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AN EFFICIENT HAAR WAVELET COLLOCATION METHOD FOR SOLVING PENNES BIOHEAT TRANSFER MODEL

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Abstract. The aim of this paper is to develop an efficient numerical technique based on Haar wavelets for solving the Pennes bioheat transfer model (PBHTM)-which is a prime model for studying the mechanism of heat transfer in living tissues. The properties of the Haar wavelet expansions together with operational matrix of integration are used to convert the underlying problem into a system of algebraic equations which in turn speeds up the computational process. The numerical outcomes suggest that the proposed method is in a reasonable agreement with the exact solution of the Pennes bioheat transfer model.

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Keywords: Bioheat transfer model, Wavelet, Collocation method, Haar wavelet, Operational matrix.

1. Introduction

It is well known that a natural instinct of the human body is to use heat in order to fight diseases. The use of heat to destroy unwanted tissue in therapeutic purposes has found immense applications, for instance, laser, microwave and magnetic fluid hyperthermia. As any unwanted tissue is surrounded by a healthy tissue, therefore, the effectiveness of the thermal therapy is completely dependent upon the precision, prediction and regulation of temperature in the undesirable tissue. To prevent any damage to the healthy tissue in a therapeutic process, it is essential to acquire a profound knowledge of the temperature distribution in the entire treatment region. However, the accurate determination of the temperature field over the region of treatment during any invasive temperature probe is limited due to the pain tolerance of the patient. Therefore, an appropriate analysis and modeling of the entire thermal procedure plays a vital role in optimizing the temperature distribution in the region of treatment. Hence, for the significant evaluation of the extend of thermal damage an appropriate bioheat model plays a crucial role during the development of equipment and pre-planning purposes. Among several available mathematical models in the open literature, the Pennes bio-heat transfer model [1] has been vastly applied mainly due to its simplicity and lucid nature. Mathematically, the Pennes bioheat transfer model is governed by a second order partial differential equation

$$\rho c \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \rho_b \omega_b c_b (u_b - u) + Q_m + Q_r. \tag{1.1}$$

where the parameters $\rho, c, k, \omega_b, c_b, u_b, \rho_b$ and u involved in the equation (1.1) denote density of the tissue, specific heat of the tissue, thermal conductivity of the tissue, blood perfusion rate, specific heat of the blood, temperature of arterial blood, density of blood and tissue temperature, respectively. The terms Q_m and Q_r represent the heat generated by the metabolic process and heat generated per unit volume of the tissue due to electromagnetic radiations.

It is worth to note that equation (1.1) can be rewritten in a relatively compact form as

$$\frac{\partial u}{\partial t} = ka \frac{\partial^2 u}{\partial x^2} + b(u_b - u) + a\left(Q_m + Q_r\right), \tag{1.2}$$

where, $a = 1/\rho c$ and $b = a\rho_b \omega_b c_b$. The one-dimensional form of (1.2) is given by

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - bu(x,t) + \theta(x,t) \tag{1.3}$$

where $\alpha = ka$, and $\theta(x,t) = bu_b + a(Q_m + Q_r(x,t))$.

In recent times, several numerical and analytical techniques have been successfully applied for obtaining the solution of Pennes bioheat transfer model (1.3). Some of the methods invoked in the recent literature include, Boundary element method [2], Monte Carlo method [3], Finite difference method [4], [5], [6], Finite difference-decomposition method [7], Variational iteration method [8], Bessel functions [9], Laplace transform method [10], Daubechies Wavelet-finite difference method [11], Galerkin Finite Element Method [12], Homotopy perturbation method [13, 14].

Over the last couple of decades, wavelets have been studied extensively and have emerged as a powerful computational tool for attaining exact and/or numerical solutions for a wide range of problems including algebraic, differential, partial differential, functional delay, and integro-differential equations. Wavelets, in essence, are continuously oscillatory functions which possess some attractive features: zero-mean, fast decay, short life, time-frequency representation, multiresolution, etc. Wavelets have the ability to detect information at different scales and at different locations throughout a computational domain. Wavelets can provide a basis set in which the basis functions are constructed by dilating and translating a fixed function known as the *mother wavelet*. Different types of wavelets and approximating functions have been used in the numerical solution of boundary-value problems such as Daubechies, Battle-Lemarie, B-spline, Chebyshev, Legendre, and Haar wavelets. Among all the wavelet families, the Haar wavelets have gained considerable popularity among researchers' mainly due to some of their outstanding properties such as simple applicability, orthogonality, and compact support. Compact support of the Haar wavelet basis permits straight inclusion of the different types of boundary conditions in the numeric algorithms.

