Acta Universitatis Apulensis No. 61/2020
ISSN: 1582-5329 pp. 65-80
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.61.06

ON SOME SUBMANIFOLDS OF (ϵ)-LP-SASAKIAN MANIFOLDS

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Abstract. The object of this paper is to study invariant and totally umbilical submanifolds of (ϵ)-LP-Sasakian manifolds.

2010 *Mathematics Subject Classification*: 53 C 25, 53 D 15.

Keywords: Submanifolds, (ϵ)-LP-Sasakian, invariant submanifolds, totally geodesic, totally umbilical.

1. Introduction

The notion of (ϵ)-Sasakian manifolds was introduced by A. Bejancu and K. L. Duggal [1]. Also in [14], the authors studied (ϵ)-Sasakian manifolds. Sasakian manifolds with indefinite metric play important role in physics, so our natural trend is to study various contact manifolds with indefinite metric. In 2009, the authors U. C. De and A. Sarkar [2] studied (ϵ)-Kenmotsu manifolds. The notion of Lorentzian Para-Sasakian manifolds was introduced by K. Matsumoto [7]. In 2012, R. Prasad and V. Srivastava [11] defined and studied (ϵ)-LP-Sasakian manifolds. Totally umbilical submanifolds of almost contact manifolds have been studied in the papers [3], [4], [5], [8]. In [3], the authors characterized totally umbilical submanifolds of Sasakian manifolds using theory of differential equations [3], [9], [10].

In the present paper we would like to study invariant submanifolds and totally umbilical submanifolds of (ϵ)-LP-Sasakian manifolds.

The present paper is organized as follows :

Section 1 is introductory. Section 2 contains preliminaries. After introduction and preliminaries we study invariant submanifolds of (ϵ)-LP-Sasakian manifolds in Section 3. In Section 4, we study totally umbilical submanifolds of (ϵ)-LP-Sasakian manifolds. In the last section we give an example.

2. Preliminaries

A differentiable manifold is called (ϵ)-LP-Sasakian manifold if the following conditions hold

$$\phi^2 X = X + \eta(X)\xi, \tag{1}$$

$$g(\xi,\xi) = -\epsilon, \qquad \eta(X) = \epsilon g(X,\xi), \tag{2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y), \tag{3}$$

where, ϕ is a (1,1) tensor, η ia 1-form, g is associated metric, ϵ is 1 or -1 according as ξ is space-like or time-like vector field and X, Y are arbitrary vector fields on the manifold. In (ϵ)-LP-Sasakian manifolds we have [11]

$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{4}$$

where ∇ denotes the covariant derivative with respect to the Lorentzian metric g.

An (ϵ)-contact metric manifold is an (ϵ)-LP-Sasakian manifold if and only if [11]

$$\bar{\nabla}_X \xi = \epsilon\phi X. \tag{5}$$

Let M^{2m+1} $(m < n)$ be a submanifold of a differentiable manifold $\bar{M}^{2n+1}$. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of M and $\bar{M}$, respectively. Then for any vector fields $X, Y \in \chi(M)$, the second fundamental form h is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y). \tag{6}$$

A submanifold M of an (ϵ)-LP-Sasakian manifold $\bar{M}$ is called totally geodesic if

$$h(X,Y) = 0, \qquad \text{for} \qquad X, Y \in \chi(M).$$

Furthermore, for any section N of normal bundle $T^{\perp}M$, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp} N, \tag{7}$$

where $\nabla^{\perp}$ denotes the normal bundle connection of M. The second fundamental form h and shape operator A_N are related by

$$g(A_N X, Y) = g(h(X,Y), N). \tag{8}$$

The Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z). \tag{9}$$

In an (ϵ)-LP-Sasakian manifolds $\bar{M}$, we also have [11]

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$S(X,\xi) = 2n\eta(X) \tag{11}$$

for any $X, Y \in \chi(\bar{M})$, where R, S, are curvature tensor, Ricci tensor respectively.

On a Riemannian manifold $\bar{M}$, for a $(0,k)$-type tensor field $T(k \geq 1)$ and a $(0,2)$-type tensor field E, we denote by $Q(E,T)$ a $(0,k+2)$-type tensor field [13] defined as follows

$$\begin{aligned} Q(E,T)(X_1, X_2, ..., X_k; X, Y) = & - T((X \wedge_E Y)X_1, X_2, ..., X_n) \\ & - T(X_1, (X \wedge_E Y)X_2, ..., X_k) - ... \\ & - T(X_1, ..., (X \wedge_E Y)X_k), \end{aligned} \tag{12}$$

where $(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y$.

