Acta Universitatis Apulensis No. 60/2019
ISSN: 1582-5329 pp. 37-52
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.60.03

SOME COMMON FIXED POINT THEOREMS IN CONE B_2-METRIC SPACES OVER BANACH ALGEBRA

V.H. Badshah, Prakash Bhagat and Satish Shukla

Abstract. In this paper, some common fixed point theorems for generalized (λ, μ)-Reich pairs in a complete cone b_2-metric spaces over Banach algebra are proved. Our results generalize and extend some well-known results from 2-metric, b-metric and cone metric spaces. An example is presented which illustrate the main result of this paper.

2010 *Mathematics Subject Classification*: 47H10; 54H25

Keywords: Cone b_2-metric space, Picard sequence, contraction, fixed point.

1. Introduction and Preliminaries

The concept of 2-metric space has been investigated by S. Gahler in a series of papers [10, 11, 12]. In 2007, Huang and Zhang [7] introduced the notion of cone metric spaces as a generalization of metric spaces and proved some fixed point results for contractive mappings. In the papers [3, 18, 19, 21] authors proved the equivalency of some notions in cone metric spaces and fixed point results in cone metric spaces; with their ordinary metric versions. Liu and Xu [4] defined the cone metric spaces over Banach algebra and proved some fixed point results for the contractive mappings with vector contractive constants. Liu and Xu [4] showed that the conclusions of the papers [3, 18, 19, 21] are not applicable if the cone metric spaces are taken over Banach algebra. Singh et al. [1] introduced cone 2-metric spaces which unifies both the concepts of cone metric and 2-metric spaces.

On the other hand, Bakhtin [5] and Czerwik [13] introduced b-metric spaces as a generalization of metric spaces. Hussain and Shah [9] introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. Mustfa et al. [20] unified and generalized the notions of 2-metric spaces and b-metric spaces by introducing the notion of b_2-metric spaces. Recently, Fernandez et al. [6] combined the concepts of b_2-metric spaces and cone metric spaces and introduced the notion of cone b_2-metric spaces over Banach algebra. They proved some fixed point theorems in this new setting. In this paper, we improve and generalize the fixed point result of Fernandez et al. [6] and prove some common fixed point results for a pair of mappings called generalized (λ, μ)-Reich pair on cone b_2-metric spaces over Banach algebra. We also point out the non-feasibility of contractive conditions of Fernandez et al. [6].

We first state some well-known definitions and concepts which will be needed in the sequel.

Let A always be a real Banach algebra with a multiplicative unit e, that is, $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1} (see [10]).

The following proposition can be found, e.g., in [10].

Proposition 1. *Let A be a Banach algebra with the unit e, and $x \in A$. If the spectral radius $\rho(x)$ of x is less than 1, that is,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n} < 1$$

then $e - x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if:

(1) P is nonempty closed and $\{\theta, e\} \subset P$;

(2) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β;

(3) $P^2 = PP \subset P$;

(4) $P \cap (-P) = \{\theta\}$

where θ and e are respectively the zero vector and unit of A.

Given a cone $P \subset A$, we define a partial ordering $\preceq$ in A with respect to P by $x \preceq y$ (or equivalently $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (or equivalently $y \succ x$) to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ (or equivalently $y \gg x$) will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

The cone is called normal if there exists a number $K > 0$ such that for all $x, y \in P$

$$x \preceq y \implies \|x\| \leq K\|y\|.$$

The least number K satisfying the above inequality is called the normal constant of P. The cone P is called solid if $\operatorname{int}P \neq \emptyset$.

In the following, we always assume that the cone P is solid cone in Banach algebra A and $\preceq$ is partial ordering with respect to P.

Proposition 2 ([15]). *Let P be a cone in a Banach algebra A, $a \in P$ and $b, c \in A$ are such that $b \preceq c$, then $ab \preceq ac$.*

Lemma 1 ([2, 14, 22]). *Let A be a Banach algebra with a solid cone P. Then:*

(a) If $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(b) If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.

(c) If $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n > n_0$.

(d) If $a, b, c \in P$ such that $a \preceq b$ and $b \ll c$, then $a \ll c$.

(e) If $a, b, c \in P$ such that $a \ll b$ and $b \preceq c$, then $a \ll c$.

(f) If $a, b, c \in P$ such that $a \ll b$ and $b \ll c$, then $a \ll c$.

Remark 1 ([14]). *If $\rho(x) < 1$ then $\|x^n\| \to 0$ as $n \to \infty$.*

Definition 1 ([8, 23]). *Let P be a solid cone in a Banach algebra A. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n > n_0$.*

Proposition 3 ([15]). *Let P be a solid cone in a Banach algebra A and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

Proposition 4 ([15]). *Let A be a Banach algebra with a unit e, P be a cone in A. Then, for any $a, b \in A$, $c \in P$ with $a \preceq b$ we have $ac \preceq bc$.*

Lemma 2 ([15]). *Let A be a Banach algebra and let x, y be vectors in A. If x and y commute, then the following hold:*

(i) $\rho(xy) \leq \rho(x)\rho(y)$;

(ii) $\rho(x + y) \leq \rho(x) + \rho(y)$;

(iii) $|\rho(x) - \rho(y)| \leq \rho(x - y)$.