The main goal of this article is to employ a collocation method based on Haar wavelets for the numerical treatment of the Pennes bio-heat transfer model (1.3). The motivation and philosophy behind this approach is that it converts the underlying problem to a set of algebraic equations by expanding the term, which has maximum derivative, given in the equation as Haar functions with unknown coefficients and thus, simplifying the solution process of the problem to a significant extent. The numerical outcomes suggest that the proposed method is in a reasonable agreement with the exact solution of the Pennes bio-heat transfer model (1.3).

The outline of the paper is as follows: In Section 2, we introduce some preliminaries including the Haar wavelets followed by the construction of operational matrix of integration. In Section 3, we propose a method of solution for the Pennes bioheat transfer model (1.3). Section 4 is completely devoted to explicitly illustrate the obtained results and discuss the accuracy and efficiency of the proposed method by comparing the obtained results with the exact solution. Finally, a conclusion is drawn in Section 5.

2. Haar Wavelets and Operational Matrix of Integration

Haar wavelets have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar[15]. The Haar wavelet, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. The Haar wavelet family for $x \in [0, 1]$ is defined as follows:

$$h_i(x) = \begin{cases} 1, & \text{for } x \in [\alpha, \beta) \\ -1, & \text{for } x \in [\beta, \gamma) \\ 0, & \text{elsewhere.} \end{cases} \tag{2.1}$$

where

$$\alpha = \frac{k}{m}, \quad \beta = \frac{k+0.5}{m}, \quad \gamma = \frac{k+1}{m}.$$

Here m and k have integer values as $m = 2^j, j = 0, 1, \ldots, J$ and J shows the resolution of the wavelet and $k = 0, 1, \ldots, m-1$ is the translation parameter. Maximal level of resolution is J. The index of h_i in Eq.(2.1) is calculated by $i = m + k + 1$. In the case with minimal values $m = 1, k = 0$, we have $i = 2$, the maximal value of i is $2M = 2^{j+1}$. We also have $i = 1$ corresponding to the scaling function of Haar wavelet family, i.e. $h_1(x) = 1$ in $[0, 1]$. For a detailed information regarding the Haar wavelets and their applications, we refer to the monographs[15, 16].

Now, we intend to establish an operational matrix for integration by means of Haar wavelets for which we follow the same notations as used in [17] for Haar function and their integrals as

$$P_{i,1}(x) = \int_0^x h_i(x)\,dx, \quad P_{i,n+1}(x) = \int_0^x P_{i,n}(x)\,dx, \quad \text{and}$$
$$C_{i,n}(x) = \int_0^1 P_{i,n}(x)\,dx \;\; n = 1, 2, \ldots. \tag{2.2}$$

The integrals in (2.2) can be evaluated analytically by virtue of (2.1); consequently, we obtain the following equations

$$P_{i,n}(x) = \begin{cases} 0 & \text{for } x \in [0, \alpha) \\ \dfrac{1}{n!}(x-\alpha)^n & \text{for } x \in [\alpha, \beta) \\ \dfrac{1}{n!}\big[(x-\alpha)^n \\ \qquad -2(x-\beta)^n\big] & \text{for } x \in [\beta, \gamma) \\ \dfrac{1}{n!}\big[(x-\alpha)^n \\ \qquad -2(x-\beta)^n + (x-\gamma)^n\big], & \text{for } x \in [\gamma, 1), \end{cases} \tag{2.3}$$

where $i = 2, 3, \ldots$ and $n = 1, 2, \ldots$. Note that

$$P_{1,n}(x) = \frac{x^n}{n!}, \quad C_{1,n}(x) = \frac{1}{(n+1)!}, \quad n = 1, 2, \ldots.$$

