

HARMONIC UNIVALENT CONVEX FUNCTIONS USING A QUANTUM CALCULUS APPROACH

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ABSTRACT. In this paper, we make an attempt to use quantum calculus and harmonic univalent convex functions. In particular, we introduce and investigate the properties of q -harmonic functions and q -harmonic convex functions of order α .

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1. INTRODUCTION

After the development of quantum calculus, popularly known as q -calculus, by Jackson (see [14, 15, 16]) in 1909-1910, quantum calculus has been applied in physics, quantum mechanics, analytic number theory, sobolev spaces, group representation theory, theta functions, gamma functions, operator theory and several other areas. For the definitions and properties of q -calculus, one may refer to [5, 6, 9, 10, 11, 18].

In 1990, Ismail, Merkes and Styer [13] used q -calculus in the theory of analytic univalent functions by defining a class of complex-valued functions that are analytic on the open unit disc $\mathbb{D} := \{z : |z| < 1\}$ with the normalizations $f(0) = 0$, $f'(0) = 1$, and $|f(qz)| \leq |f(z)|$ on \mathbb{D} for every q , $q \in (0, 1)$. Motivated by these authors, several researchers used the theory of analytic univalent functions and q -calculus; for example see [4, 20, 23].

Though the theory of harmonic univalent functions was discovered by Clunie and Sheil-Small in 1984 [7], but the role of q -calculus in this specialized area of research is quite new. Motivated by a method for q -calculus, outlined by Ernst [9] in 2003, we in this paper make an attempt to apply q -calculus and harmonic convex functions. In particular, we introduce and study a class of generalized harmonic univalent convex functions by using q -calculus approach.

2. PRELIMINARIES

Throughout this paper, we shall assume that q -satisfies the condition $q \in (0, 1)$.

Definition 1. Let $q \in (0, 1)$ and $\lambda \in \mathbb{C}$. The q -number, denoted by $[\lambda]_q$, is defined by

$$[\lambda]_q = \frac{1 - q^\lambda}{1 - q}.$$

When $\lambda = n \in \mathbb{N}$, we obtain $[n]_q = 1 + q + q^2 + \dots + q^{n-1}$, and when $q \rightarrow 1^-$, then $[n]_q = n$.

Applying the above q -number and motivated by Jackson [14], q -derivative is defined below.

Definition 2. The q -derivative (or q -difference operator) of a function f , defined on a subset of \mathbb{C} , is given by

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \\ f'(0), & z = 0. \end{cases} \quad (1)$$

We note that $\lim_{q \rightarrow 1^-} (D_q f)(z) = f'(z)$ if f is differentiable at z .

For definitions and properties of q -derivative and q -calculus, one may refer to [5, 10, 11, 18].

For a function $f(z) = z^n$, we observe that

$$D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1}.$$

Let \mathcal{A} denote the class of normalized functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1 \quad (2)$$

which are analytic in the open unit disk \mathbb{D} . In view of (1) and (2), it follows that for any $f \in \mathcal{A}$, we have

$$D_q f(z) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \quad \text{and} \quad D_q(z D_q f(z)) = \sum_{n=1}^{\infty} [n]_q^2 a_n z^{n-1},$$

where $q \in (0, 1)$.

Jackson [15] introduced the q -integral of a function f as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(zq^n)$$

provided the series converges.

In order to use q -calculus and harmonic univalent functions, we first need some notations and terminology of harmonic univalent functions.

Let \mathcal{H} denote the family of continuous complex-valued sense-preserving functions $f = h + \bar{g}$ in the unit disc \mathbb{D} that are harmonic and where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1. \quad (3)$$

Note that $f = h + \bar{g}$ is locally univalent and sense-preserving in \mathbb{D} if and only if $|g'(z)/h'(z)| < 1$ in \mathbb{D} (see [7]). Also, let $\mathcal{S}_{\mathcal{H}}$ be a subclass of functions f in \mathcal{H} that are univalent in \mathbb{D} . We observe that for $g(z) \equiv 0$ in \mathbb{D} , the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class \mathcal{S} of normalized analytic univalent functions in \mathbb{D} .