For a $(2n+1)$ dimensional Riemannian manifold $\bar{M}$, the concircular curvature tensor C is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{13}$$

for any vector fields $X, Y, Z \in \chi(\bar{M})$.

3. Invariant submanifolds of an (ϵ)-LP-Sasakian manifolds

Let M^{2m+1} be a submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}^{2n+1}(n > m)$. Generally a submanifold M is said to be invariant submanifold of $\bar{M}$ if ϕ(TM)$\subset TM$. On an invariant submanifold M of $\bar{M}$, it follows that $\xi \in \chi(M)$.

From the equations (5) and (6) we have

$$\nabla_X \xi + h(X,\xi) = \epsilon\phi(X).$$

Comparing normal and tangential components we have

$$h(X,\xi) = 0, \tag{14}$$

$$\nabla_X \xi = \epsilon\phi(X), \tag{15}$$

for any vector field $X \in \chi(M)$.

Now form (4) we have

$$(\nabla_X \phi)Y - h(X,\phi Y) + \phi(h(X,Y)) = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y). \tag{16}$$

Comparing the tangential and normal components we have

$$h(X, \phi Y) = \phi(h(X, Y)), \tag{17}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y). \tag{18}$$

Lemma 3.1. *An invariant submanifold of an (ϵ)-LP-Sasakian manifold is an (ϵ)-LP-Sasakian manifold*

Proof. Form (5) and (6) we have

$$\nabla_X \xi + h(X, \xi) = \epsilon\phi(X).$$

So, $\nabla_X \xi = \epsilon\phi(X)$. By [11], the submanifold is (ϵ)-LP-Sasakian manifold.

Lemma 3.2. *[12] M be a three dimensional invariant submanifold of an (ϵ)-LP-Sasakian manifold then, then there exists two differentiable distributions D and $D^{\perp}$ on M such that*

$$TM = D \oplus D^{\perp} \oplus < \xi >, \tag{19}$$

$$\phi(D) \subset D^{\perp}, \qquad \phi(D^{\perp}) \subset D.$$

Theorem 3.1. *Every three dimensional invariant submanifolds of an (ϵ)-LP-Sasakian manifold is totally geodesic.*

Proof. Let M be an invariant submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}$, then by Lemma 3.2 there are two orthogonal distributions D and $D^{\perp}$ satisfying the equation (19). Let $X_1, Y_1 \in D$, and $\phi(X_1), \phi(Y_1) \in D^{\perp}$. Using (1) and (17) we obtain

$$\begin{aligned} h(\phi X_1, \phi Y_1) &= \phi^2 h(X_1, Y_1) \\ &= -h(X_1, Y_1). \end{aligned}$$

Let $\phi X_1 = X_2$, $\phi Y_1 = Y_2$. We see that $X_2 \in D^{\perp}$ and $Y_2 \in D^{\perp}$. Therefore

$$h(X_2, Y_2) = -h(X_1, Y_1) \tag{20}$$

for any $X_1, Y_1 \in D$ and $X_2, Y_2 \in D^{\perp}$. By the property of bilinearity of h we have

$$h(X_1 + X_2 + \xi, Y_1) = h(X_1, Y_1) + h(X_2, Y_1) + h(\xi, Y_1), \tag{21}$$

$$h(X_1 + X_2 + \xi, Y_2) = h(X_1, Y_2) + h(X_2, Y_2) + h(\xi, Y_2), \tag{22}$$

$$h(X_1 + X_2 + \xi, \xi) = h(X_1, \xi) + h(X_2, \xi) + h(\xi, \xi). \tag{23}$$

Keeping in mind $h(X, \xi) = 0$, and from (21), (22), (23) we obtain

$$h(X_1 + X_2 + \xi, Y_1 + Y_2 + \xi) = h(X_1, Y_2) + h(X_2, Y_1), \tag{24}$$

$$h(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = h(X_1, Y_2) - h(X_2, Y_1). \tag{25}$$

Since
$$TM = D \oplus D^{\perp} \oplus < \xi >,$$
so any arbitrary vector fields U, V can be taken as $U = X_1 + X_2 + \xi$, and $V = Y_1 + Y_2 + \xi$. Then form (24) and (25) we have
$$\begin{aligned} h(U, V) &= h(X_2, Y_1) - h(X_1, Y_2) \\ &= h(\phi X_1, Y_1) - h(X_1, \phi Y_2) = 0. \end{aligned}$$

Hence the submanifold M is totally geodesic.