By invoking the Haar basis functions, any square integrable function $f(x)$ defined on $[0, 1]$ can be expressed as

$$f(x) = a_1 h_1(x) + a_2 h_2(x) + \cdots = \sum_{i=1}^{\infty} a_i h_i(x), \quad x \in [0, 1] \tag{2.4}$$

where the Haar coefficients a_i, $i = 1, 2, \dots$, are determined by

$$a_i = \langle f, h_i \rangle = 2^j \int_0^1 f(x) h_i(x)\, dx. \tag{2.5}$$

Even though the series expansion (2.4) is an infinite sum, we can reasonably approximate $f(x)$ by using finitely many terms, provided $f(x)$ is a piecewise constant or it may be approximated as a piecewise constant for each sub-interval; that is,

$$f(x) \backsimeq f_m(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{2.6}$$

Analogously, we can rewrite the expansion (2.6) in matrix form as

$$\mathbf{F}^T = \mathbf{A}_m^T \mathbf{H}_m, \tag{2.7}$$

where $\mathbf{F}$ is the discrete form of the continuous function $f(x_\ell)$ and $\mathbf{A}_m^T = [a_1, a_2, \dots, a_m]$ is the m-dimensional row vector. Moreover, $\mathbf{H}_m$ denotes the Haar wavelet matrix of order m and is given by $\mathbf{H}_m = [\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_m]^T$; that is,

$$\mathbf{H}_m = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \vdots \\ \mathbf{h}_m \end{pmatrix} = \begin{pmatrix} h_{1,1} & h_{1,2} & \dots & h_{1,m} \\ h_{2,1} & h_{2,2} & \dots & h_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m,1} & h_{m,2} & \dots & h_{m,m} \end{pmatrix}, \tag{2.8}$$

where $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_m$ denote the discrete versions of the Haar wavelet basis. For the Haar wavelet approximations, we rely on the following collocation points:

$$x_\ell = \frac{\ell - 0.5}{2M}, \quad \ell = 1, 2, \dots, m. \tag{2.9}$$

For example, if $j = 2;$ that is, $2M = 8$, the Haar matrix $\mathbf{H}$ and the operational matrix **P** can be expressed as

$$\mathbf{H}_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$\mathbf{P}_8 = \frac{1}{64}\begin{pmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Assume that $f(x)$ satisfies a Lipschitz condition on $[0,1]$, there exist positive number $K > 0$, such that $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$, $\forall\, x_1, x_2 \in [0,1]$, where K is the Lipschitz constant. Therefore, the Haar approximation $f_m(x)$ of $f(x)$ is given by

$$f_m(x) = \sum_{i=0}^{m-1} a_i h_i(x), \quad m = 2^{p+1},\ p = 0, 1, 2, \ldots, M. \tag{2.10}$$

Then, the corresponding error at mth level may be defined as

$$\left\| f(x) - f_m(x) \right\|_2 = \left\| f(x) - \sum_{i=0}^{m-1} a_i h_i(x) \right\|_2 = \left\| \sum_{i=2^{p+1}}^{\infty} a_i h_i(x) \right\|_2. \tag{2.11}$$

With the exact solution of the model (1.3) at hand, we can obtain an upper bound of the error for the solution of Pennes bioheat transfer model (1.3). Convergence of the proposed method may be discussed on the same lines as given in Yi and Huang [19].

Theorem 2.1.[19] *Suppose $f(x)$ satisfies the Lipschitz condition on $[0,1]$ with Lipschitz constant K and $f_m(x)$ are the Haar approximations of $f(x)$, then we have the error bound as follows*

$$\left\| f(x) - f_m(x) \right\|_2 \leq \frac{K}{\sqrt{3}\, m^2}. \tag{2.12}$$

3. Solution of the Problem

In this section, we shall formally investigate the Pennes bioheat transfer model (1.3) for its approximate solution by employing the Haar wavelet operational matrices. The main advantage of the proposed method is that it converts the whole problem into a system of algebraic equations for which the computation is easy and simple. To facilitate the method of solution, we recall the one-dimensional form of the model (1.3)

$$\dot{u}(x,t) = \alpha u''(x,t) - bu(x,t) + \theta(x,t), \tag{3.1}$$

where $\cdot$ and $'$ indicate the differentiation with respect to t and x, respectively. Moreover, $\alpha, b, \theta(x,t)$ have their usual meanings and

$$Q_r(x,t) = \Big(-399800 - \frac{\mu^2}{2}\Big)\exp\left\{-\frac{t(45-u_a)\sinh\Big(\mu(L-x)\Big)}{\sinh(\mu L)}\right\} - 420. \tag{3.2}$$