Lemma 3 ([15]). *Let A be a Banach algebra and let k be a vector in A. If $0 \leq \rho(k) < 1$, then we have*

$$\rho\left((e-k)^{-1}\right) \leq (1-\rho(k))^{-1}.$$

Definition 2 ([1, 16, 17]). *Let X be a nonempty set. Suppose the mapping $d\colon X \times X \times X \to P$ satisfies:*

(1) for every $x, y \in X$ with $x \neq y$ there exists $z \in X$ such that $d(x,y,z) \neq \theta$;

(2) if at least two of $x, y, z \in X$ are equal, then $d(x,y,z) = \theta$;

(3) $d(x,y,z) = d(p(x,y,z))$ for all $x, y, z \in X$, where $p(x,y,z)$ denotes all the permutations of x, y, z;

(4) $d(x,y,z) \preceq d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x, y, z, w \in X$.

Then d is called a cone 2-metric on X, and (X,d) is called a cone 2-metric space over Banach algebra A.

Definition 3 ([9]). *Let X be a nonempty set and A a real Banach algebra with cone P. A vector-valued function $d\colon X \times X \to P$ is said to be a cone b-metric function on X with the constant $s \geq 1$ if the following conditions are satisfied:*

1. *$\theta \preceq d(x,y)$, for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;*
2. *$d(x,y) = d(y,x)$ for all $x, y \in X$;*
3. *$d(x,z) \preceq s[d(x,y) + d(y,z)]$ for all $x, y, z \in X$.*

The pair (X,d) is called the cone b-metric space. Observe that if $s = 1$, then the ordinary triangle inequality in a cone metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of cone b-metric spaces is effectively larger than that of the ordinary cone metric spaces. A cone b-metric space will be called normal, if the underlying cone P is normal cone.

Definition 4 ([6]). *Let X be a nonempty set and $d_b\colon X \times X \times X \to P$ be a mapping. Suppose, there exists $s \geq 1$ and the following conditions are satisfied:*

(I) for every $x, y \in X$ with $x \neq y$ there exists $z \in X$ such that $d_b(x,y,z) \neq \theta$;

(II) if at least two of $x, y, z \in X$ are equal, then $d_b(x,y,z) = \theta$;

(III) $d_b(x,y,z) = d_b(p(x,y,z))$ for all $x, y, z \in X$, where $p(x,y,z)$ denotes all the permutations of x, y, z;

(IV) $d_b(x,y,z) \preceq s[d_b(x,y,w)+d_b(x,w,z)+d_b(w,y,z)]$ *for all* $x,y,z,w \in X$.

Then, the mapping d_b *is called a cone* b_2*-metric over* X *and the triplet* (X,d_b,s) *is called a cone* b_2*-metric space on Banach algebra* A. *If the cone* P *is normal, then* (X,d_b,s) *is called a normal cone* b_2*-metric space over Banach algebra* A.

Example 1. *Let* $X=\mathbb{R}$ *and* $A=C^1_{\mathbb{R}}[0,1]$ *be the Banach algebra with the norm* $\|x(t)\|=\|x(t)\|_\infty+\|x'(t)\|_\infty$, *the point-wise multiplication and the unit* $e(t)=1$ *for all* $t\in[0,1]$. *Let* $P=\{\psi\in C^1_{\mathbb{R}}[0,1]\colon \psi(t)\geq 0$ *for all* $t\in[0,1]\}$ *be the solid cone in* A. *Define the mapping* $d_b\colon X\times X\times X\to P$ *by*

$$d_b(x,y,z)=\min\left\{(x-y)^2,(y-z)^2,(z-x)^2\right\}e^t$$

for all $x,y,z\in X$. *Then* (X,d_b,s) *is a cone* b_2*-metric space over Banach algebra* A *with* $s=2$ *for all* $t\in[0,1]$. *On the other hand,* d_b *is not a 2-cone metric space, for example, at points* $x=1,y=5,z=15,w=2$ *we have* $d_b(x,y,z)=16e^t, d_b(x,y,w)=e^t, d_b(x,w,z)=e^t, d_b(w,y,z)=9e^t$, *and so,*

$$d_b(x,y,z)\succ d_b(x,y,w)+d_b(x,w,z)+d_b(w,y,z).$$

Example 2. *Let* $X=\{1,2,3,4\}$ *and* $A=\mathbb{R}^2$ *be the Banach algebra with the norm* $\|(x_1,x_2)\|=|x_1|+|x_2|$, *with the multiplication defined by* $(x_1,x_2)(y_1,y_2)=(x_1y_1,x_2y_1+x_1y_2)$ *and the unit* $e(t)=(1,0)$. *Let* $P=\{(x_1,x_2)\colon x_1,x_2\geq 0\}$ *be the normal cone in* A. *Define the mapping* $d_b\colon X\times X\times X\to P$ *by*

$$d_b(1,2,3)=a(1,1), d_b(1,2,4)=b(1,1), d_b(2,3,4)=c(1,1), d_b(1,3,4)=\lambda(1,1)$$