Theorem 3.2. *An invariant submanifold of an (ϵ)-LP-Sasakian manifold is totally geodesic if and only if $Q(S, \bar{\nabla} h) = 0$.*

Proof. Let M be an invariant submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}$ satisfying $Q(S, \bar{\nabla} h) = 0$, then
$$Q(S, \bar{\nabla}_X h)(W, K; U, V) = 0,$$
for the vector fields $X, W, K, U, V \in \chi(M)$. By the above equation and (12), we have
$$\begin{aligned} 0 = &- (\bar{\nabla}_X h)(S(V, W)U, K) + (\bar{\nabla}_X h)(S(U, W)V, K) \\ &- (\bar{\nabla}_X h)(W, S(V, K)U) + (\bar{\nabla}_X h)(W, S(U, K)V). \end{aligned}$$

Hence,

$$\begin{aligned} 0 = &- \nabla^{\perp}_X h(S(V, W)U, K) + h(\nabla_X S(V, W)U, K) + h(S(V, W)U, \nabla_X K) \\ &+ \nabla^{\perp}_X h(S(U, W)V, K) - h(\nabla_X S(U, W)V, K) - h(S(U, W)V, \nabla_X K) \\ &- \nabla^{\perp}_X h(W, S(V, K)U) + h(\nabla_X W, S(V, K)U) + h(W, \nabla_X S(V, K)U) \\ &+ \nabla^{\perp}_X h(W, S(U, K)V) - h(\nabla_X W, S(U, K)V) - h(W, \nabla_X S(U, K)V), \end{aligned}$$

Using equation (14) and putting $K = V = W = \xi$, in the above equation we have
$$S(\xi, \xi) h(U, \nabla_X \xi) = 0. \tag{26}$$

Using the equation (11) and (26) we have
$$(2n) h(U, \phi X) = 0. \tag{27}$$

By using (17) we obtain $h(U, X) = 0$, for any $U, X \in \chi(M)$, hence the submanifold is totally geodesic. Converse is trivially true.

Theorem 3.3. *An invariant submanifold of an (ϵ)-LP-Sasakian manifold is totally geodesic if and only if $Q(S, R.h) = 0$.*

Proof. Let M be an invariant submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}$ satisfying $Q(S, R.h) = 0$, then

$$Q(S, R(X, Y).h)(W, K; U, V) = 0$$

for the vector fields $X, Y, W, K, U, V \in \chi(M)$. Form (12) we have

$$\begin{aligned} 0 = & - S(V, W)(R(X, Y).h)(U, K) + S(U, W)(R(X, Y).h)(V, K) \\ & - S(V, K)(R(X, Y).h)(W, U) + S(U, K)(R(X, Y).h)(W, V). \end{aligned}$$

Hence,

$$\begin{aligned} 0 = & - S(V, W)[R^{\perp}(X, Y)h(U, K) - h(R(X, Y)U, K) - h(R(X, Y)K, U)] \\ & + S(U, W)[R^{\perp}(X, Y)h(V, K) - h(R(X, Y)V, K) - h(R(X, Y)K, V)] \\ & - S(V, K)[R^{\perp}(X, Y)h(W, U) - h(R(X, Y)W, U) - h(R(X, Y)U, W)] \\ & + S(U, K)[R^{\perp}(X, Y)h(W, V) - h(R(X, Y)W, V) - h(R(X, Y)V, W)]. \end{aligned}$$

Using equation (14) and putting $K = V = W = Y = \xi$, in the above equation we have

$$S(\xi, \xi)h(U, R(X, \xi)\xi) = 0. \tag{28}$$

By the equations (10), (11) and the above equation we have

$$(2n)h(U, X) = 0.$$

Consequently, we have $h(U, X) = 0$, for any $U, X \in \chi(M)$, hence the submanifold is totally geodesic. Converse is trivially true.