Equation (3.1) can be revamped into the following form

$$\dot{u}(x_\ell, t_{s+1}) = \alpha u''(x_\ell, t_{s+1}) - bu(x_\ell, t_{s+1}) + \theta(x_\ell, t_{s+1}), \tag{3.3}$$

subjected to the boundary conditions

$$u(x_\ell, 0) = 37 + \frac{(45-u_a)\sinh\Big(\mu(L-x_\ell)\Big)}{\sinh(\mu L)} = g(x_\ell), \tag{3.4}$$

$$u(0, t_{s+1}) = 37 + 8e^{t_{s+1}} = f_0(t_{s+1}), \quad u(1, t_{s+1}) = 37 = f_1(t_{s+1}) \tag{3.5}$$

which leads us from the time layer t_s to t_{s+1}, where $\mu = \sqrt{\frac{\rho_b \omega_b c_b}{k}}, L = 1.$

Next, we divide the interval (0,1] into N equal parts of length Δt=(0,1]/N and denote $t_s = (s-1)\Delta t, s = 1, 2, 3, ..., N$. By virtue of the Haar wavelet basis, we can express $\ddot{u}''(x,t)$ as:

$$\ddot{u}''(x,t) = \sum_{i=0}^{2M} a_i h_i(x). \tag{3.6}$$

The row vector a_i is constant in the sub-interval $t \in (t_s, t_{s+1}]$. Integrating (3.6) with respect to t from t_s to t and twice with respect to x from 0 to x, we obtain

$$u''(x,t) = (t-t_s)\sum_{i=0}^{2M} a_i h_i(x) + u''(x,t), \tag{3.7}$$

$$u(x,t) = (t-t_s)\sum_{i=0}^{2M} a_i P_2(x) + u(x,t_s) - u(0,t_s) + x\left[u'(0,t) - u'(0,t_s)\right] + u(0,t), \tag{3.8}$$

$$\dot{u}(x,t) = \sum_{i=0}^{2M} a_i P_2(x) + x\dot{u}'(0,t) + \dot{u}(0,t). \tag{3.9}$$

For brevity, we shall denote the boundary conditions as:

$$u(0,t_s) = f_0(t_s), \quad u(1,t_s) = f_1(t_s),$$
$$\dot{u}(0,t) = f_0'(t), \quad \dot{u}(1,t) = f_1'(t).$$

Then, in particular, for $x = 1$ in (3.8) and (3.9), we obtain

$$u'(0,t) - u'(0,t_s) = -(t-t_s)\sum_{i=0}^{2M} a_i P_2(1) + f_1(t) - f_0(t_s) + f_0(t_s), \tag{3.10}$$

$$\dot{u}'(0,t) = f_1'(t) - \sum_{i=0}^{2M} a_i P_2(1) - f_0'(t). \tag{3.11}$$

Moreover, from (2.3), we obtain

$$P_2(1) = \begin{cases} 0.5, & \text{if} \quad i{=}1; \\ \dfrac{1}{4m^2}, & \text{if} \quad i{>}1. \end{cases}$$