with symmetry in all variables and with $d(x,y,z)=(0,0)$ *when at least two of the arguments are equal, where* a,b,c *are nonnegative reals such that* $a+b+c>0$ *and* $\lambda=a+b+c+1$. *Then,* (X,d_b,s) *is a cone* b_2*-metric space over Banach algebra* A *with* $s\geq\dfrac{\lambda}{\lambda-1}$. *On the other hand,* d_b *is not a 2-cone metric space, for example, at points* $x=1,y=3,z=4,w=2$ *we have* $d_b(x,y,z)=\lambda(1,1), d_b(x,y,w)=a(1,1), d_b(x,w,z)=b(1,1), d_b(w,y,z)=c(1,1)$, *and so,*

$$d_b(x,y,z)\succ d_b(x,y,w)+d_b(x,w,z)+d_b(w,y,z).$$

Definition 5 ([6])**.** *Let* (X,d_b,s) *be a cone* b_2*-metric space over the Banach algebra* A. *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if for every* $c\in A$ *with* $\theta\ll c$ *there exists* $n_0\in\mathbb{N}$ *such that* $d_b(x_n,x_m,a)\ll c$ *for all* $n,m>n_0$ *and* $a\in X$.

Definition 6 ([6])**.** *Let* (X,d_b,s) *be a cone* b_2*-metric space over the Banach algebra* A. *A sequence* $\{x_n\}$ *in* X *is called a convergent sequence and converges to* $x\in X$ *if for every* $c\in A$ *with* $\theta\ll c$ *there exists* $n_0\in\mathbb{N}$ *such that* $d_b(x_n,x,a)\ll c$ *for all* $n>n_0$ *and* $a\in X$. *We denote this fact by* $x_n\to x$ *as* $n\to\infty$ *and* x *is called the limit of the sequence* $\{x_n\}$.

Definition 7 ([6]). *Let (X, d_b, s) be a cone b_2-metric space over the Banach algebra A. Then, (X, d_b, s) is called complete if every Cauchy sequence in X converges to some $x \in X$.*

Remark 2. *Limit of a convergent sequence in a cone b_2-metric space over a Banach algebra is unique. Indeed, if $\{x_n\}$ converges to two distinct limits $x, y \in X$, then for every given $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d_b(x_n, x, a) \ll \frac{c}{3s}, d_b(x_n, y, a) \ll \frac{c}{3s}$ for all $n > n_0$ and $a \in X$, therefore*

$$\begin{aligned} d_b(x, y, a) &\preceq s[d_b(x, y, x_n) + d_b(x, x_n, a) + d_b(x_n, y, a)] \\ &\ll s\left[\frac{c}{3s} + \frac{c}{3s} + \frac{c}{3s}\right] \\ &= c. \end{aligned}$$

The above inequality with Lemma 1 yields $d_b(x, y, a) = \theta$ for all $a \in X$, i.e., $x = y$. This contradiction proves the result.

The proof of the following remark is similar to the above remark.

Remark 3. *Every convergent sequence in a cone b_2-metric space over a Banach algebra is a Cauchy sequence.*

2. Fixed point theorems

In this section, we introduce some new concepts and prove a common fixed point theorem.

Definition 8. *Let (X, d_b, s) be a cone b_2-metric space over Banach algebra A and $T: X \to X$ be a mapping. Then, the mapping T is called a generalized λ-Banach contraction if there exists $\lambda \in P$ such that $\rho(\lambda) < \frac{1}{s}$ and the following condition is satisfied:*

$$d_b(Tx, Ty, a) \preceq \lambda d_b(x, y, a) \tag{1}$$

for all $x, y, a \in X$. The mapping T is said to be a generalized λ-Kannan contraction if there exists $\lambda \in P$ such that $\rho(\lambda) < \frac{1}{2s}$ and the following condition is satisfied:

$$d_b(Tx, Ty, a) \preceq \lambda \left[d_b(x, Tx, a) + d_b(y, Ty, a)\right] \tag{2}$$

for all $x, y, a \in X$.

Definition 9. *Let (X, d_b, s) be a cone b_2-metric space over Banach algebra A and $T: X \to X$ be a mapping. Then, the mapping T is called a generalized (λ, μ)-Reich contraction if there exist $\lambda, \mu \in P$ such that $\rho(\lambda) + 2\rho(\mu) < \dfrac{1}{s}$ and the following condition is satisfied:*

$$d_b(Tx, Ty, a) \preceq \lambda d_b(x, y, a) + \mu[d_b(x, Tx, a) + d_b(y, Ty, a)] \tag{3}$$

for all $x, y, a \in X$.

It is easy to see that the class of generalized (λ, μ)-Reich contractions is a unification and generalization of classes of generalized λ-Banach and generalized μ-Kannan contractions.

Remark 4. *In [6], Fernandez et al. used the following condition:*

$$d_b(Tx, Ty, a) \preceq \kappa d_b(x, y, a) + \lambda d_b(x, Tx, a) + \mu d_b(y, Ty, a) \tag{4}$$

for all $x, y, a \in X$, where $s\rho(\kappa) + s\rho(\lambda) + \rho(\mu) < 1$ and $s^2 + s\rho(\lambda) < 1$. Observe that, since $s \geq 1$ and $\rho(\lambda) \geq 0$, therefore, the condition $s^2 + s\rho(\lambda) < 1$ is not feasible. Therefore, the fixed point results of Fernandez et al. [6] are not consistent for any given mapping and for any given cone b_2-metric space.