Theorem 3.4. *An invariant submanifold of an (ϵ)-LP-Sasakian manifold is totally geodesic if and only if $Q(g, R.h) = 0$.*

Proof. Let M be an invariant submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}$ satisfying $Q(g, R.h) = 0$, then

$$Q(g, R(X, Y).h)(W, K; U, V) = 0$$

for the vector fields $X, Y, W, K, U, V \in \chi(M)$. Form (12) we have

$$\begin{aligned} 0 = & - g(V, W)(R(X, Y).h)(U, K) + g(U, W)(R(X, Y).h)(V, K) \\ & - g(V, K)(R(X, Y).h)(W, U) + g(U, K)(R(X, Y).h)(W, V). \end{aligned}$$

Hence,

$$\begin{aligned} 0 = & - g(V,W)[R^{\perp}(X,Y)h(U,K) - h(R(X,Y)U,K) - h(R(X,Y)K,U)] \\ & + g(U,W)[R^{\perp}(X,Y)h(V,K) - h(R(X,Y)V,K) - h(R(X,Y)K,V)] \\ & - g(V,K)[R^{\perp}(X,Y)h(W,U) - h(R(X,Y)W,U) - h(R(X,Y)U,W)] \\ & + g(U,K)[R^{\perp}(X,Y)h(W,V) - h(R(X,Y)W,V) - h(R(X,Y)V,W)]. \end{aligned}$$

Using equation (14) and putting $K = V = W = Y = \xi$, in the above equation we have

$$h(R(X,\xi)\xi,U) = 0. \tag{29}$$

By the equations (10) and (29) we have

$$h(U,X) = 0. \tag{30}$$

We have $h(U,X) = 0$, for any $U, X \in \chi(M)$, hence the submanifold is totally geodesic. Converse is trivially true.

Theorem 3.5. *An invariant submanifold of an (ϵ)-LP-Sasakian manifold is totally geodesic if and only if $Q(g, C.h) = 0$, provided that $r \neq 2n(2n+1)$.*

Proof. Let M be an invariant submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}$ satisfying $Q(g, R.h) = 0$, then

$$Q(g, R(X,Y).h)(W,K;U,V) = 0,$$

for the vector fields $X, Y, W, K, U, V \in \chi(M)$. Form (12) we have

$$\begin{aligned} 0 = & - g(V,W)(R(X,Y).h)(U,K) + g(U,W)(R(X,Y).h)(V,K) \\ & - g(V,K)(R(X,Y).h)(W,U) + g(U,K)(R(X,Y).h)(W,V). \end{aligned}$$

Hence,

$$\begin{aligned} = & - g(V,W)[R^{\perp}(X,Y)h(U,K) - h(R(X,Y)U,K) - h(R(X,Y)K,U)] \\ & + g(U,W)[R^{\perp}(X,Y)h(V,K) - h(R(X,Y)V,K) - h(R(X,Y)K,V)] \\ & - g(V,K)[R^{\perp}(X,Y)h(W,U) - h(R(X,Y)W,U) - h(R(X,Y)U,W)] \\ & + g(U,K)[R^{\perp}(X,Y)h(W,V) - h(R(X,Y)W,V) - h(R(X,Y)V,W)]. \end{aligned}$$

Using equation (14) and putting $K = V = W = Y = \xi$, in the above equation we have

$$h(C(X,\xi)\xi,U) = 0. \tag{31}$$

By the the equations (10), (13) and (31) we have

$$\{1 - \frac{r}{2n(2n+1)}\}h(U,X) = 0. \tag{32}$$

$h(U,X) = 0$, for any $U, X \in \chi(M)$,provided $r \neq 2n(2n+1)$. Hence the submanifold is totally geodesic. Converse is trivially true.

Theorem 3.6. *An invariant submanifold of an (ϵ)-LP-Sasakian manifold is totally geodesic if and only if $Q(S, C.h) = 0$, provided that $r \neq 2n(2n+1)$.*

Proof. Let M be an invariant submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}$ satisfying $Q(S, C.h) = 0$, then

$$Q(S, C(X,Y).h)(W,K;U,V) = 0,$$

for the vector fields $X, Y, W, K, U, V \in \chi(M)$. Form (12) we have

$$\begin{aligned} 0 = & - S(V,W)(C(X,Y).h)(U,K) + S(U,W)(C(X,Y).h)(V,K) \\ & - S(V,K)(C(X,Y).h)(W,U) + S(U,K)(C(X,Y).h)(W,V). \end{aligned}$$