Implementing (3.10) and (3.11) into the equations (3.7)-(3.9), and discretizising the

the result by assuming $x \to x_l, t \to t_{s+1}$ leads to

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s) \sum_{i=0}^{2M} a_i h_i(x_l) + u''(x_l, t_s), \tag{3.12}$$

$$\begin{aligned} u(x_l, t_{s+1}) &= (t_{s+1} - t_s) \sum_{i=0}^{2M} a_i P_2(x_l) + u(x_l, t_s) - f_0(t_s) + f_0(t_{s+1}) \\ &\quad + x_l \left[-(t_{s+1} - t_s) \sum_{i=0}^{2M} a_i P_2(1) + f_1(t_{s+1}) - f_0(t_{s+1}) - f_1(t_s) + f_0(t_s) \right], \end{aligned} \tag{3.13}$$

$$\begin{aligned} \dot{u}(x_l, t_{s+1}) &= (t_{s+1} - t_s) \sum_{i=0}^{2M} a_i P_2(x_l) + f_0'(t_{s+1}) \\ &\quad + x_l \left[-\sum_{i=0}^{2M} a_i P_2(1) + f_1'(t_{s+1}) - f_0'(t_{s+1}) \right]. \end{aligned} \tag{3.14}$$

Upon substituting the values from (3.12)-(3.14) into the Pennes bio-heat transfer equation (3.3), we get

$$\begin{aligned} &(t_{s+1} - t_s) \sum_{i=0}^{2M} a_i P_2(x_l) - x_l (t_{s+1} - t_s) \sum_{i=0}^{2M} a_i P_2(1) - \frac{\alpha}{2} (t_{s+1} - t_s)^2 \sum_{i=0}^{2M} a_i h_i(x_l) \\ &\quad + \frac{b}{2} (t_{s+1} - t_s)^2 \sum_{i=0}^{2M} a_i P_2(x_l) - \frac{b x_l}{2} (t_{s+1} - t_s)^2 \sum_{i=0}^{2M} a_i P_2(1) \\ &= -\left[\dot{u}(x_l, t_s) - \dot{u}(0, t_s)\right] + x_l \left[f_1'(t_s) - f_0'(t_s)\right] \\ &\quad + x_l \left[f_0'(t_{s+1}) - f_1'(t_{s+1})\right] - f_0'(t_{s+1}) \\ &\quad + \alpha (t_{s+1} - t_s) \dot{u}''(x_l, t_s) + \alpha u''(x_l, t_s) - b(t_{s+1} - t_s) \left[\dot{u}(x_l, t_s) - \dot{u}(0, t_s)\right] \\ &\quad - b u(x_l - t_s) + b u(0, t_s) + b x_l (t_{s+1} - t_s) \left[f_1'(t_s) - f_0'(t_s)\right] \\ &\quad + b x_l \left[f_1(t_s) - f_0(t_s) + f_0(t_{s+1}) - f_1(t_{s+1})\right] - b f_0(t_{s+1}) + \theta(x_l, t_{s+1}) \end{aligned} \tag{3.13}$$

Equation (3.15) can be rewritten in the matrix form as

$$\mathbf{S}\mathbf{A}_m^T = \mathbf{B}_m, \tag{3.16}$$

where $\mathbf{A}_m^T, \mathbf{S}, \mathbf{B}_m$ are row matrix ,square matrix of order $2M \times 2M$ and column matrix of order m, respectively, and are given by

$$\mathbf{A}_m^T = \left[a_1, a_2, a_3, \dots a_m\right], \quad \mathbf{B}_m = \left[b_1, b_2, b_3, \dots b_m\right],$$

$$\mathbf{S} = \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} \cdots & s_{1,m} \\ s_{2,1} & s_{2,2} & s_{2,3} \cdots & s_{2,m} \\ s_{3,1} & s_{3,2} & s_{3,3} \cdots & s_{3,m} \\ \vdots & \vdots & \vdots & \vdots \\ s_{m,1} & s_{m,2} & s_{m,3} \cdots & s_{m,m} \end{bmatrix}.$$

The system of equations in (3.15) represents the algebraic form of Pennes bioheat transfer equation, which upon solving yield the Haar coefficients $a_i's$. In sequel, from (3.13), we obtain the value of u, which is reasonably close to the exact solution of the Pennes bio-heat transfer equation.

4. Numerical Results and Discussion

In this section, numerical and graphical results obtained from HWCM for the solution of 1D Pennes bioheat transfer equation are presented in order to demonstrate the accuracy and efficiency of the proposed method. The model under consideration uses the parameters $\rho, \rho_b, c, \omega_b, u_a, k$ and Q_m whose corresponding values are given in Table 1.

