

RATE OF CONVERGENCE OF WAVELET SERIES BY CESARO MEANS

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ABSTRACT. Wavelet frames have become a useful tool in time frequency analysis and image processing. Many authors worked in the field of wavelet frames and obtained various necessary and sufficient conditions. Ron and Shen [17] gave a characterization of wavelet frames. Benedetto and Treiber [3], Ron and Shen [17] presented different presentations to the wavelet frames. Any function $f \in L^2(\mathbb{R})$ can be expanded as an orthonormal wavelet series and pointwise convergence and uniform convergence of series have been discussed extensively by various authors [9, 17]. In this paper we investigate the pointwise convergence of orthogonal wavelet series in Pringsheim's sense. Furthermore, we investigate cesaro $|C, 1, 1|$ summability and the strong cesaro $|C, 1, 1|$ summability of wavelet series.

2010 *Mathematics Subject Classification:* 46A45, 46A35.

Keywords: Wavelet series, summability, cesaro sums.

1. INTRODUCTION

In order to analyze seismic data, wavelets with local support in the time and frequency domains were defined by A. Grossman and J. Morlet [6]. However, the prototypes of wavelets can be found in the work of A. Haar [8]. S. Mallat [12] and Y. Meyer developed the framework of multiresolution analysis in order to identify the underlying structure and to generate interesting examples of orthonormal bases for $L^2(\mathbb{R})$. P.G. Lemarie and Y. Meyer [10] constructed wavelets in $S(\mathbb{R}^n)$, the space of rapidly decreasing smooth functions. J.O. Stromberg [17] was looking for unconditional bases for Hardy spaces and developed spline wavelets. In the context of wavelet theory, G. Battle [2] and P.G. Lemari [11] developed these bases. Spline wavelets have exponential decay, but only C^N smoothness (for a finite N depending on the order of the associated splines). I. Daubechies [4] constructed compactly supported wavelets with C^N smoothness. The support of these wavelets increased

with the smoothness; in general, to have C^∞ smoothness, wavelets must have infinite support. The convergence results for wavelet expansions was first studied by Y. Meyer [14] then he was followed by G. Walter ([25],[26]). Kon and Repheal [9] gave definite conditions for the convergence of wavelet series. Under some regularity assumptions on the wavelets, Meyer proved that wavelet expansions of continuous functions converge everywhere. In comparison of these results, the pointwise convergence results give almost everywhere convergence (and convergence on the Lebesgue set) for expansions of general $L^p(1 \leq p \leq \infty)$ functions. More results about convergence of wavelet series on different spaces have been obtained by Firdous A. Shah, Neyaz A. Sheikh [23]. In this paper we investigate the pointwise convergence of orthogonal wavelet series in Pringsheim's sense. Furthermore, we study cesaro $|C, 1, 1|$ summability and the strong cesaro $|C, 1, 1|$ summability of wavelet series.

Let $\psi_j(x) : j = 1, 2, \dots$ be orthonormal system of real-valued functions on X . We consider the orthogonal series

$$\sum_{j=1}^{\infty} c_j \psi_j(x), \tag{1.1}$$

where c_j is sequence of real numbers (so-called co-efficients)satisfying the condition

$$\sum_{j=1}^{\infty} c_j^2 \leq \infty. \tag{1.2}$$

By Riesz-Fischer theorem, there exists a function $f(x) \in L^2 = L^2(X, \mathcal{F}, \mu)$ such that (1.1) is the generalized Fourier series of $f(x)$ with respect to the system $\psi_j(x)$. The partial sums

$$s_m(x) = \sum_{j=1}^m c_j \psi_j(x), m = 1, 2, \dots$$

converges to $f(x)$ in L^2 norm:

$$\lim_{m \rightarrow \infty} \int |s_m(x) - f(x)|^2 d\mu(x) = 0 \tag{1.3}$$

Here and in the sequel, the integrals are taken over the entire space X . It is well known that the condition (1.2) does not ensure the pointwise convergence of the partial sums $s_m(x)$ to $f(x)$ as $m \rightarrow \infty$. If

