u)(x) - (\mathcal{K}_n u)(x)| &= |\langle k(x,.), u(.)\rangle - \langle k(x,.)u(.)\rangle_n| \\ &\leq cn^{-m}|k(x,.)|_{H^m_{w^{\alpha,\beta}}[-1,1]}\|u(.)\|_{w^{\alpha,\beta}}. \end{aligned}$$

Taking supremum over $x \in [-1,1]$, we obtain

$$\|\mathcal{K}u - \mathcal{K}_n u\|_\infty \leq cn^{-m}|k(x,.)|_{H^m_{w^{\alpha,\beta}}[-1,1]}\|u\|_{w^{\alpha,\beta}}.$$

This completes the proof.

Theorem 13. *$\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n$ and $\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}$ is ν-convergent to $\mathcal{K}$ in weighted L^2 norm.*

Proof. We have $\|\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n\|_{w^{\alpha,\beta}} \leq p_1\|\mathcal{K}_n\|_\infty$.
From Theorem-12, we see that $\{\mathcal{K}_n\}$ converges to $\mathcal{K}$ pointwise. Hence, $\{\mathcal{K}_n\}$ is pointwise bounded and since $\mathbb{X}$ is Banach space, and by Uniform Boundedness principle we have $\{\mathcal{K}_n\}$ is uniformly bounded. This shows that $\|\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n\|_{w^{\alpha,\beta}}$ is uniformly bounded.
By using the Theorem-12 and Lemma-1, we obtain

$$\begin{aligned} \|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})u\|_{w^{\alpha,\beta}} &\leq \|\mathcal{P}_n^{\alpha,\beta}(\mathcal{K}_n - \mathcal{K})u\|_{w^{\alpha,\beta}} + \|(\mathcal{P}_n^{\alpha,\beta} - \mathcal{I})\mathcal{K}u\|_{w^{\alpha,\beta}} \\ &\leq \zeta\|(\mathcal{K}_n - \mathcal{K})u\|_\infty + \gamma_2 n^{-m}\|(\mathcal{K}u)^{(r)}\| \\ &\leq cn^{-m}\|u\|_{m,w^{\alpha,\beta}} + cn^{-m}\|D_x^1 k\|_{m,\infty}\|u\|_\infty \to 0, \quad as \;\; n \to \infty. \end{aligned}$$

This gives that $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n$ is pointwise convergent to $\mathcal{K}$.

Let $B = \{u \in \mathbb{X} : \|u\| \leq 1\}$ be a closed unit ball in $\mathcal{C}[-1,1]$. Since $\mathcal{K}$ is compact operator, the set $S = \{\mathcal{K}u : u \in B\}$ is a relatively compact set in $\mathcal{C}[-1,1]$. Then we have

$$\begin{aligned} \|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} &= \sup\{\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{K}u\|_{w^{\alpha,\beta}} : u \in B\} \\ &= \sup\{\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})u\|_{w^{\alpha,\beta}} : u \in S\} \to 0 \; as \; n \to \infty. \end{aligned}$$

Since $\mathcal{P}_n^{\alpha,\beta}$ is bounded in weighted L^2 norm and $\mathcal{K}_n$ is compact, $S_1 = \{\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n u : u \in B\}$ is a relatively compact set. Thus

$$\begin{aligned} \|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n\|_{w^{\alpha,\beta}} &= \sup\{\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n u\|_{w^{\alpha,\beta}} : u \in B\} \\ &= \sup\{\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})u\|_{w^{\alpha,\beta}} : u \in S\} \to 0 \; as \; n \to \infty. \end{aligned}$$

Combining all these results leads to the first result that $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n$ is ν-convergent to $\mathcal{K}$.
For the proof of the second result, using Theorem-12 and Lemma-1, we obtain

$$\|(\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}-\mathcal{K})u\|_{w^{\alpha,\beta}} \leq \|\mathcal{K}_n\|_{w^{\alpha,\beta}}\|(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})u\|_{w^{\alpha,\beta}}+\|(\mathcal{K}_n-\mathcal{K})u\|_{w^{\alpha,\beta}} \to 0 \ as \ n \to \infty.$$