We define a more general class as follows:

Definition 10. *Let (X, d_b, s) be a cone b_2-metric space over Banach algebra A and $T, S: X \to X$ be two mappings. Then, the pair (T, S) is called a generalized (λ, μ)-Reich pair if there exist $\lambda, \mu \in P$ such that $\rho(\lambda) + 2\rho(\mu) < \dfrac{1}{s}$ and the following condition is satisfied:*

$$d_b(Tx, Sy, a) \preceq \lambda d_b(x, y, a) + \mu[d_b(x, Tx, a) + d_b(y, Sy, a)] \tag{5}$$

for all $x, y, a \in X$.

It is obvious that every generalized (λ, μ)-Reich contraction T is actually a generalized (λ, μ)-Reich pair with $S = I$.

Next, we discuss the nature and some results about common fixed points of a generalized (λ, μ)-Reich pair in a cone b_2-metric space over Banach algebra.

Proposition 5. *Let (X, d_b, s) be a cone b_2-metric space over Banach algebra A and $T, S: X \to X$ be two mappings such that the pair (T, S) is a generalized (λ, μ)-Reich pair. If $x^* \in X$ is a fixed point of T (or of S), then x^* is a unique common fixed point of the pair (T, S).*

Proof. Suppose, x^* is a fixed point of T, i.e., $Tx^* = x^*$. Since the pair (T, S) is a generalized (λ, μ)-Reich pair we have

$$\begin{aligned} d_b(x^*, Sx^*, a) &= d_b(Tx^*, Sx^*, a) \\ &\preceq \lambda d_b(x^*, x^*, a) + \mu[d_b(x^*, Tx^*, a) + d_b(x^*, Sx^*, a)] \\ &= \lambda \cdot \theta + \mu[\theta + d_b(x^*, Sx^*, a)] \end{aligned}$$

i.e., $(e - \mu)d_b(x^*, Sx^*, a) \preceq \theta$. Since $\rho(\mu) < 1$, we have $e - \mu \in P$ is invertible, and so, the last inequality yields $d_b(x^*, Sx^*, a) = \theta$ for all $a \in X$. Therefore, $Sx^* = x^*$. Thus, x^* is a common fixed point of the pair (T, S).

For uniqueness, suppose x^*, y^* are two distinct common fixed points of the pair (T, S), i.e., $Tx^* = Sx^* = x^*$, $Ty^* = Sy^* = y^*$ and $x^* \neq y^*$. Then, we have

$$\begin{aligned} d_b(x^*, y^*, a) &= d_b(Tx^*, Sy^*, a) \\ &\preceq \lambda d_b(x^*, y^*, a) + \mu[d_b(x^*, Tx^*, a) + d_b(y^*, Sy^*, a)] \\ &= \lambda d_b(x^*, y^*, a) + \mu[\theta + \theta] \end{aligned}$$

i.e., $(e - \lambda)d_b(x^*, y^*, a) \preceq \theta$. Again, it shows that $d_b(x^*, y^*, a) = \theta$ for all $a \in X$, i.e., $x^* = y^*$. This contradiction proves the uniqueness.

If x^* is a fixed point of the mapping S. Then by similar process one can find the desired result.

Remark 5. *It is clear from the above remark that if (T, S) is a generalized (λ, μ)-Reich pair, then T cannot have more than one fixed point; and similar is true for the mapping S.*

The following lemmas will be used in establishing the common fixed point results for a generalized (λ, μ)-Reich pair.

Lemma 4. *Let (X, d_b, s) be a cone b_2-metric space over Banach algebra A and $T, S \colon X \to X$ be two mappings such that the pair (T, S) is a generalized (λ, μ)-Reich pair and λ, μ commute. If the sequence $\{x_n\} \subset X$ is defined by $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$, $n \geq 0$ and $x_0 \in X$ is arbitrary, then there exists $\alpha \in P$ such that $\rho(\alpha) < \frac{1}{s}$ and*

$$d_b(x_n, x_{n+1}, a) \preceq \alpha^n d_b(x_0, x_1, a) \text{ for all } a \in X.$$

Furthermore, for the sequence $\{x_n\}$ with initial value $x_0 \in X$, we have

$$d_b(x_k, x_{k-1}, x_t) = \theta \text{ for all } k > t.$$

Proof. Let $x_0 \in X$ be arbitrary point. Then since the pair (T, S) is a generalized (λ, μ)-Reich pair we obtain:

$$\begin{aligned} & d_b(x_{2n+1}, x_{2n+2}, a) \\ = \; & d_b(Tx_{2n}, Sx_{2n+1}, a) \\ \preceq \; & \lambda d_b(x_{2n}, x_{2n+1}, a) + \mu[d_b(x_{2n}, Tx_{2n}, a) + d_b(x_{2n+1}, Sx_{2n+1}, a)] \\ = \; & \lambda d_b(x_{2n}, x_{2n+1}, a) + \mu[d_b(x_{2n}, x_{2n+1}, a) + d_b(x_{2n+1}, x_{2n+2}, a)]. \end{aligned}$$

The above inequality shows that

$$(e - \mu)d_b(x_{2n+1}, x_{2n+2}, a) \preceq (\lambda + \mu)d_b(x_{2n}, x_{2n+1}, a).$$