Hence,

$$\begin{aligned} 0 = & - S(V,W)[C^{\perp}(X,Y)h(U,K) - h(C(X,Y)U,K) - h(C(X,Y)K,U)] \\ & + S(U,W)[C^{\perp}(X,Y)h(V,K) - h(C(X,Y)V,K) - h(C(X,Y)K,V)] \\ & - S(V,K)[C^{\perp}(X,Y)h(W,U) - h(C(X,Y)W,U) - h(C(X,Y)U,W)] \\ & + S(U,K)[C^{\perp}(X,Y)h(W,V) - h(C(X,Y)W,V) - h(C(X,Y)V,W)]. \end{aligned}$$

Using equation (14) and putting $K = V = W = Y = \xi$, in the above equation we have

$$S(\xi,\xi)h(U, C(X,\xi)\xi) = 0. \tag{33}$$

By the the equations (10), (11), (13) and (33) we have

$$2n\{1 - \frac{r}{2n(2n+1)}\}h(U,X) = 0. \tag{34}$$

$h(U,X) = 0$, for any $U, X \in \chi(M)$, provided that $r \neq 2n(2n+1)$. Hence the submanifold is totally geodesic. Converse is trivially true.

4. **Totally umbilical submanifolds of (ϵ)-LP-Sasakian manifolds.**

Let M be a totally umbilical submanifold of (ϵ)-LP-Sasakian manifolds $\bar{M}$. Then the second fundamental form h of M is given by $h(X,Y) = g(X,Y)H$ [3], where $X, Y \in \chi(M)$ and H is mean curvature vector.

If we set $\alpha = ||H||^2$, then for the totally umbilical submanifold M with mean curvature parallel in the normal bundle, we have $X.\alpha = 0$ for any $X \in \chi(M)$, that is, α is constant.

If $\alpha \neq 0$, define a unit vector $e \in \nu$ in the normal bundle, by setting $H = \sqrt{\alpha}e$. The normal bundle can be split into the direct sum $\alpha = \{e\} \oplus \{e\}^\perp$, where $\{e\}^\perp$ is the orthogonal compliment of the line sub-bundle $\{e\}$ spanned by e. For each $X \in \chi(M)$. Set

$$\phi X = \psi(X) - A(X)e + P(X), \qquad \phi e = t + F, \tag{35}$$

where $\psi(x)$ is the tangential components of ϕX, while $A(X)$ and $P(X)$ are the $\{e\}$ and $\{e\}^\perp$ components, respectively. t and F are the $\{e\}$ and $\{e\}^\perp$ components of ϕe, respectively, in view of the skew-symmetry of ϕ.

Lemma 4.1. *Let M be a totally umbilical submanifold of an (ϵ)-LP-Sasakian manifolds $\bar{M}$ with curvature vector parallel to the normal bundle. If $\alpha \neq 0$, then for any $X \in \chi(M)$ following hold:*
i) $\bar{\nabla}_X e = -\sqrt{\alpha}X$,
ii) $\nabla_X t = -\sqrt{\alpha}\psi(X) + \epsilon g(e,\xi)X + 2\epsilon g(X,\xi)g(e,\xi)\xi$,
iii)$\nabla^\perp_X F = -\sqrt{\alpha}P(X)$.

Proof. Taking inner product with respect to Y in both sides of equation (7), we obtain

$$\bar{\nabla}_X N = -g(H,N)X + \nabla^\perp_X N.$$

Putting $N = e$, in the above equation we obtain

$$\bar{\nabla}_X e = -\sqrt{\alpha}X.$$

Thus (i) is proved.

Next put $Y = e$ in the equation (4), and the equation (35) we obtain

$$\begin{aligned}\nabla_X t + \nabla^\perp_X F \quad &+ \quad \sqrt{\alpha}(\psi(X) - A(X)e + P(X)) \\ &= \quad \epsilon g(e,\xi)X + 2\epsilon g(X,\xi)g(e.\xi)\xi.\end{aligned}$$

Now comparing the tangential part we have

$$\nabla_X t = -\sqrt{\alpha}\psi(X) + \epsilon g(e,\xi)X + 2\epsilon g(X,\xi)g(e,\xi)\xi.$$

Thus (ii) is proved. Now comparing $\{e\}^{\perp}$ component and using the result $A(X) = g(X,t)$ we obtain

$$\nabla^{\perp}_X F = -\sqrt{\alpha} P(X).$$

Thus (iii) is proved.