This shows that $\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}$ is pointwise converges to $\mathcal{K}$ on $\mathcal{C}[-1,1]$. Now, we have

$$\begin{aligned} \|(\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} &= \sup\{\|(\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}u\|_{w^{\alpha,\beta}} : u \in B\} \\ &= \sup\{\|(\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})u\|_{w^{\alpha,\beta}} : u \in S\} \to 0 \ as \ n \to \infty. \end{aligned}$$

Again for a relatively compact set $S_2 = \{\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}u : u \in B\}$, we conclude that

$$\begin{aligned} \|(\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}\|_{w^{\alpha,\beta}} &= \sup\{\|(\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}u\|_{w^{\alpha,\beta}} : u \in B\} \\ &= \sup\{\|(\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta} - \mathcal{K})u\|_{w^{\alpha,\beta}} : u \in S_2\} \to 0 \ as \ n \to \infty. \end{aligned}$$

This gives $\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}$ is ν-convergent to $\mathcal{K}$. This completes the proof.

Since $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n$ and $\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}$ are ν-convergent to $\mathcal{K}$, for all small n, the spectrum of $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n$ and $\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta}$ inside Γ consists of m eigenvalues, say $\tilde{\lambda}_{n,1}, \tilde{\lambda}_{n,2}, \cdots, \tilde{\lambda}_{n,p}$ counted accordingly to their algebraic multiplicities inside Γ. Let

$$\tilde{\tilde{\lambda}}_n = \frac{\tilde{\lambda}_{n,1} + \tilde{\lambda}_{n,2} + \cdots + \tilde{\lambda}_{n,p}}{p}$$

denote their arithmetic mean and we approximate λ by $\widetilde{\lambda}_n$. Let $\widetilde{\mathcal{P}}_n^S$ be the spectral projections of $\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n$ associated with their corresponding spectral inside Γ.

Theorem 14. [1, 9] Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\widetilde{\mathcal{P}}_n^S)$ be the ranges of the spectral projections $\mathcal{P}^S$ and $\widetilde{\mathcal{P}}_n^S$, respectively. Then for sufficiently large n, there exists a constant c independent of n such that

$$\begin{aligned} &\widehat{\delta}_{w^{\alpha,\beta}}(\mathcal{R}(\widetilde{\mathcal{P}}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq c\|(\mathcal{K} - \mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_{w^{\alpha,\beta}}, \\ &\widehat{\delta}_{\infty}(\mathcal{R}(\widetilde{\mathcal{P}}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq c\|(\mathcal{K} - \mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_{\infty}. \end{aligned}$$

In particular for any $\widetilde{u}_n \in \mathcal{R}(\widetilde{\mathcal{P}}_n^S)$, we have

$$\begin{aligned} &\|\widetilde{u}_n - \mathcal{P}^S\widetilde{u}_n\|_{w^{\alpha,\beta}} \leq c\|(\mathcal{K} - \mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_{w^{\alpha,\beta}}, \\ &\|\widetilde{u}_n - \mathcal{P}^S\widetilde{u}_n\|_{\infty} \leq c\|(\mathcal{K} - \mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_{\infty}. \end{aligned}$$

Theorem 15. [1, 9] Then for sufficiently large n, there exists a constant c independent of n such that

$$\begin{aligned} &|\lambda - \tilde{\tilde{\lambda}}_n| \leq c\|(\mathcal{K} - \mathcal{K}_n\mathcal{P}_n^{\alpha,\beta})\mathcal{K}\|_{w^{\alpha,\beta}}, \\ &|\lambda - \tilde{\lambda}_{n,i}|^l \leq c\|(\mathcal{K} - \mathcal{K}_n\mathcal{P}_n^{\alpha,\beta})\mathcal{K}\|_{w^{\alpha,\beta}}. \end{aligned}$$

To discuss the convergence rates for the approximated eigenvalues and eigenvectors to the exact eigenvalues and eigenvectors, we need the following theorems.