Since $\rho(\mu) < 1$, therefore the vector $e - \mu$ is invertible, and so, we have

$$d_b(x_{2n+1}, x_{2n+2}, a) \preceq (\lambda + \mu)(e - \mu)^{-1} d_b(x_{2n}, x_{2n+1}, a).$$

Set $\alpha = (\lambda + \mu)(e - \mu)^{-1}$ in the above inequality we have

$$d_b(x_{2n+1}, x_{2n+2}, a) \preceq \alpha d_b(x_{2n}, x_{2n+1}, a). \tag{6}$$

Following similar process as the above and using the above inequality we obtain:

$$\begin{aligned} d_b(x_{2n+2}, x_{2n+3}, a) &= d_b(Sx_{2n+1}, Tx_{2n+2}, a) \\ &\preceq \alpha d_b(x_{2n+1}, x_{2n+2}, a) \\ &\preceq \alpha^2 d_b(x_{2n}, x_{2n+1}, a). \end{aligned} \tag{7}$$

Successive use of the inequalities (6) and (7) yields:

$$d_b(x_{2n+1}, x_{2n+2}, a) \preceq \alpha^{2n+1} d_b(x_0, x_1, a). \tag{8}$$

Similarly, we can prove:

$$d_b(x_{2n+2}, x_{2n+3}, a) \preceq \alpha^{2n+2} d_b(x_0, x_1, a). \tag{9}$$

It follows from the inequalities (8) and (9) that

$$d_b(x_n, x_{n+1}, a) \preceq \alpha^n d_b(x_0, x_1, a) \text{ for all } a \in X. \tag{10}$$

We shall show that $\rho(\alpha) < 1$. Since λ, μ commute we have:

$$\begin{aligned} (\lambda + \mu)(e - \mu)^{-1} &= (\lambda + \mu)\left[\sum_{i=0}^{\infty} \mu^i\right] = \sum_{i=0}^{\infty} (\lambda + \mu)\mu^i \\ &= \left[\sum_{i=0}^{\infty} \mu^i\right](\lambda + \mu) = (e - \mu)^{-1}(\lambda + \mu). \end{aligned}$$

Therefore, $\lambda+\mu$ and $(e-\mu)^{-1}$ commute. Now using Lemma 2 and Lemma 3 and the fact that $\rho(\lambda)+2\rho(\mu)<\dfrac{1}{s}<1$ we obtain

$$\begin{aligned}\rho(\alpha) &= \rho\left((\lambda+\mu)(e-\mu)^{-1}\right)\\ &\leq \rho\left(\lambda+\mu\right)\rho\left((e-\mu)^{-1}\right)\\ &\leq \left(\rho(\lambda)+\rho(\mu)\right)\left(1-\rho(\mu)\right)^{-1}\\ &\leq \left(\frac{1}{s}-\rho(\mu)\right)\left(1-\rho(\mu)\right)^{-1}\\ &< \frac{1}{s}.\end{aligned}$$

If $\{x_n\}$ is the sequence defined as above and $k>t$. We construct a sequence $\{y_n\}$ defined by $y_0=x_t$, $y_n=x_{n+t}$. Then for $k>t$ from the inequality (10) we have:

$$\begin{aligned}d_b(x_{k-1},x_k,x_t) &= d_b(y_{k-t-1},y_{k-t},y_0)\\ &\preceq \alpha^{k-t-1}d_b(y_0,y_1,y_0)\\ &= \alpha^{k-t-1}\cdot\theta\\ &= \theta\end{aligned}$$

which completes the proof.

Lemma 5. *Let (X,d_b,s) be a cone-b_2-metric space over Banach algebra A and suppose for any sequence $\{x_n\}$ the following condition is satisfied:*

$$d_b(x_n,x_{n+1},a)\preceq\alpha^n d_b(x_0,x_1,a) \text{ for all } a\in X$$

where $\rho(\alpha)<\dfrac{1}{s}$; and $d_b(x_{k-1},x_k,x_t)=\theta$ for $k>t, a\in X$. Then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. Suppose, $x_0\in X$ is arbitrary and $\{x_n\}$ be the sequence with initial value x_0. We shall show that $\{x_n\}$ is a Cauchy sequence. Then, by given condition we have

$$d_b(x_n,x_{n+1},a)\preceq\alpha^n d_b(x_0,x_1,a) \text{ for all } a\in X. \tag{11}$$

Suppose $n,m\in\mathbb{N}$ and $m>n$. Then, we have:

$$d_b(x_n,x_m,a) \preceq s[d_b(x_n,x_m,x_{m-1})+d_b(x_n,x_{m-1},a)+d_b(x_{m-1},x_m,a)].$$

Since $d_b(x_{k-1}, x_k, x_t) = \theta$ for $k > t, a \in X$ using the inequality (11) in the above inequality we obtain:

$$\begin{aligned} d_b(x_n, x_m, a) &\preceq s[\theta + d_b(x_n, x_{m-1}, a) + \alpha^{m-1} d_b(x_0, x_1, a)] \\ &\preceq s[\theta + \alpha^{m-1} d_b(x_0, x_1, a) + d_b(x_n, x_{m-1}, a)] \\ &\preceq s\alpha^{m-1} d_b(x_0, x_1, a) + s^2[d_b(x_n, x_{m-1}, x_{m-2}) \\ &\quad + d_b(x_n, x_{m-2}, a) + d_b(x_{m-2}, x_{m-1}, a)] \\ &\preceq s\alpha^{m-1} d_b(x_0, x_1, a) + s^2\alpha^{m-2} d_b(x_0, x_1, a) + s^2 d_b(x_n, x_{m-2}, a). \end{aligned}$$