We also recall that convolution of two complex-valued harmonic functions

$$f_1(z) = z + \sum_{n=2}^{\infty} a_{1n} z^n + \sum_{n=1}^{\infty} \overline{b_{1n} z^n} \quad \text{and} \quad f_2(z) = z + \sum_{n=2}^{\infty} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b_{2n} z^n}$$

is defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1n} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b_{1n} b_{2n} z^n}, \quad (z \in \mathbb{D}).$$

A comprehensive study for the theory of harmonic univalent functions may be found in Duren [8]. One may also refer to the survey articles by the first author [1] and [2].

We now define the class \mathcal{H}_q consisting of q -harmonic functions in \mathbb{D} .

Definition 3. A harmonic function $f = h + \bar{g}$ defined by (3) is said to be q -harmonic, locally univalent and sense-preserving in \mathbb{D} denoted by \mathcal{H}_q , if and only if the second dilatation w_q satisfies the condition

$$|w_q(z)| = \left| \frac{D_q g(z)}{D_q h(z)} \right| < 1, \quad (4)$$

where $q \in (0, 1)$ and $z \in \mathbb{D}$. Note that as $q \rightarrow 1^-$, \mathcal{H}_q reduces to the family \mathcal{H} .

We also define a new operator and we call it as q -Bernardi harmonic integral operator.

Definition 4. Let $f = h + \bar{g} \in \mathcal{H}_q$ be defined by (3). Then the operator $L : \mathcal{H} \rightarrow \mathcal{H}$ is said to be q -Bernardi harmonic integral operator denoted by $L(f) = F$ defined as

$$F(z) = \frac{[c+1]_q}{z^c} \int_0^z t^{c-1} h(t) d_q t + \overline{\frac{[c+1]_q}{z^c} \int_0^z t^{c-1} g(t) d_q t}, \quad c = 1, 2, \dots \quad (5)$$

Note that for $q \rightarrow 1^-$, this operator reduces to the operator studied for a class defined by fractional operator in [21]. Letting $g(z) \equiv 0$ for all $z \in \mathbb{D}$, the operator (5) gives the corresponding operator introduced in [20]. The operator defined by (5) is named after Bernardi because by letting $q \rightarrow 1^-$ and $g(z) \equiv 0$, we obtain the operator $I : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$I(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt.$$

In fact, for $q \rightarrow 1^-$, $g(z) \equiv 0$ and $c = 1$, our operator (5) reduces to the Libera integral operator defined and studied by Libera [19].

In this paper, we introduce and study the class $\mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ defined as below.

Definition 5. A function $f = h + \bar{g}$ in \mathcal{H}_q is said to be in $\mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ if

$$\operatorname{Re} \left(\frac{z D_q(z D_q h(z)) + \overline{z D_q(z D_q g(z))}}{z D_q h(z) - \overline{z D_q g(z)}} \right) > \alpha, \quad |z| = r < 1, \quad (6)$$

where $q \in (0, 1)$ and $\alpha \in [0, 1)$. A function f in $\mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ is called q -harmonic convex function of order α .

Remark 1. When $q \rightarrow 1^-$, then the class $\mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ reduces to the traditional class $\mathcal{S}_{\mathcal{H}}^{\mathcal{C}}(\alpha)$, known as harmonic convex functions of order α (see for example [1, 2, 8]). Also, note that when $q \rightarrow 1^-$ and $g(z) \equiv 0$ in \mathbb{D} , then the class $\mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ reduces to the traditional class $\mathcal{S}^{\mathcal{C}}(\alpha)$, known as convex functions of order α , was studied by Robertson in 1936 [22]. Moreover, if $g(z) \equiv 0$ in \mathbb{D} , then $\mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha) \equiv \mathcal{S}_q^{\mathcal{C}}(\alpha)$, [23].

3. COEFFICIENT BOUNDS AND UNIVALENCE CRITERIA

We begin with a theorem that relates the modulus of the coefficients to the q -harmonic class \mathcal{H}_q .

Theorem 1. Let $f = h + \bar{g}$ be given by (3). Also, suppose

$$\sum_{n=1}^{\infty} \left([n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| + [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right) \leq 2, \quad (7)$$

where $a_1 = 1$, $\alpha \in [0, 1)$ and $q \in (0, 1)$. Then f is locally univalent and sense-preserving in \mathbb{D} ; that is $f \in \mathcal{H}_q$.