Theorem 16. *Let $\mathcal{K}$ be a compact integral operator with kernel $k(x,t) \in H^m_{w^{\alpha,\beta}}[-1,1]\times[-1,1]$. Then the followings hold.*

$$\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}\|_\infty = \mathcal{O}(n^{-m}),$$
$$\|(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}\|_\infty = \mathcal{O}(n^{\frac{3}{4}-m}),$$
$$\|(\mathcal{P}_n^{\alpha,\beta}-\mathcal{I})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}).$$

Proof. Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|\mathcal{K}u\|_\infty &= \sup_x \Big|\int_{-1}^1 k(x,t)u(t)\mathrm{d}t\Big| \\
&= \sup_x \Big|\int_{-1}^1 w^{\frac{-\alpha}{2},\frac{-\beta}{2}}(t)w^{\frac{\alpha}{2},\frac{\beta}{2}}(t)k(x,t)u(t)\mathrm{d}t\Big| \\
&\leq \|1\|_{w^{-\alpha,-\beta}} \sup_x \Big(\int_{-1}^1 w^{\alpha,\beta}(t)|k(x,t)u(t)|^2\mathrm{d}t\Big)^{1/2} \\
&\leq c\|1\|_{w^{-\alpha,-\beta}}\|u\|_{w^{\alpha,\beta}} \leq c\|u\|_\infty.
\end{aligned}$$

Replacing u by $\mathcal{K}u$ in equation (16) and using the above inequality, we obtain

$$\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}u\|_\infty \leq cn^{-m}\|\mathcal{K}u\|_{\omega^{\alpha,\beta}} \leq cn^{-m}\|1\|_{\omega^{\alpha,\beta}}\|\mathcal{K}u\|_\infty \leq cn^{-m}\|u\|_\infty.$$

This completes the proof of first estimate. Now replacing v by $\mathcal{K}u$ in Lemma-1 for $\alpha,\beta > -1$ and then using estimate (7)

$$\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}u\|_\infty \leq c\,n^{\frac{3}{4}-m}|\mathcal{K}u|_{H^{m,n}_{w^{\alpha,\beta}}[-1,1]} \leq cn^{\frac{3}{4}-m}|\|u\|_{\omega^{\alpha,\beta}}.$$

Thus, we obtain the required second estimate. The third estimate can be proved similarly. This completes the proof.

Theorem 17. *Let $\mathcal{K}$ be a compact operator with a kernel $k(.,.) \in H^m_{w^{\alpha,\beta}}[-1,1]\times[-1,1]$ with $\alpha,\beta > -1$ and $m \geq 1$. Then the followings hold:*

$$\|(\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_\infty \leq \begin{cases} cn^{-m}\log n, & -1<\alpha,\beta<\frac{-1}{2}, \\ n^{\nu+\frac{1}{2}-m}, \nu=\max(\alpha,\beta), & \textit{otherwise}. \end{cases}$$
$$\|(\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}).$$

Proof. Using Theorem-16, we obtain

$$\begin{aligned}
\|(\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_\infty &\le \|(\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}+\mathcal{P}_n^{\alpha,\beta}\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_\infty \\
&\le \|(\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K})\mathcal{K}\|_\infty + \|\mathcal{P}_n^{\alpha,\beta}\|_\infty\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}\|_\infty \\
&\le c\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}\|_\infty + c\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}\|_\infty \\
&\qquad\qquad \times \begin{cases} c\log n, & -1<\alpha,\beta<\frac{-1}{2}, \\ cn^{\nu+\frac{1}{2}}, \nu=\max(\alpha,\beta), & otherwise \end{cases} \\
&\le cn^{-m} + cn^{-m}\begin{cases} c\log n, & -1<\alpha,\beta<\frac{-1}{2}, \\ cn^{\nu+\frac{1}{2}}, \nu=\max(\alpha,\beta), & otherwise \end{cases} \\
&\le \begin{cases} cn^{-m}\log n, & -1<\alpha,\beta<\frac{-1}{2}, \\ n^{\nu+\frac{1}{2}-m}, \nu=\max(\alpha,\beta), & otherwise. \end{cases}
\end{aligned}$$