By repeating this process we obtain:

$$\begin{aligned} d_b(x_n, x_m, a) &\preceq s\alpha^{m-1} d_b(x_0, x_1, a) + s^2\alpha^{m-2} d_b(x_0, x_1, a) \\ &\quad + \cdots + s^{m-n}\alpha^n d_b(x_0, x_1, a) \\ &\preceq s^{m-n}\alpha^n d_b(x_0, x_1, a) + s^{m-n-1}\alpha^{n+1} d_b(x_0, x_1, a) + \cdots \\ &\quad + s^2\alpha^{m-2} d_b(x_0, x_1, a) + s\alpha^{m-1} d_b(x_0, x_1, a) \\ &= s^{m-n}\alpha^n[e + s^{-1}\alpha + \cdots + s^{-(m-n-1)}\alpha^{m-n-1}] d_b(x_0, x_1, a) \\ &\preceq s^{m-n}\alpha^n[e + s^{-1}\alpha + s^{-2}\alpha^2 + \cdots] d_b(x_0, x_1, a). \end{aligned}$$

Since $\rho(s^{-1}\alpha) = \dfrac{1}{s}\rho(\alpha) < \dfrac{1}{s^2} < 1$, then the vector $e - s^{-1}\alpha$ is invertible and

$$(e - s^{-1}\alpha)^{-1} = e + s^{-1}\alpha + s^{-2}\alpha^2 + \cdots .$$

Therefore, the above inequality yields

$$d_b(x_n, x_m, a) \preceq s\alpha^n(e - s^{-1}\alpha)^{-1} d_b(x_0, x_1, a). \tag{12}$$

Since $\rho(\alpha) < \dfrac{1}{s} < 1$, therefore $\|\alpha^n\| \to 0$ as $n \to \infty$, and so, for every $c \in P^\circ$ there exists $n_0 \in \mathbb{N}$ such that $\alpha^n \ll c$ for all $n > n_0$. It shows that the sequence $\{\alpha^n\}$ is a c-sequence. By Proposition 3 the sequence $\{s\alpha^n(e - s^{-1}\alpha)^{-1} d_b(x_0, x_1, a)\}$ is also a c-sequence. Therefore, it follows from the above inequality that for every $c \in A$ with $\theta \ll c$, there exists $n_1 \in \mathbb{N}$ such that, $d_b(x_n, x_m, a) \ll c$ for all $n > n_1$. Thus, $\{x_n\}$ is a Cauchy sequence.

The next theorem gives a sufficient condition for the existence and uniqueness of the common fixed point of a generalized (λ, μ)-Reich pair.

Theorem 6. *Let (X, d_b, s) be a complete cone b_2-metric space over Banach algebra A and $T, S\colon X \to X$ be two mappings such that the pair (T, S) is a generalized (λ, μ)-Reich pair. Then the pair (T, S) has unique common fixed point in X.*

Proof. Suppose, $x_0 \in X$ be arbitrary and $\{x_n\}$ be defined by $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$. Then, by Lemma 4 and Lemma 5 the sequence $\{x_n\}$ is a Cauchy sequence.

By completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We shall show that x^* is the unique common fixed point of the pair (T, S). Then, for any $n \in \mathbb{N}$ and for all $a \in X$ we have

$$\begin{aligned} d_b(x_{2n}, Tx^*, a) &= d_b(Sx_{2n-1}, Tx^*, a) \\ &\preceq \lambda d_b(x_{2n-1}, x^*, a) + \mu[d_b(Sx_{2n-1}, x_{2n-1}, a) + d_b(x^*, Tx^*, a)] \\ &\preceq \lambda d_b(x_{2n-1}, x^*, a) + \mu[d_b(x_{2n}, x_{2n-1}, a) + d_b(x^*, Tx^*, a)]. \end{aligned}$$

Using the above inequality we obtain

$$\begin{aligned} d_b(x^*, Tx^*, a) &\preceq s[d_b(x^*, Tx^*, x_{2n}) + d_b(x^*, x_{2n}, a) + d_b(x_{2n}, Tx^*, a)] \\ &= s\big[d_b(x^*, x_{2n}, Tx^*) + d_b(x^*, x_{2n}, a) \\ &\quad + \lambda d_b(x_{2n-1}, x^*, a) + \mu[d_b(x_{2n}, x_{2n-1}, a) + d_b(x^*, Tx^*, a)\big]. \end{aligned}$$

The above inequality implies that

$$\begin{aligned} (e - s\mu) d_b(x^*, Tx^*, a) &\preceq s\big[d_b(x^*, x_{2n}, Tx^*) + d_b(x^*, x_{2n}, a) \\ &\quad + \lambda d_b(x_{2n-1}, x^*, a) + \mu d_b(x_{2n}, x_{2n-1}, a)\big]. \end{aligned}$$