$$\sum_{j=1}^{\infty} c_j^2 [\log(j+1)]^2 < \infty, \tag{1.4}$$

where the logarithm is to be the base 2, then

$$\lim_{m \rightarrow \infty} s_m(x) = f(x) \text{ a.e.}, \quad (1.5)$$

where $f(x)$ is the sum of (1.1) in the L^2 - norm (see in (1.3)). We note that the condition (1.4) is not only sufficient, but also necessary in certain cases for the fulfillment of (1.5), as the following theorem of Tandori. If

$$|c_1| \geq |c_2| \geq \dots \geq |c_j| \geq \dots$$

and

$$\sum_{j=1}^{\infty} c_j^2 [\log(j+1)]^2 = \infty.$$

then one can construct an ONS $\psi_j(x) : j = 1, 2, \dots$ on the unit interval $[0,1]$ endowed with the ordinary Lebesgue measure so that the orthogonal series (1.1) diverges at each point $x \in [0, 1]$. The cesaro (C,1) summability of the orthogonal series (1.1) defined by the arithmetic means

$$\sigma_M(x) = \frac{1}{M} \sum_{m=1}^M s_m(x), M = 1, 2, \dots$$

of the partial sums is guaranteed by a weaker condition than (1.1). The Menshov-Kaczmarz theorem (see, e.g., [1, Theorem 2.8.1, p.125]) reads as follows: If

$$\sum_{j=1}^{\infty} c_j^2 [\log \log(j+3)]^2 < \infty, \quad (1.6)$$

. then

$$\lim_{M \rightarrow \infty} \sigma_M(x) = f(x) \text{ a.e.} \quad (1.7)$$

The following theorem on the strong cesaro (C,1) summability of orthogonal series (1.1) was proved by Borgen: Under the condition (1.6), we even have

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M |s_m(x) - f(x)|^2 = 0 \text{ a.e.}$$

We note that in the case of single series of numbers, the notion of strong cesaro $|C, 1|$ summability is due to G. H. Hardy (where it is called H_2 summability). Here our main goal in this paper is to prove an analogous result for wavelet orthogonal series on the strong cesaro $|C, 1, 1|$ summability. We also note that the following

generalization of Borgen's theorem was proved by Tandori [24]: Let $1 \leq v_1 < v_2 < \dots v_m < \dots$ be an arbitrary sequence of natural numbers and set

$$\sigma_M(v : x) = \frac{1}{M} \sum_{m=1}^M s_{v_m}(x), M = 1, 2, \dots$$

If condition (1.6) is satisfied, then we have

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M |s_{v_m}(x) - f(x)|^2 = 0 \text{ a.e.}$$

2. BACKGROUND: WAVELET SERIES

Let $\psi_{j,k} : j, k = 1, 2, \dots$ be an ONS wavelet on a finite positive measure space (X, F, μ) . We consider the wavelet series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k} \psi_{j,k}(x), \quad (2.1)$$

where $c_{j,k} : j, k = 1, 2, \dots$ are wavelet coefficients satisfying the condition

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}^2 < \infty. \quad (2.2)$$

By the Riesz-Fischer theorem, there exists a function $g(x)$ in $L^2 = L^2(X, F, \mu)$ such that (2.1) is the wavelet series of $g(x)$ with respect to the system $\psi_{j,k}(x)$ and the rectangular partial sums

$$s_{m,n}(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k} \psi_{j,k}(x), \quad m, n = 1, 2, \dots \quad (2.3);$$

of the wavelet series (2.1) converge to $g(x)$ in L^2 -norm:

$$\lim_{m,n \rightarrow \infty} \int |s_{m,n}(x) - g(x)|^2 d\mu(x) = 0. \quad (2.4)$$

It is clear that condition (2.2) does not ensure the pointwise convergence of the rectangular partial sums $s_{m,n}(x) \rightarrow g(x)$ as $m, n \rightarrow \infty$. The extension of Redemacher-Menshov theorem proved by a number of authors reads as follows: If