Similarly

$$\begin{aligned}
\|(\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n)\mathcal{K}\|_{\omega^{\alpha,\beta}} &\le \|(\mathcal{K}-\mathcal{P}_n^{\alpha,\beta}\mathcal{K})\mathcal{K}\|_{\omega^{\alpha,\beta}} + \|\mathcal{P}_n^{\alpha,\beta}\|_{\omega^{\alpha,\beta}}\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}\|_{\omega^{\alpha,\beta}} \\
&\le c\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})\mathcal{K}\|_{\omega^{\alpha,\beta}} + c\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}\|_{\omega^{\alpha,\beta}} \\
&\le cn^{-m}\|1\|_{\omega^{\alpha,\beta}}\|u\|_{\omega^{\alpha,\beta}} + c\|1\|_{\omega^{\alpha,\beta}}\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}\|_\infty \\
&\le cn^{-m}\|u\|_{\omega^{\alpha,\beta}} + cn^{-m}\|1\|_{\omega^{\alpha,\beta}}\|1\|^2_{\omega^{\alpha,\beta}}\|1\|_{\omega^{-\alpha,-\beta}}\|u\|_\infty
\end{aligned}$$

This completes the proof.

Theorem 18. *The following holds:*

$$\|(\mathcal{K}-\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta})u\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}).$$

Proof. By replacing u by $(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})u$ in the equation (14), we obtain

$$\begin{aligned}
\|(\mathcal{K}-\mathcal{K}_n\mathcal{P}_n^{\alpha,\beta})u\|_\infty &\le \|(\mathcal{K}-\mathcal{K}_n)u\|_\infty + \|\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})u\|_\infty \\
&\le n^{-m}|k(x,.)|_{H^m_{w^{\alpha,\beta}}[-1,1]}\|u\|_{w^{\alpha,\beta}} + c\|(\mathcal{I}-\mathcal{P}_n^{\alpha,\beta})u\|_{w^{\alpha,\beta}} \\
&\le cn^{-m}\|u\|_{w^{\alpha,\beta}} + n^{-m}|u|_{H^m_{w^{\alpha,\beta}}[-1,1]}.
\end{aligned}$$

This completes the proof.

In the following theorems, we give the superconvergence rates for eigenvalues and eigenvectors.

Theorem 19. *Then for sufficiently large n and for $i=1,2,\cdots,p$, there exists a constant c independent of n such that*

$$|\lambda-\tilde{\tilde{\lambda}}_n| = \mathcal{O}(n^{-m}),$$
$$|\lambda-\tilde{\lambda}_{n,i}|^l = \mathcal{O}(n^{-m}).$$

Proof. By using Theorem-18, for $i = 1, 2, \cdots, p$, it follows that

$$|\lambda - \tilde{\hat{\lambda}}_n| \leq c\|(\mathcal{K} - \mathcal{K}_n\mathcal{P}_n^{\alpha,\beta})u\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}),$$
$$|\lambda - \tilde{\lambda}_{n,i}|^l \leq c\|(\mathcal{K} - \mathcal{K}_n\mathcal{P}_n^{\alpha,\beta})u\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}).$$

This completes the proof.

In the following theorem we give the error bound for iterated eigenvectors both in weighted norm and infinity norm.