Lemma 4.2. *Let M be a totally umbilical submanifold of an (ϵ)-LP-Sasakian manifold $\bar{M}$ with mean curvature vector parallel in the normal bundle. If $\alpha \neq 0$, and $\xi \perp e$, then, setting $\xi = \xi_1 + \xi_2$, where ξ_1 is the tangential component and ξ_2 is the $\{e\}^{\perp}$-component of ξ, we have*
(i) $\nabla_X \xi_1 = \epsilon\psi(X)$,
(ii)$(\nabla_X \psi)Y = (1+\epsilon\alpha)[g(X,Y)\xi + g(Y,\xi)X] + \epsilon\eta(X)\eta(Y)\xi_1$.

Proof. Putting $\xi = \xi_1 + \xi_2$ in the equation (12) and (35) we have

$$\nabla_X \xi_1 + \nabla_X \xi_2 + h(X,\xi) = \epsilon(\psi(X) - A(X)e + P(X)).$$

Comparing tangential part we have (i), and comparing e component, we have $h(X,\xi) = -\epsilon A(X)e$ i.e.,

$$\sqrt{\alpha}\eta(X) = -A(X), \qquad \sqrt{\alpha}\xi_1 = \epsilon t. \tag{36}$$

Now using the equations (4) and (35) we have

$$(\nabla_X \psi Y) - \nabla_X (AY)e - A(Y)(\nabla_X e) + A(\nabla_X Y)e + (\nabla_X P)Y$$
$$= g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

Using the Lemma 4.1, we obtain from the above equation

$$(\nabla_X \psi)Y + (\nabla_X P)Y + \sqrt{\alpha} g(X,Y)(t+F) - \alpha\eta(Y)X$$
$$= g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

Comparing the tangential part we obtain (ii).

Theorem 4.1. *Let M be a n dimensional totally umbilical submanifold of an (ϵ)-LP-Sasakian manifold with mean curvature vector parallel in the normal bundle. Then one of the following holds :*
(i) M is totally geodesic,
(ii) M is isometric to a sphere,
(iii) M is homothetic to a Sasakian manifold.

Proof. Since H is parallel in the normal bundle, μ is a constant. If $\alpha = 0$, then $H = 0$, and consequently $h(X,Y) = 0$, $X, Y \in \chi(M)$. Thus the submanifold M is totally geodesic, which proves the first part of the theorem.

Next we assume that $\alpha \neq 0$. Define a smooth function $f : M \to R$ by $f = g(e,\xi)$, for $X, Y \in \chi(M)$. Then Lemma 4.1, and equations (5), (6), (7), imply that

$$\begin{aligned} Yf &= g(\nabla_X \xi, e) + g(\xi, \nabla_X e) \\ &= -\epsilon g(Y,t) - \sqrt{\alpha} g(\xi, Y). \end{aligned}$$

So, by using the equation (5), (35) and the Lemma 4.2, we have,

$$XYf - (\nabla_X Y)f = -\epsilon^2 f g(X,Y),$$

$$g(\nabla_X \mathrm{grad} f, Y) = -\epsilon^2 f g(X,Y). \tag{37}$$

Taking trace of this equation we have

$$\Delta f = -\epsilon^2 n f. \tag{38}$$

Then, if f is non-constant function, then the equation (38) is the differential equation in [8], which is necessary and sufficient condition for M to be isometric to a sphere of radius $\frac{1}{\epsilon}$.

If f is a constant, then equation (38) gives $-n\epsilon^2 f = 0$, consequently $f = 0$, that is $\xi \perp e$.

Now define a smooth function $G : M \to R$ by

$$G = \frac{1}{2} tr.\psi^2. \tag{39}$$

Note that (35) gives $g(\psi Y, X) = -g(\psi X, Y), \quad X, Y \in \chi(M)$.

Let ω be a 1-form defined by $\omega = dG$. For each $p \in M$ we can choose a local orthonormal frame $\{e_1, ..., e_n\}$ of M such that $\nabla e_i(p) = 0$. Thus, for any $Z \in \chi(M)$, we have

$$\omega(Z) = ZG = \sum_{i=1}^{n} g((\nabla_Z \psi)(e_i), \psi(e_i)). \tag{40}$$

Using the Lemma 4.2, we obtain

$$\omega(Z) = -2N g(\xi, \psi Z) - 2g(Z, \xi), \tag{41}$$

where $N = (1 + \epsilon\alpha)$.