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}^2 [\log(j+1)]^2 [\log(k+1)]^2 < \infty, \quad (2.5)$$

then

$$\lim_{m,n \rightarrow \infty} s_{m,n}(x) = g(x) \text{ a.e.},$$

where $g(x)$ occurs in (2.4). We note that condition (2.5) is also the best possible one that guarantees the a.e. convergence of the wavelet series (2.1). Indeed, as it was proved in [16] that if the double sequence $c_{j,k}$ is such that

$$|c_{j,k}| \geq |c_{j_1,k_1}| \text{ for all } 1 \leq j \leq j_1, 1 \leq k \leq k_1.$$

and

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}^2 [\log(j+1)]^2 [\log(k+1)]^2 < \infty, \quad (2.6)$$

then there exists an ONS $\psi_{j,k}(x)$ endowed with the plane Lebesgue measure such that the wavelet series (2.1) diverges a.e. The a.e. cesaro (C, 1, 1) summability of the wavelet series (2.1) defined by the arithmetic means

$$\sigma_{M,N}(x) = \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N s_{m,n}(x), M, N = 1, 2, \dots \quad (2.7)$$

of the rectangular partial sums, can be guaranteed under a weaker condition than (2.5). The extension of the Menshov-Kaczmarz theorem reads as follows: If

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}^2 [\log \log(j+3)]^2 [\log \log(k+3)]^2 < \infty, \quad (2.8)$$

then

$$\lim_{M,N \rightarrow \infty} \sigma_{M,N}(x) = g(x) \text{ a.e.} \quad (2.9)$$

A double series of real (or complex) numbers

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{j,k} \quad (2.10)$$

is said to converge in Pringsheim's sense to the sum s if for every $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathcal{N}$ such that

$$\left| \sum_{j=1}^m \sum_{k=1}^n u_{j,k} - s \right| < \epsilon \text{ if } \min(m, n) > K.$$

Hardy [7] introduced the notion of regular convergence as follows. The double series (2.10) is said to converge regularly if it converges in Pringsheim's sense and, in addition, if each of its so-called row (single)subseries defined by

$$\sum_{j=1}^{\infty} u_{j,k}$$

converges for each fixed $k \in N$, as well as each of its so-called column (single)subseries defined by

$$\sum_{k=1}^{\infty} u_{j,k}$$

converges for each fixed $j \in N$. Without knowing Hardy's definition, the present order rediscovered the notion of regular convergence (where it was called as "convergence in a restricted sense") as follows: for every $\mathcal{E} > 0$ there exists $\mathcal{K} = \mathcal{K}(\mathcal{E}) \in \mathcal{N}$ such that

$$\left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{j,k} \right| < \epsilon \text{ if } \max(j_1, k_1) > K \text{ and } 1 \leq j_1 \leq j_2, 1 \leq k_1 \leq k_2.$$

It is clear that if the double series (2.10) converges absolutely, that is, if

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |u_{j,k}| < \infty,$$

then it converges regularly, as well: but the converse implication is not true in general.

3. THE KRONECKER LEMMA FOR WAVELET SERIES

In the proofs of our Theorems 1 and 2 we will need the extension of the familiar Kronecker lemma from single to double series of numbers. For the readers convenience, first we formulate the Kronecker lemma for single series as follows: If a non-decreasing sequence $\lambda_j : j = 1, 2, \dots$ of positive numbers tends to ∞ as $j \rightarrow \infty$, and the sequence $u_j : j = 1, 2, \dots$ of real numbers is such that the series

$$\sum_{j=1}^{\infty} \frac{u_j}{\lambda_j}$$

converges, then

$$\lim_{m \rightarrow \infty} \frac{1}{\lambda_m} \sum_{j=1}^m u_j = 0.$$

We recall that the Kronecker lemma for double series reads as follows: Let $\lambda_{j,k} : j, k = 1, 2, \dots$ be a double sequence of positive numbers satisfying the following conditions:

$$\delta_{1,0} \lambda_{j,k} = \lambda_{j+1,k} - \lambda_{j,k} \geq 0. \quad \delta_{0,1} \lambda_{j,k} = \lambda_{j,k+1} - \lambda_{j,k} \geq 0.$$

$$\delta_{1,1} \lambda_{j,k} = \lambda_{j+1,k+1} - \lambda_{j+1,k} - \lambda_{j,k+1} + \lambda_{j,k} \geq 0 \text{ for all } j, k = 1, 2, \dots \quad (3.1)$$

and

$$\lambda_{j,k} \rightarrow \infty \text{ as } \max(m, n) \rightarrow \infty \text{ (or } \min(m, n) \rightarrow \infty \text{)}.$$

If the double series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{u_{j,k}}{\lambda_{j,k}} \text{ converges regularly.} \quad (3.2)$$

where the $u_{j,k}$ are real numbers, then

$$\frac{1}{\lambda_{m,n}} \sum_{j=1}^m \sum_{k=1}^n u_{j,k} \rightarrow 0 \text{ as } \max(m, n) \rightarrow \infty \text{ (or } \min(m, n) \rightarrow \infty \text{)}. \quad (3.3)$$

4. CONVERGENCE OF WAVELET SERIES

Theorem 1. *If the wavelet series (2.1) is such that condition (2.2) is satisfied, then*

$$\frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N |s_{m,n}(x) - \sigma_{m,n}(x)|^2 \rightarrow 0 \text{ a.e. as } \max(M, N) \rightarrow \infty. \quad (4.1)$$

Proof. By invoking (2.3) and (2.7), for $m, n \geq 2$ we have

$$s_{m,n}(x) - \sigma_{m,n}(x) = \sum_{j=2}^m \sum_{k=2}^n \frac{(j-1)(k-1)}{mn} c_{j,k} \psi_{j,k}(x),$$

for $m \geq 2$ we have

$$s_{m,1}(x) - \sigma_{m,1}(x) = \sum_{j=2}^m \frac{j-1}{m} c_{j,1} \psi_{j,1}(x),$$

for $n \geq 2$

$$s_{1,n}(x) - \sigma_{1,n}(x) = \sum_{k=2}^{\infty} \frac{k-1}{n} c_{1,k} \psi_{1,k}(x)$$

Due to the orthonormality of the system $\psi_{j,k}(x)$, for $m, n = 1, 2, \dots$ we have,

$$\frac{1}{mn} \int |s_{m,n}(x) - \sigma_{m,n}(x)|^2 d\mu(x) = \sum_{j=2}^m \sum_{k=2}^n \frac{(j-1)^2 (k-1)^2}{m^3 n^3} c_{j,k}^2 \quad (4.2)$$

for $m \geq 2$ we have,

$$\frac{1}{m} \int |s_{m,1}(x) - \sigma_{m,1}(x)|^2 d\mu(x) = \sum_{j=2}^{\infty} \frac{(j-1)^2}{m^3} c_{j,1}^2;$$

and for $n \geq 2$ we have,

$$\frac{1}{n} \int |s_{1,n}(x) - \sigma_{1,n}(x)|^2 d\mu(x) = \sum_{k=2}^{\infty} \frac{(k-1)^2}{n^3} c_{1,k}^2$$

Hence it follows that

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \int |s_{m,n}(x) - \sigma_{m,n}(x)|^2 d\mu(x) \\ &= \sum_{j=2}^{\infty} \sum_{k=2}^{\infty} (j-1)^2 (k-1)^2 c_{j,k}^2 \sum_{m=j}^{\infty} \sum_{n=k}^{\infty} \frac{1}{m^3 n^3} \\ &+ \sum_{j=2}^{\infty} (j-1)^2 c_{j,1}^2 \sum_{m=j}^{\infty} \frac{1}{m^3} + \sum_{k=2}^{\infty} (k-1)^2 c_{1,k}^2 \sum_{n=k}^{\infty} \frac{1}{n^3}. \end{aligned} \quad (4.3)$$