Theorem 20. *Let $\mathcal{R}(\mathcal{P}^S)$ and $\mathcal{R}(\widehat{\mathcal{P}}_n^S)$ be the ranges of the spectral projections $\mathcal{P}^S$ and $\widehat{\mathcal{P}}_n^S$, respectively. Then for sufficiently large n, there exists a constant c independent of n such that*

$$\widehat{\delta}_\infty(\mathcal{R}(\widetilde{\mathcal{P}}_n^S), \mathcal{R}(\mathcal{P}^S)) = \begin{cases} cn^{-m}\log n, & -1 < \alpha, \beta < \frac{-1}{2}, \\ n^{\nu+\frac{1}{2}-m}, \nu = \max(\alpha,\beta), & otherwise. \end{cases}$$
$$\widehat{\delta}_{w^{\alpha,\beta}}(\mathcal{R}(\widetilde{\mathcal{P}}_n^S), \mathcal{R}(\mathcal{P}^S)) = \mathcal{O}(n^{-m}).$$

In particular, for any $\tilde{u}_n \in \mathcal{R}(\widetilde{\mathcal{P}}_n^S)$, we have

$$\|\tilde{u}_n - \mathcal{P}^S\tilde{u}_n\|_\infty = \begin{cases} cn^{-m}\log n, & -1 < \alpha, \beta < \frac{-1}{2}, \\ n^{\nu+\frac{1}{2}-m}, \nu = \max(\alpha,\beta), & otherwise. \end{cases}$$
$$\|\tilde{u}_n - \mathcal{P}^S\tilde{u}_n\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}).$$

Proof. It follows from Theorem-17 that,

$$\widehat{\delta}_\infty(\mathcal{R}(\widetilde{\mathcal{P}}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq c\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{K}\|_\infty \begin{cases} cn^{-m}\log n, & -1 < \alpha, \beta < \frac{-1}{2} \\ n^{\nu+\frac{1}{2}-m}, \nu = \max(\alpha,\beta), & otherwise. \end{cases}$$
$$\widehat{\delta}_{w^{\alpha,\beta}}(\mathcal{R}(\widetilde{\mathcal{P}}_n^S), \mathcal{R}(\mathcal{P}^S)) \leq c\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}).$$

In particular, for any $\tilde{u}_n \in \mathcal{R}(\widetilde{\mathcal{P}}_n^S)$, we have

$$\|\tilde{u}_n - \mathcal{P}^S\tilde{u}_n\|_\infty \leq c\,\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{K}\|_\infty = \begin{cases} cn^{-m}\log n, & -1 < \alpha, \beta < \frac{-1}{2}, \\ n^{\nu+\frac{1}{2}-m}, \nu = \max(\alpha,\beta), & otherwise. \end{cases}$$
$$\|\tilde{u}_n - \mathcal{P}^S\tilde{u}_n\|_{w^{\alpha,\beta}} \leq c\,\|(\mathcal{P}_n^{\alpha,\beta}\mathcal{K}_n - \mathcal{K})\mathcal{K}\|_{w^{\alpha,\beta}} = \mathcal{O}(n^{-m}).$$

This completes the proof.

5. Numerical results

In this section we discuss the numerical results. Consider the eigenvalue problem,

$$\mathcal{K}u = \lambda u, \;\; 0 \neq \lambda \in \mathbb{C}, \;\; \|u\|_\infty = 1,$$

for the following integral operator

$$\mathcal{K}u(s) = \int_{-1}^{1} k(s,t)u(t)\;\mathrm{d}t, \quad s \in [-1,1],$$

for various smooth kernels $k(s,t)$.

Kernel-1: $k(s,t) = e^{st}$

Table 1: Jacobi spectral Galerkin Method:

n	$\lvert\lambda - \widehat{\lambda}_n\rvert$	$\|u_n - \mathcal{P}^S u_n\|_{\omega^{\alpha,\beta}}$	$\|u_n - \mathcal{P}^S u_n\|_\infty$	$\|\widehat{u}_n - \mathcal{P}^S \widehat{u}_n\|_{\omega^{\alpha,\beta}}$	$\|\widehat{u}_n - \mathcal{P}^S \widehat{u}_n\|_\infty$
2	$9.1979e-04$	$9.2876e-04$	$1.9686e-03$	$9.4567e-04$	$4.3768e-03$
3	$3.3402e-06$	$6.1603e-06$	$1.5495e-05$	$1.2365e-06$	$1.3567e-05$
4	$3.3401e-08$	$6.7236e-06$	$1.6913e-05$	$7.4584e-08$	$3.4879e-07$
5	$8.8817e-10$	$2.5101e-08$	$7.1200e-08$	$5.2346e-10$	$9.4628e-09$
6	$5.7731e-11$	$2.6765e-08$	$7.5920e-08$	$3.1987e-11$	$1.3267e-10$
7	$3.5527e-12$	$6.4303e-11$	$1.9935e-10$	$1.6548e-12$	$7.3498e-11$

Table 2: Discrete Jacobi spectral Galerkin Method:

n	$\lvert\lambda - \widetilde{\widehat{\lambda}}_n\rvert$	$\|\tilde{u}_n - \mathcal{P}^S \tilde{u}_n\|_{\omega^{\alpha,\beta}}$	$\|\tilde{u}_n - \mathcal{P}^S \tilde{u}_n\|_\infty$
2	$3.9433e-04$	$2.6037e-03$	$5.2672e-03$
3	$1.5924e-06$	$1.7873e-05$	$4.4062e-05$
4	$1.5923e-08$	$1.9378e-05$	$4.7771e-05$
5	$2.8866e-09$	$6.9078e-08$	$1.9338e-07$
6	$4.4409e-10$	$7.3415e-08$	$2.0552e-07$
7	$6.6613e-11$	$3.2738e-10$	$8.0601e-10$

Kernel-2: $k(s,t) = \frac{1}{\sqrt{1+|s-t|^2}}$

Table 3: Jacobi spectral Galerkin Method:

n	$\lvert\lambda - \widehat{\lambda}_n\rvert$	$\|u_n - \mathcal{P}^S u_n\|_{\omega^{\alpha,\beta}}$	$\|u_n - \mathcal{P}^S u_n\|_\infty$	$\|\widehat{u}_n - \mathcal{P}^S \widehat{u}_n\|_{\omega^{\alpha,\beta}}$	$\|\widehat{u}_n - \mathcal{P}^S \widehat{u}_n\|_\infty$
2	$1.4378e-03$	$1.4363e-02$	$2.8451e-02$	$5.5127e-02$	$1.0987e-02$
3	$1.5382e-05$	$5.1162e-04$	$1.2761e-03$	$1.2536e-04$	$4.5785e-03$
4	$1.5383e-06$	$5.5428e-04$	$1.3825e-03$	$9.2516e-06$	$8.4582e-05$
5	$1.7868e-07$	$2.0182e-06$	$7.6691e-06$	$1.2637e-07$	$1.3208e-06$
6	$3.7866e-08$	$2.1439e-06$	$8.1470e-06$	$4.1698e-08$	$5.6812e-07$
7	$9.3129e-09$	$9.7241e-07$	$2.8152e-06$	$1.3546e-09$	$1.8653e-08$

Table 4: Discrete Jacobi spectral Galerkin Method:

n	$\lvert\lambda - \widetilde{\tilde{\lambda}}_n\rvert$	$\lVert\tilde{u}_n - \mathcal{P}^S\tilde{u}_n\rVert_{\omega^{\alpha,\beta}}$	$\lVert\tilde{u}_n - \mathcal{P}^S\tilde{u}_n\rVert_\infty$
2	$8.0874e-04$	$3.0659e-03$	$6.5684e-03$
3	$5.8207e-05$	$8.9986e-05$	$1.7609e-04$
4	$5.8071e-06$	$9.77486e-05$	$1.9127e-04$
5	$1.8754e-07$	$1.8742e-05$	$5.2990e-05$
6	$9.8755e-08$	$1.9938e-05$	$5.6370e-05$
7	$1.4869e-08$	$2.4150e-07$	$2.7581e-07$

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