Proof. Since

$$\begin{aligned} |D_q h(z)| &= \left| 1 - \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \right| \\ &> 1 - \sum_{n=1}^{\infty} [n]_q |a_n| \\ &\geq 1 - \sum_{n=1}^{\infty} [n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| \\ &\geq \sum_{n=1}^{\infty} [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \\ &\geq \sum_{n=1}^{\infty} [n]_q |b_n| \\ &> \sum_{n=1}^{\infty} [n]_q |b_n| |z|^{n-1} \geq |D_q g(z)|, \end{aligned}$$

it follows that $f \in \mathcal{H}_q$, by Definition 3.

The following theorem gives necessary and sufficient convolution condition for q -harmonic convex functions of order α .

Theorem 2. Let $f = h + \bar{g}$ with h and g of the form (3), where f belongs to \mathcal{H}_q . Then $f \in \mathcal{S}_{\mathcal{H}_q}^c(\alpha)$ if and only if

$$H_1(z) + \overline{G_1(z)} \neq 0, \quad (8)$$

where

$$\begin{aligned} H_1(z) &= h(z) * \frac{(2 - 2\alpha)z + [2 - 2\alpha + (2\alpha + \zeta - 1)(1 + q)]qz^2}{(1 - z)(1 - qz)(1 - q^2z)}, \\ \overline{G_1(z)} &= \overline{g(z)} * \frac{(2\zeta + 2\alpha)\bar{z} + [2\zeta + 2\alpha - (2\alpha + \zeta - 1)(1 + q)]q\bar{z}^2}{(1 - \bar{z})(1 - q\bar{z})(1 - q^2\bar{z})}, \end{aligned}$$

$|\zeta| = 1$, $\zeta \neq -1$, $\alpha \in [0, 1)$, $q \in (0, 1)$ and $0 < |z| < 1$.

Proof. Since at $z = 0$

$$\frac{zD_q(zD_qh(z)) + \overline{zD_q(zD_qg(z))}}{zD_qh(z) - \overline{zD_qg(z)}} = 1,$$

the required condition (6) is equivalent to

$$\frac{\frac{zD_q(zD_qh(z)) + \overline{zD_q(zD_qg(z))}}{zD_qh(z) - \overline{zD_qg(z)}} - \alpha}{1 - \alpha} \neq \frac{\zeta - 1}{\zeta + 1}. \quad (9)$$

It is straight forward to verify that

$$f(z) * \frac{z}{1-z} = f(z) \quad \text{and} \quad f(z) * \frac{z}{(1-z)(1-qz)} = zD_qf(z). \quad (10)$$

Using (9), (10) and algebraic manipulations, we get

$$0 \neq (\zeta + 1)[zD_q(zD_qh(z)) + \overline{zD_q(zD_qg(z))} - \alpha zD_qh(z) + \alpha \overline{zD_qg(z)}] \\ - (\zeta - 1)[zD_qh(z) - \overline{zD_qg(z)} - \alpha zD_qh(z) + \alpha \overline{zD_qg(z)}] := H_2(z) + \overline{G_2(z)},$$

where

$$H_2(z) = zD_qh(z) * \left(\frac{(\zeta + 1)z}{(1-z)(1-qz)} - \frac{\alpha(\zeta + 1)z}{1-z} - \frac{(\zeta - 1)z}{1-z} + \frac{\alpha(\zeta - 1)z}{1-z} \right) \\ = zD_qh(z) * \left(\frac{(2 - 2\alpha)z + (2\alpha + \zeta - 1)qz^2}{(1-z)(1-qz)} \right) \\ = h(z) * zD_q \left(\frac{(2 - 2\alpha)z + (2\alpha + \zeta - 1)qz^2}{(1-z)(1-qz)} \right) \\ = h(z) * \frac{(2 - 2\alpha)z + [2 - 2\alpha + (2\alpha + \zeta - 1)(1 + q)]qz^2}{(1-z)(1-qz)(1-q^2z)} = H_1(z)$$

and

$$\overline{G_2(z)} = \overline{zD_qg(z)} * \left(\frac{(\overline{\zeta} + 1)z}{(1-z)(1-qz)} + \frac{\alpha(\overline{\zeta} + 1)z}{1-z} + \frac{(\overline{\zeta} - 1)z}{1-z} - \frac{\alpha(\overline{\zeta} - 1)z}{1-z} \right) \\ = \overline{zD_qg(z)} * \left(\frac{(2\overline{\zeta} + 2\alpha)z - (2\alpha + \overline{\zeta} - 1)qz^2}{(1-z)(1-qz)} \right) \\ = \overline{g(z)} * zD_q \left(\frac{(2\overline{\zeta} + 2\alpha)z - (2\alpha + \overline{\zeta} - 1)qz^2}{(1-z)(1-qz)} \right) \\ = \overline{g(z)} * \frac{(2\overline{\zeta} + 2\alpha)z + [2\overline{\zeta} + 2\alpha - (2\alpha + \overline{\zeta} - 1)(1 + q)]qz^2}{(1-z)(1-qz)(1-q^2z)} = \overline{G_1(z)}.$$

This completes the proof.