The first covariant derivative of (41) is

$$(\nabla\omega)(Y,Z) = -2Ng(\xi_1, (\nabla_Y)Z) - 2Ng(\nabla_Y\xi_1, \psi Z) - 2g(Z, \nabla_Y\xi_1).$$

Consequently using the equation (41) and the above equation we have

$$(\nabla^2\omega)\epsilon(2g(Y,Z)\omega(X) + g(X,Y)\omega(Z) + g(X,Z)\omega(Y)) = 0. \tag{42}$$

Equation (42) is the differential equation in [4] which, G being non-constant, is the necessary and sufficient condition for M to be isometric to a sphere. This again leads to case (ii). Suppose G is constant function. Then equation (40) gives $\psi(\xi_1) = 0$. Define a smooth function $G_1 : M \to R$ by

$$G_1 = g(\xi_1, \xi_1).$$

Then using the Lemma 4.2, we get $X\alpha = 0$, $X \in \chi(M)$. In others words ξ_1 has constant length. Taking the covariant derivative in (i) of Lemma 4.2 and using (ii), we get

$$\nabla_X\nabla_Y\xi_1 - \nabla_{\nabla_X Y}\xi_1 = N\epsilon(g(X,Y)\xi_1 - g(Y,\xi_1)X). \tag{43}$$

Further more, from (i) of the Lemma 4.2, it follows that ξ_1 is a killing vector field. Since $N \neq 0$ and ξ_1 is a killing vector field of constant length, which satisfies (43), a result of Okumura [9] states that, if $\xi_1 \neq 0$, then M is homothetic to a Sasakian manifold. which is (iii). Thus to complete the proof we have only to show that $\xi_1 = 0$, cannot happen.

We see that if $\xi_1 = 0$, then $\xi \in \{e\}^\perp$, since $\xi \perp e$. Lemma 4.2 gives $\psi(X) = 0$, thus ϕX is normal to M for all $X \in \chi(M)$. Again, equation (36) gives $t = 0$, consequently $\phi e = F \in \{e\}^\perp$, and $g(\phi X, \phi e) = g(X, e) - \eta(X)\eta(e) = 0$, $X \in \chi(M)$, $g(\phi e, \xi) = 0$. Thus the dim of

$$\nu \geq \dim\{M\} + \dim\{\xi\} + \dim\{e\} + \dim\{\phi e\} - 1,$$

which is impossible as $\dim\{\bar{M}\} = 2n + 1$. This complete the proof.

5. **Example**

In the following we give an example of invariant submanifold of an (ϵ)-LP-Sasakian manifolds. We consider the 5-dimensional manifold [12] $\bar{M} = \{(x, y, z, u, v) \in \mathbb{R}^5 : (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$, where (x, y, z, u, v) are the standard coordinates of $\mathbb{R}^5$. We consider the vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = -2\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z}, \quad e_5 = \frac{\partial}{\partial v}$$

which are linearly independent at each point of $\bar{M}$. Now we define the metric g by

$$\begin{aligned} g(e_i, e_j) &= \epsilon, \quad \text{if} \quad i = j \neq 3, \\ &= 0, \quad \text{if} \quad i \neq j, \\ &= -\epsilon, \quad \text{if} \quad i = j = 3. \end{aligned}$$

Here i, j runs form 1 to 5. We consider an 1-form η defined by

$$\eta(X) = \epsilon g(X, e_3), \quad X \in \chi(\bar{M}).$$

i.e., we choose $e_3 = \xi$.

We define the (1.1) tensor field ϕ by

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \phi e_4 = e_5, \quad \phi e_5 = e_4.$$

The linear property of g and ϕ shows that

$$\eta(e_3) = -\epsilon, \quad \phi^2(X) = X + \eta(X)e_3,$$

$$g(\phi X, \phi Y) = g(X, Y) + \epsilon\eta(X)\eta(Y)$$

for any vector fields X, Y on $\chi(\bar{M})$. So (ϕ, ξ, η, g) defines an almost contact manifold with $e_5 = \xi$. Moreover, let $\bar{\nabla}$ is the Levi-Civita connection with respect to metric g of $\bar{M}$. Then we have

$$[e_1, e_2] = -2\epsilon e_3, \quad [e_4, e_5] = -2\epsilon e_3, \quad [e_i, e_j] = 0, \quad \text{otherwise}.$$