Again, since $\rho(\mu) < \dfrac{1}{s}$ we have

$$\begin{aligned} d_b(x^*, Tx^*, a) &\preceq s(e - s\mu)^{-1}\big[d_b(x^*, x_{2n}, Tx^*) + d_b(x^*, x_{2n}, a) \\ &\quad + \lambda d_b(x_{2n-1}, x^*, a) + \mu d_b(x_{2n}, x_{2n-1}, a)\big]. \end{aligned}$$

Since $x_n \to x^*$ as $n \to \infty$, the sequences:

$$\{d_b(x_n, x^*, a)\} \text{ and } \{d_b(x_{2n}, x_{2n-1}, a)\}$$

are c-sequences for all $a \in X$. Therefore, by Proposition 3 the sequence formed by the right hand side of the above inequality is also a c-sequence. Therefore, it follows from the last inequality that $\{d_b(x^*, Tx^*, a)\}$ is a c-sequence for all $a \in X$, and so, there exists $n_2 \in \mathbb{N}$ such that $d_b(x^*, Tx^*, a) \ll c$ for all $n > n_2$ and for all $a \in X$. It shows that $d_b(x^*, Tx^*, a) = \theta$ for all $a \in X$. Thus, $Tx^* = x^*$, i.e., x^* is a fixed point of T.

By Proposition 5 it follows that x^* is the unique common fixed point of the pair (T, S).

Example 3. *Let $X = \{(a, 0) : a \in [0, \infty)\} \cup \{(0, b)\}$, where $b > 0$ is fixed; and $A = C^1_{\mathbb{R}}[0, 1]$ be the Banach algebra with the norm $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$, the point-wise multiplication and the unit $e(t) = 1$ for all $t \in [0, 1]$. Let $P = \{\psi \in C^1_{\mathbb{R}}[0, 1] : \psi(t) \geq 0$ for all $t \in [0, 1]\}$ be the solid cone in A. Define the mapping $d_b : X \times X \times X \to P$ as the $\frac{4}{b^2}e^t$ times the square of the area of triangle formed by the vertices $x, y, z \in X$ in $\mathbb{R}^2$, where $t \in [0, 1]$, e.g.*

$$d_b((a, 0), (c, 0), (0, b)) = (a - c)^2 e^t.$$

Then (X, d_b, s) is a complete cone b_2-metric space over Banach algebra A with $s = 2$. Define two mapping $T, S : X \to X$ by:

$$T(a,0) = \begin{cases} \frac{1}{4}(a,0), & \text{if } a \in \mathbb{Q}; \\ (0,0), & \text{otherwise;} \end{cases} \quad S(a,0) = \begin{cases} \frac{1}{4}(a,0), & \text{if } a \in \mathbb{R} \setminus \mathbb{Q}; \\ (0,0), & \text{otherwise} \end{cases}$$

and $T(0, b) = S(0, b) = (0, 0)$. Then by some routine calculations one can see that the pair (T, S) is a generalized (λ, μ)-Reich pair with $\lambda = \frac{1}{16}, \mu = \frac{1}{9}$. Thus, all the conditions of Theorem 6 are satisfied, and so, there exists a unique common fixed point of the pair (T, S). Indeed, $(0, 0)$ is the unique common fixed point of the pair (T, S).

Corollary 7. *Let (X, d_b, s) be a complete cone b_2-metric space over Banach algebra A and $T : X \to X$ be a generalized (λ, μ)-Reich contraction. Then the mapping T has unique fixed point in X.*

Proof. Taking $S = T$ in Theorem 6, the result follows.

The following corollary is an improvement of the fixed point result of Fernandez et al. [6] (see Remark 4).

Corollary 8. *Let (X, d_b, s) be a complete cone b_2-metric space over Banach algebra A and $T : X \to X$ be a mapping satisfying the following condition:*

$$d_b(Tx, Ty, a) \preceq \kappa d_b(x, y, a) + \lambda d(Tx, x, a) + \mu d_b(Ty, y, a) \tag{13}$$

for all $x, y, a \in X$, where $\kappa, \mu, \rho \in P$ such that $s\rho(\kappa) + s\rho(\lambda) + s\rho(\mu) < 1$ and κ, λ, μ commute. Then the mapping T has unique fixed point in X.

Proof. For any fixed pair x, y in X, since d_b is symmetric, interchange the role of x and y in (13) we obtain

$$d_b(Tx, Ty, a) \preceq \kappa d_b(x, y, a) + \lambda d(Ty, y, a) + \mu d_b(Tx, x, a). \tag{14}$$

It follows from (13) and (14) that:

$$\begin{aligned} d_b(Tx,Ty,a) &\preceq \kappa d_b(x,y,a) + \frac{\lambda+\mu}{2}[d(Tx,x,a)+d_b(Ty,y,a)] \\ &= \kappa d_b(x,y,a) + \nu[d(Tx,x,a)+d_b(Ty,y,a)] \end{aligned}$$

where $\nu = \dfrac{\lambda+\mu}{2}$. Since $s\rho(\kappa)+s\rho(\mu)+s\rho(\mu)<1$, we have

$$\begin{aligned} \rho(\kappa)+2\rho(\nu) &= \rho(\kappa)+2\rho\left(\frac{\lambda+\mu}{2}\right) \\ &= \rho(\kappa)+\rho\left(\lambda+\mu\right) \\ &\leq \rho(\kappa)+\rho\left(\lambda\right)+\rho\left(\mu\right) \\ &< \frac{1}{s}. \end{aligned}$$

Thus, T is a generalized (κ,ν)-Reich contraction. Now result follows from Corollary 7.