Remark 2. When $q \rightarrow 1^-$ and $\alpha = 0$, Theorem 2 reduces to the corresponding convolution condition obtained in [3].

Theorem 3. Let $f = h + \bar{g}$ with h and g of the form (3). If

$$\sum_{n=1}^{\infty} \left([n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| + [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right) \leq 2, \quad (11)$$

where $a_1 = 1$, $q \in (0, 1)$ and $\alpha \in [0, 1)$, then f is univalent in \mathbb{D} and $f \in \mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$.

Proof. In order to prove that f is univalent, we use a technique that was first used by Jahangiri [17] for the univalence of harmonic functions. To this end, we need to show that $f(z_1) \neq f(z_2)$ when $z_1 \neq z_2$. Suppose $z_1, z_2 \in \mathbb{D}$ so that $z_1 \neq z_2$. Since \mathbb{D} is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in \mathbb{D}$, where $0 \leq t \leq 1$. Then we can write

$$f(z_2) - f(z_1) = \int_0^1 \left[(z_2 - z_1)D_q h(z(t)) + \overline{(z_2 - z_1)D_q g(z(t))} \right] dt.$$

Dividing the above equation by $z_2 - z_1 \neq 0$ and taking the real parts, we obtain

$$\begin{aligned} \operatorname{Re} \frac{f(z_2) - f(z_1)}{z_2 - z_1} &= \int_0^1 \operatorname{Re} \left[D_q h(z(t)) + \frac{\overline{(z_2 - z_1)}}{z_2 - z_1} D_q g(z(t)) \right] dt \\ &> \int_0^1 \left[\operatorname{Re} D_q h(z(t)) - |D_q g(z(t))| \right] dt \\ &\geq \int_0^1 \left[\operatorname{Re} D_q h(z(t)) - \sum_{n=1}^{\infty} [n]_q |b_n| \right] dt \\ &\geq \int_0^1 \left[1 - \sum_{n=2}^{\infty} [n]_q |a_n| - \sum_{n=1}^{\infty} [n]_q |b_n| \right] dt \\ &\geq \int_0^1 \left[1 - \sum_{n=2}^{\infty} [n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| - \sum_{n=1}^{\infty} [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right] dt. \end{aligned}$$

The last inequality is non-negative by the condition (11). This proves that $f(z_1) \neq f(z_2)$. This leads to the univalence of f .

In order to prove that $f \in \mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, from (8) we notice

$$\left| h(z) * \left(\frac{(2 - 2\alpha)z + [2 - 2\alpha + (2\alpha + \zeta - 1)(1 + q)]qz^2}{(1 - z)(1 - qz)(1 - q^2z)} \right) + \overline{g(z)} * \left(\frac{(2\bar{\zeta} + 2\alpha)z + [2\bar{\zeta} + 2\alpha - (2\alpha + \bar{\zeta} - 1)(1 + q)]qz^2}{(1 - z)(1 - qz)(1 - q^2z)} \right) \right|$$