By Koszul formula, we obtain the following

$$\begin{aligned} &\bar{\nabla}_{e_1} e_3 = \epsilon e_2, \quad \bar{\nabla}_{e_1} e_2 = -\epsilon e_3, \quad \bar{\nabla}_{e_2} e_3 = \epsilon e_1, \quad \bar{\nabla}_{e_2} e_1 = \epsilon e_3, \\ &\bar{\nabla}_{e_3} e_5 = \epsilon e_4, \quad \bar{\nabla}_{e_3} e_4 = \epsilon e_5, \quad \bar{\nabla}_{e_3} e_2 = \epsilon e_1, \quad \bar{\nabla}_{e_3} e_1 = \epsilon e_2, \\ &\bar{\nabla}_{e_4} e_5 = -\epsilon e_3, \quad \bar{\nabla}_{e_4} e_3 = \epsilon e_5, \quad \bar{\nabla}_{e_5} e_4 = \epsilon e_3, \quad \bar{\nabla}_{e_5} e_3 = \epsilon e_4, \\ &\bar{\nabla}_{e_i} e_j = 0, \quad \text{otherwise}. \end{aligned}$$

Thus we see that $\bar{M}$ is an (ϵ)-LP-Sasakian manifold.

Let M be a subset of $\bar{M}$ and consider the isometric immersion $f : M \to \bar{M}$ defined by

$$f(x, y, z) = (x, y, z, 0, 0).$$

It is easy to prove that $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$ is a submanifold of $\bar{M}$, where (x, y, z) are the standard co-ordinate of $\mathbb{R}^3$. We choose the vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. We define g_1 by

$$\begin{aligned} g(e_i, e_j) &= \epsilon, \quad \text{if} \quad i = j \neq 3, \\ &= 0, \quad \text{if} \quad i \neq j, \\ &= -\epsilon, \quad \text{if} \quad i = j = 3. \end{aligned}$$

Here i, j runs form 1 to 3.

We define 1-form η_1 and (1,1) tensor ϕ_1 respectively by

$$\eta_1 = g_1(X, e_5),$$

$$\phi_1(e_1) = e_2, \quad \phi_1(e_2) = e_1, \quad \phi_1(e_3) = 0.$$

The linear property of g_1 and ϕ_1 shows that

$$\eta_1(e_3) = 1, \quad \phi_1^2(X) = X + \eta_1(X)e_5,$$

$$g_1(\phi_1 X, \phi_1 Y) = g_1(X, Y) - \eta_1(X)\eta_1(Y)$$

for any vector fields X, Y on $M(\phi_1, \xi, \eta_1, g_1)$. It is seen that M is an invariant submanifold of $\bar{M}$ with $e_3 = \xi$. Moreover, let ∇ be the Levi-Civita connection with respect to the metric g_1. Then we have

$$[e_1, e_2] = -2\epsilon e_3, \quad [e_i, e_j] = 0, \quad \text{otherwise}.$$

By using Kouszul formula, we obtain

$$\begin{gathered} \nabla_{e_1} e_3 = \epsilon e_2, \quad \nabla_{e_1} e_2 = -\epsilon e_3, \quad \nabla_{e_2} e_3 = \epsilon e_1, \quad \nabla_{e_2} e_1 = \epsilon e_3, \\ \nabla_{e_3} e_2 = \epsilon e_1, \quad \nabla_{e_3} e_1 = \epsilon e_2, \quad \nabla_{e_i} e_j = 0, \quad \text{otherwise}. \end{gathered}$$

Hence $M(\phi, \xi, \eta, g)$ is a three dimensional (ϵ)-LP-Sasakian submanifold of the manifold $\bar{M}$.

If we take $D =< e_1 >, D^{\perp} =< e_2 >$. Then we can write

$$TM = D \oplus D^{\perp} \oplus < \xi > .$$

So the Lemma 3.2 is verified.

Now from the value of $\bar{\nabla}_{e_i} e_j$, and $\nabla_{e_i} e_j$ and the form the relation $h(e_i, e_j) = \bar{\nabla}_{e_i} e_j - \nabla_{e_i} e_j$, we can obtain $h(X, Y) = 0$, for any vector fields X, Y. So the submanifold M is totally geodesic. Hence the Theorem 3.1 is verified.

Acknowledgment. The second author is supported by UGC, India, ID-421642

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