Symbol(unit)	Value
ρ and $\rho_b(kg/m^3)$	1000
c and $c_b(j/kgc)$	4000
$k(W/mc)$	0.5
$\omega_b(m^3/s/m^3$	0.0005
$u_a(c)$	37
$Q_m(W/m^3)$	420

Table 1: The Value of the parameters.

The exact solution corresponding to the Pennes bioheat equation (1.3) is given by:

$$u(x,t) = u_a + e^{-t}\frac{(45-u_a)\sinh\big(\mu(L-x)\big)}{\sinh(\mu L)}. \tag{4.1}$$

Upon setting $N = 512, J = 8$, the obtained results are compared with the exact solution. It is worth noticing that the obtained and exact solutions are significantly close to each other as is suggested by the absolute error. This in turn indicates that the proposed method is quite effective in studying the temperature distribution in the living tissues during thermal therapy. The comparison results are shown in the Table 2 and 3. Moreover, the plots of computed solutions presented in figures $1, 2, 3$

indicate that the hyperthermia position in a living tissue is varying with change of time. Here it is worth mentioning that the numerical results derived in this study can be used to several heat transfer problems arising in different biological systems.

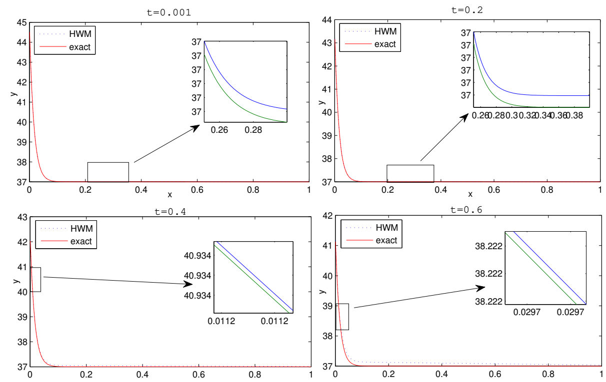

Figure 1: Comparison between Haar and Exact for $t = 0.001, t = 0.2, t = 0.4, t = 0.6$

Table 2: Comparison between exact and HWCM solution

Time(t)	$x_1 = 0.00097$			$x_2 = 0.0029$			$x_3 = 0.0049$			$x_4 = 0.0068$		
	HWCM	*Exact*	*Ab.Error*	*HWCM*	*Exact*	*Ab.Error*	*HWCM*	*Exact*	*Ab.Error*	*HWCM*	*Exact*	*Ab.Error*
0.001	44.51336	44.51336	1.1280e-07	43.64037	43.64037	1.2161e-07	42.86881	42.86881	1.2939e-07	42.18690	42.18690	1.3626e-07
0.1	43.80627	43.80627	0.00109	43.01564	43.01446	0.00117	42.31687	42.31563	0.00123	41.69928	41.69799	0.00128
0.2	43.16194	43.15757	0.00436	42.44706	42.44211	0.00495	41.81518	41.80978	0.00540	41.25673	41.25092	0.00580
0.3	42.58134	42.57160	0.00973	41.93587	41.92423	0.01164	41.36515	41.35207	0.01307	40.86077	40.84639	0.01437
0.4	42.05855	42.04139	0.01715	41.47709	41.45562	0.02146	40.96261	40.93791	0.02465	40.50800	40.48036	0.02763
0.5	41.58819	41.56164	0.02655	41.06618	41.03161	0.03456	40.60376	40.56317	0.04058	40.19522	40.14916	0.04606
0.6	41.16682	41.12754	0.03927	40.70043	40.64795	0.05247	40.28649	40.22409	0.06240	39.92090	39.84948	0.07141

5. Conclusion

In this paper, we employed the Haar wavelet collocation method for the numerical solution of Pennes bioheat transfer model (PBHTM). The main advantage of the proposed method is that it transfers the whole scheme into a system of algebraic equations for which the computation is easy and simple. In addition, other nice