$$\begin{aligned}
 &= \left| (2 - 2\alpha)[qz^2D_q(D_qh(z)) + zD_qh(z)] + (2\alpha + \zeta - 1)(qz^2D_q(D_qh(z))) \right. \\
 &\quad \left. + \overline{(2\bar{\zeta} + 2\alpha)[qz^2D_q(D_qg(z)) + zD_qg(z)] - (2\alpha + \bar{\zeta} - 1)(qz^2D_q(D_qg(z)))} \right| \\
 &= \left| (2 - 2\alpha)z + \sum_{n=2}^{\infty} [n]_q \left((\zeta + 1)[n]_q - 2\alpha - (\zeta - 1) \right) a_n z^n \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} [n]_q \overline{\left((\bar{\zeta} + 1)[n]_q + 2\alpha + (\bar{\zeta} - 1) \right) b_n z^n} \right| \\
 &\geq (2 - 2\alpha)|z| \left[1 - \sum_{n=2}^{\infty} [n]_q \left| \frac{(\zeta + 1)[n]_q - 2\alpha - (\zeta - 1)}{2 - 2\alpha} \right| |a_n| |z|^{n-1} \right. \\
 &\quad \left. - \sum_{n=1}^{\infty} [n]_q \left| \frac{(\bar{\zeta} + 1)[n]_q + 2\alpha + (\bar{\zeta} - 1)}{2 - 2\alpha} \right| |b_n| |z|^{n-1} \right] \\
 &> (2 - 2\alpha)|z| \left[1 - \sum_{n=2}^{\infty} [n]_q \left| \frac{(\zeta + 1)[n]_q - 2\alpha - (\zeta - 1)}{2 - 2\alpha} \right| |a_n| \right. \\
 &\quad \left. - \sum_{n=1}^{\infty} [n]_q \left| \frac{(\bar{\zeta} + 1)[n]_q + 2\alpha + (\bar{\zeta} - 1)}{2 - 2\alpha} \right| |b_n| \right], \quad \text{for } |\zeta| = 1 \\
 &= 2(1 - \alpha)|z| \left[1 - \sum_{n=2}^{\infty} [n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| - \sum_{n=1}^{\infty} [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right].
 \end{aligned}$$

This last expression is non-negative because of (11). In view of Theorem 2, it follows that $f \in \mathcal{S}_{\mathcal{H}_q}^c(\alpha)$.

The q -harmonic convex mappings

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q([n]_q - \alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1 - \alpha}{[n]_q([n]_q + \alpha)} \overline{y_n z^n}, \quad (12)$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given by (11) is sharp.

Remark 3. When $q \rightarrow 1^-$, (11) reduces to the corresponding result obtained by Jahangiri [17]. Also, when $q \rightarrow 1^-$ and $g(z) \equiv 0$, equation (11) yields to the corresponding results for analytic convex functions of order α obtained by Goodman [12].

In order to establish that (7) is also necessary condition, we need to define a

class \mathcal{TH}_q consisting of functions $f = h + \bar{g}$, where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad \text{and} \quad g(z) = - \sum_{n=1}^{\infty} |b_n| z^n, \quad (z \in \mathbb{D}). \quad (13)$$

The condition (11) is also necessary for the class $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ defined by $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha) := \mathcal{TH}_q \cap \mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$. In fact, we prove the following coefficient characterization.

Theorem 4. *Let $f = h + \bar{g}$ be defined by (13), where f belongs to \mathcal{TH}_q . Then $f \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left([n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| + [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right) \leq 2, \quad (14)$$

where $a_1 = 1$, $q \in (0, 1)$ and $\alpha \in [0, 1)$.

Proof. Since $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha) \subset \mathcal{S}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, we only need to prove the "only if" part of this theorem. To this end, for functions $f = h + \bar{g}$ of the form (13), we notice that the condition (6) is equivalent to

$$\operatorname{Re} \left\{ \frac{(1 - \alpha)z - \sum_{n=2}^{\infty} [n]_q ([n]_q - \alpha) |a_n| z^n - \sum_{n=1}^{\infty} [n]_q ([n]_q + \alpha) |b_n| \bar{z}^n}{z - \sum_{n=2}^{\infty} [n]_q |a_n| z^n + \sum_{n=1}^{\infty} [n]_q |b_n| \bar{z}^n} \right\} \geq 0. \quad (15)$$

The above required condition (15) must hold for all values of $z \in \mathbb{D}$, $|z| = r < 1$. By choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, we must have

$$\frac{(1 - \alpha) - \sum_{n=2}^{\infty} [n]_q ([n]_q - \alpha) |a_n| r^{n-1} - \sum_{n=1}^{\infty} [n]_q ([n]_q + \alpha) |b_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q |a_n| r^{n-1} + \sum_{n=1}^{\infty} [n]_q |b_n| r^{n-1}} \geq 0. \quad (16)$$

If the condition (14) does not hold, then the numerator in (16) is negative for r sufficiently close to 1. Thus there exists a point $z_0 = r_0$ in $(0, 1)$ for which the quotient in (16) is negative. This contradicts the required condition for $f \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ and so the proof is complete.

4. EXTREME POINTS, CONVOLUTION AND CONVEX COMBINATIONS

In this section, we first determine the extreme points of the closed convex hulls of $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, denoted by $clco\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$.