Corollary 9. *Let (X,d_b,s) be a complete cone b_2-metric space over Banach algebra A and $T\colon X\to X$ be a generalized λ-Banach contraction. Then the mapping T has unique fixed point in X.*

Proof. Taking $\mu=\theta$ and $S=T$ in Theorem 6, the result follows.

Corollary 10. *Let (X,d_b,s) be a complete cone b_2-metric space over Banach algebra A and $T\colon X\to X$ be a generalized μ-Kannan contraction. Then the mapping T has unique fixed point in X.*

Proof. Taking $\lambda=\theta$ and $S=T$ in Theorem 6, the result follows.

Conflict of Interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] B. Singh, S. Jain and P. Bhagat, Cone 2-metric space and fixed point theorem of contractive mappings. Comment. Math., Vol. **52**, no. 2, (2012), 143-151.

[2] G. Jungck, S. Radenović, S. Radojević, V. Rakócević, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed Point Theory Appl. Vol. **57**, (2009), 13 pages.

[3] H. Çakallı, A. Sönmez, Ç. Genç, On an equivalence of topological vector space valued cone metric spaces and metric spaces, Appl. Math. Lett., Vol. **25**, (2012), 429-433.

[4] H. Liu and S.-Y. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory Appl., Vol. **2013**, (2013), 10 pages.

[5] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst., Vol. **30**, (1989), 26-37.

[6] J. Fernandez, N. Malviya, K. Saxena, Cone b_2-metric spaces over Banach algebra with applications, São Paulo J. Math. Sci., Vol. **11**, no. 1, (2017), 221–239.

[7] L.G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., Vol. **332**, (2007), 1468-1476.

[8] M. Dordević, D. Dorić, Z. Kadelburg, S. Radenović, D. Spasić, Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory Appl. Vol. **2011**, (2011), 9 pages.

[9] N. Hussian, M.H. Shah, KKM mappings in cone b-metric spaces, Comput. Math. Appl., Vol. **62**, no. 4, (2011) 1677-1684.

[10] S.Gähler, 2-metricsche Räume und ihre topologische strukture, Math. Nachr., Vol. **26**, (1963), 115-148.

[11] S.Gähler,Über die Uniformisierbarkeit 2-metricsche Räume, Math. Nachr., **28**, (1965), 235-244.

[12] S.Gähler, Zur geometric 2-metricsche Räume, Revne Roumaine der Mathem. Pures et Appliques, **11**, (1966), 665-667.

[13] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica Et Informatica Universitatis Ostraviensis Vol. **1**, no. 1, (1993), 5-11.

[14] S. Radenović, B.E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, Comput. Math. Appl., **57**, (2009), 1701-1707.

[15] S. Xu, S.Radenovic, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach, algebras without assumption of normality, Fixed Point Theory Appl., Vol. **2014**, (2014), 12 pages.

[16] T. Wang, J. Yin, Q. Yan, Fixed point theorems on cone 2-metric spaces over Banach algebras and an application, Fixed Point Theory Appl., **2015**, (2015), 13 pages.

[17] V.H. Badshah, P. Bhagat, S. Shukla, Some fixed point results on $(\varphi, L, \mathfrak{m})$-weak contraction in cone 2-metric spaces, Asia Pacific Journal of Mathematics, Vol. **3**, no. 1, (2016), 24-37.

[18] W.S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal., Vol. **72**, no. 5, (2010) 2259-2261.

[19] Y. Feng, W. Mao, The equivalence of cone metric spaces and metric spaces, Fixed Point Theory, Vol. **11**, no. 2, (2010) 259-264.

[20] Z. Mustafa, V. Parvaneh, J.R. Roshan, Z. Kadelburg, b_2-Metric spaces and some fixed point theorems, Fixed Point Theory and Applications, Vol. **2014**, (2014), 24 pages.

[21] Z. Kadelburg, S. Radenović, V. Rakočević, A note on the equivalence of some metric and cone metric fixed point results, Appl. Math. Lett., Vol. **24**, (2011), 370-374.

[22] Z. Kadelburg, M. Pavlović, S. Radenović, Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces, Comput. Math. Appl., Vol. **59**, (2010) 3148-3159.

[23] Z. Kadelburg, S. Radenović, A note on various types of cones and fixed point results in cone metric spaces. Asian J. Math. Appl., Vol. **2013**, (2013), 7 pages.

V. H. Badshah
School of Studies in Mathematics,
Vikram University,
Ujjain, (M.P.), India
email: *vhbadshah@yahoo.co.in*

Prakash Bhagat
Department of Applied Mathematics,,
NMIMS, Shirpur,
India
email: *prakash1175@yahoo.com*

Satish Shukla
Department of Applied Mathematics, Shri Vaishnav Institute of Technology & Science(M.P.)
Gram Baroli, Sanwer Road, Indore ,
453331 India
email: *satishmathematics@yahoo.co.in*