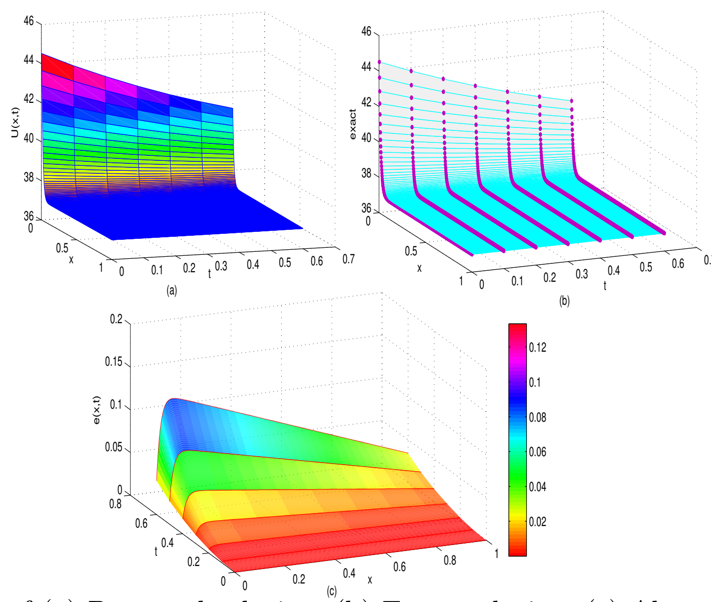

Figure 2: Physical behavior of (a) Proposed solution, (b) Exact solution, (c) Absolute error.

features of this scheme are its simplicity, applicability, and less computational effort. To assure the efficiency of the proposed method, comparisons between the obtained numerical and the exact solutions of the problem under consideration are presented in an illustrative manner. We would like to stress that the numerical solution includes not only time information but also frequency information due to the localization property of wavelet basis; with some change we can apply this method with the help of other orthonormal systems. The computational work was carried out in MATLAB (R2018b) software.

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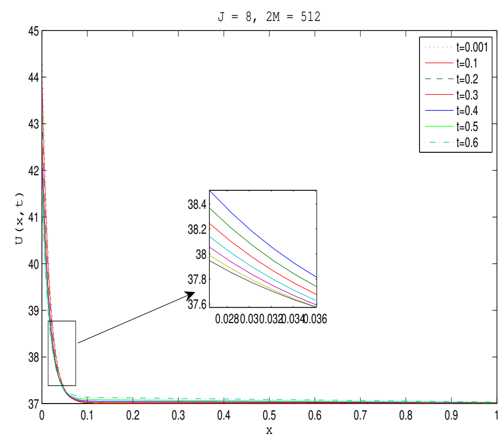

Figure 3: Temperature Distribution $u(x, t)$ for different value of t.

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Table 3: Comparison between exact and HWCM solution

Time(t)	$x_5 = 0.0088$			$x_6 = 0.0107$			$x_7 = 0.0127$			$x_8 = 0.0146$		
	HWCM	*Exact*	*Ab.Error*	*HWCM*	*Exact*	*Ab.Error*	*HWCM*	*Exact*	*Ab.Error*	*HWCM*	*Exact*	*Ab.Error*
0.001	41.58422	41.58422	1.4233e-07	41.05157	41.05157	1.4770e-07	40.58081	40.58081	1.5245e-07	40.16475	40.16475	1.5664e-07
0.1	41.15346	41.15212	0.00133	40.67106	40.66968	0.00137	40.24471	40.24329	0.00141	39.86789	39.86645	0.00144
0.2	40.76316	40.75700	0.00616	40.32694	40.32046	0.00647	39.94140	39.93465	0.00675	39.60066	39.59367	0.00699
0.3	40.41499	40.39947	0.01552	40.02100	40.00448	0.01652	39.67279	39.65538	0.01741	39.36504	39.34685	0.01819
0.4	40.10618	40.07597	0.03021	39.75105	39.71856	0.03248	39.43717	39.40269	0.03448	39.15975	39.12351	0.03624
0.5	39.83411	39.77325	0.05085	39.51494	39.45986	0.05508	39.23284	39.17404	0.05880	38.98351	38.92143	0.06207
0.6	39.59771	39.51839	0.07932	39.31206	39.22577	0.08628	39.05957	38.96715	0.09242	38.83640	38.73858	0.09781

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