Theorem 5. *Let f be given by (13). Then $f \in clco\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ if and only if $f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$, where*

$$h_1(z) = z, \quad h_n(z) = z - \frac{1 - \alpha}{[n]_q([n]_q - \alpha)} z^n, \quad (n \geq 2)$$

,

$$g_n(z) = z - \frac{1 - \alpha}{[n]_q([n]_q + \alpha)} \bar{z}^n, \quad (n \geq 1)$$

and $\sum_{n=1}^{\infty} (x_n + y_n) = 1$ where $x_n \geq 0$ and $y_n \geq 0$. In particular, the extreme points of $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For a function f of the form $f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$, where $\sum_{n=1}^{\infty} (x_n + y_n) = 1$, we obtain

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q([n]_q - \alpha)} x_n z^n - \sum_{n=1}^{\infty} \frac{1 - \alpha}{[n]_q([n]_q + \alpha)} y_n \bar{z}^n.$$

Then $f \in clco\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ because

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{[n]_q([n]_q - \alpha)}{1 - \alpha} \left(\frac{1 - \alpha}{[n]_q([n]_q - \alpha)} x_n \right) + \sum_{n=1}^{\infty} \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} \left(\frac{1 - \alpha}{[n]_q([n]_q + \alpha)} y_n \right) \\ &= \sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = 1 - x_1 \leq 1. \end{aligned}$$

Conversely, suppose $f \in clco\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$. In view of (14), we have $|a_n| \leq \frac{1 - \alpha}{[n]_q([n]_q - \alpha)}$ and $|b_n| \leq \frac{1 - \alpha}{[n]_q([n]_q + \alpha)}$. Set

$$x_n = \frac{[n]_q([n]_q - \alpha)}{1 - \alpha} |a_n|, \quad (n \geq 2) \quad \text{and} \quad y_n = \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} |b_n|, \quad (n \geq 1).$$

By Theorem 4, $\sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n \leq 1$. Therefore we define $x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n \geq 0$.

Consequently, we obtain $f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$ as required.

Using definition of convolution, we show that the class $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ is closed under convolution.

Theorem 6. For $0 \leq \beta < \alpha < 1$, suppose $f \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ and $F \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\beta)$. Then $f * F \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha) \subset \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\beta)$.

Proof. Let $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n - \sum_{n=1}^{\infty} |b_n|\bar{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n|z^n - \sum_{n=1}^{\infty} |B_n|\bar{z}^n$, then

$$(f * F)(z) = z + \sum_{n=2}^{\infty} |a_n||A_n|z^n + \sum_{n=1}^{\infty} |b_n||B_n|\bar{z}^n.$$

Since of $F \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\beta)$, it follows from Theorem 4 that $|A_n| \leq 1$ and $|B_n| \leq 1$. Therefore, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{[n]_q([n]_q - \alpha)}{1 - \alpha} |a_n||A_n| + \sum_{n=1}^{\infty} \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} |b_n||B_n| \\ & \leq \sum_{n=2}^{\infty} \frac{[n]_q([n]_q - \alpha)}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} |b_n| \leq 1. \end{aligned}$$

In view of Theorem 4, it follows that $f * F \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha) \subset \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\beta)$.

We now prove that $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ is closed under convex combination of its members.

Theorem 7. The class $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ is closed under convex combination.

Proof. For $j = 1, 2, 3, \dots$, let $f_j \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, where f_j is given by

$$f_j(z) = z - \sum_{n=2}^{\infty} |a_{n_j}|z^n - \sum_{n=1}^{\infty} |b_{n_j}|\bar{z}^n.$$

Then, by Theorem 4

$$\sum_{n=1}^{\infty} \left(\frac{[n]_q([n]_q - \alpha)}{1 - \alpha} |a_{n_j}| + \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} |b_{n_j}| \right) \leq 2. \quad (17)$$

For $\sum_{j=1}^{\infty} t_j = 1, 0 \leq t_j \leq 1$, the convex combination of f_j may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z - \sum_{n=2}^{\infty} \sum_{j=1}^{\infty} t_j |a_{n_j}|z^n - \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} t_j |b_{n_j}|\bar{z}^n.$$

Using (17), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{[n]_q([n]_q - \alpha)}{1 - \alpha} \sum_{j=1}^{\infty} t_j |a_{n_j}| + \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} \sum_{j=1}^{\infty} t_j |b_{n_j}| \right) \\ &= \sum_{j=1}^{\infty} t_j \sum_{n=1}^{\infty} \left(\frac{[n]_q([n]_q - \alpha)}{1 - \alpha} |a_{n_j}| + \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} |b_{n_j}| \right) \\ &\leq 2 \sum_{j=1}^{\infty} t_j = 2, \end{aligned}$$

and so by Theorem 4, we have $\sum_{j=1}^{\infty} t_j f_j(z) \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$.