where we interchanged the order of summations with respect to j and m , as well as with respect to k and n on the right-hand side in (4.2). For $j \geq 2$, we have,

$$\sum_{m=j}^{\infty} \frac{1}{m^3} \leq \int_{j-1}^{\infty} \frac{dt}{t^3} = \frac{1}{2(j-1)^2}.$$

and for $k \geq 2$, we also have an analogous inequality. Taking into account these inequalities, the right-hand side in (4.3) does not exceed

$$\frac{1}{4} \sum_{j=2}^{\infty} \sum_{k=2}^{\infty} c_{j,k}^2 \leq \infty.$$

thanks to (2.2). To sum up, we have proved that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \int |s_{m,n}(x) - \sigma_{m,n}(x)|^2 d\mu(x) \leq \infty. \quad (4.4)$$

By the monotone convergence theorem of the Lebesgue integral, it follows from (4.4) that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} |s_{m,n}(x) - \sigma_{m,n}(x)|^2 \leq \infty, a.e. \quad (4.5)$$

that is, the double series in (4.5) converges absolutely. Consequently, it converges regularly for almost every $x \in X$. Furthermore, the conditions in(3.1) are clearly satisfied by $\lambda_{j,k} := jk$. Thus, condition (3.2) is satisfied and we may apply the extension of the Kronecker lemma for the double series in (4.5) (cf. (3.3)) yielding (4.1) to be proved.

Our second new result gives answer to the problem of the strong cesaro $|C, 1, 1|$ summability of wavelet series.

Theorem 2. *If the double orthogonal series (2.1) is such that condition (2.8) is satisfied, then it is strong Ces'aro $|C, 1, 1|$ summable a.e., that is,*

$$\frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N |s_{m,n}(x) - g(x)|^2 \rightarrow 0 \text{ a.e. as } M, N \rightarrow \infty. \quad (4.6)$$

where $g(x)$ is the sum of (2.1) in the L^2 -norm (see in (2.4)).

Proof. We start with the elementary inequality

$$|s_{m,n}(x) - g(x)|^2 \leq 2(|s_{m,n}(x) - \sigma_{m,n}(x)|^2 + |\sigma_{m,n}(x) - g(x)|^2).$$

hence it follows that for $M, N \geq 2$ we have

$$\begin{aligned} & \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N |s_{m,n}(x) - g(x)|^2 \\ & \leq \frac{2}{MN} \sum_{m=1}^M \sum_{n=1}^N |s_{m,n}(x) - \sigma_{m,n}(x)|^2 + \frac{2}{MN} \sum_{m=1}^M \sum_{n=1}^N |\sigma_{m,n}(x) - g(x)|^2. \end{aligned} \quad (4.7)$$

By Theorem 1, the first term on the right-hand side of (4.7) converges to 0 a.e. as $\max\{M, N\} \rightarrow \infty$. As to the second term there, due to condition(2.8), we may apply the extended MenshovKaczmarcz theorem for double orthogonal series to obtain

(2.9), hence we conclude that the second term on the right-hand side of (4.7) also converges to 0 a.e. as $M, N \rightarrow \infty$. The proof of (4.6) is complete.

We note that the cesaro $(C, 1, 1)$ summability of the wavelet series (2.1) follows from its strong cesaro $|C, 1, 1|$ summability. Indeed, by the definition (2.7) and the familiar Cauchy inequality, we may estimate as follows:

$$\begin{aligned} |g(x)| &\leq \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N |s_{m,n}(x) - g(x)| \\ &\leq \frac{1}{MN} \left(\sum_{m=1}^M \sum_{n=1}^N 1^2 \right)^{\frac{1}{2}} \left(\frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N |s_{m,n}(x) - g(x)|^2 \right)^{\frac{1}{2}} \\ &= \left(\frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N |s_{m,n}(x) - g(x)|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Now, the implication (3.2) \implies (2.9) is clear.

Acknowledgements. Work is supported by DST-SERB Project No.: SR/S4/MS/818/13.

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