5. DISTORTION BOUNDS AND COVERING THEOREM

The following theorem gives distortion bounds for functions in $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, which yields covering result for this class.

Theorem 8. *If $f \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, then for $|z| = r < 1$ we have*

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{[2]_q} \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha} |b_1| \right) r^2 \quad (18)$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{1}{[2]_q} \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha} |b_1| \right) r^2. \quad (19)$$

Proof. Let $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$. Taking the absolute value of f , we obtain

$$\begin{aligned} |f(z)| &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \\ &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \alpha}{[2]_q([2]_q - \alpha)} \sum_{n=2}^{\infty} \left(\frac{[2]_q([2]_q - \alpha)}{1 - \alpha} |a_n| + \frac{[2]_q([2]_q - \alpha)}{1 - \alpha} |b_n| \right) r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \alpha}{[2]_q([2]_q - \alpha)} \sum_{n=2}^{\infty} \left(\frac{[n]_q([n]_q - \alpha)}{1 - \alpha} |a_n| + \frac{[n]_q([n]_q + \alpha)}{1 - \alpha} |b_n| \right) r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \alpha}{[2]_q([2]_q - \alpha)} \left(1 - \frac{1 + \alpha}{1 - \alpha} |b_1| \right) r^2. \end{aligned}$$

This proves (18). We omit the proof of (19) because its proof similar to (18).

The following covering result follows from the inequality (19).

Corollary 9. *If $f \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, then*

$$\left\{ w : |w| < \frac{[2]_q([2]_q - \alpha) - 1 + \alpha + (1 + \alpha - [2]_q([2]_q - \alpha))|b_1|}{[2]_q([2]_q - \alpha)} \right\} \subset f(\mathbb{D}).$$

Remark 4. *For $q \rightarrow 1^-$, the covering theorem in Corollary 9 yields the corresponding traditional result for harmonic convex functions of order α obtained in [17].*

Moreover, for $q \rightarrow 1^-$, $\alpha = b_1 = 0$, Corollary 9 yields the following result given in [17].

Remark 5. *If $f \in \mathcal{T}^0\mathcal{S}_{\mathcal{H}}^{\mathcal{C}}(0)$, then $\{w : |w| < 3/4\} \subset f(\mathbb{D})$.*

6. q - BERNARDI HARMONIC INTEGRAL OPERATOR

In this section, we prove that the class $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$ is closed under q -Bernardi harmonic integral operator given by (5).

Theorem 10. *Let $f = h + \bar{g} \in \mathcal{TH}_q$ be given by (13) and $f \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, then F defined by (5) also belongs to $\mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$.*

Proof. From (5), it follows that

$$F(z) = z - \sum_{n=2}^{\infty} \left(\frac{[c+1]_q}{[c+n]_q} \right) |a_n| z^n - \sum_{n=1}^{\infty} \left(\frac{[c+1]_q}{[c+n]_q} \right) |b_n| \bar{z}^n.$$

Since $f \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, it follows that the inequality (14) is equivalent to

$$\sum_{n=2}^{\infty} [n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \leq 1.$$

Therefore

$$\begin{aligned} & \sum_{n=2}^{\infty} [n]_q \frac{[n]_q - \alpha}{1 - \alpha} \left(\frac{[c+1]_q}{[c+n]_q} |a_n| \right) + \sum_{n=1}^{\infty} [n]_q \frac{[n]_q + \alpha}{1 - \alpha} \left(\frac{[c+1]_q}{[c+n]_q} |b_n| \right) \\ & \leq \sum_{n=2}^{\infty} [n]_q \frac{[n]_q - \alpha}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} [n]_q \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \\ & \leq 1. \end{aligned}$$

Thus $F \in \mathcal{TS}_{\mathcal{H}_q}^{\mathcal{C}}(\alpha)$, by Theorem 4.

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