

**OTHER FIXED POINT THEOREMS FOR TWO PAIRS OF
ABSORBING MAPPINGS SATISFYING AN IMPLICIT RELATION
IN PARTIAL METRIC SPACES**

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ABSTRACT. In this paper some fixed point theorems for two pairs of absorbing mappings satisfying an implicit relation are proved, which extend Theorem 5 [21] to two pairs of mappings. As application, a general fixed point theorem for a sequence of mappings in partial metric spaces is obtained.

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1. INTRODUCTION

In 1994, Matthews [15] introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks and proved the Banach contraction principle in such spaces.

The partial metric spaces play an important role in construction models in theory of computation.

Many authors studied the fixed points for mappings satisfying some contractive condition in partial metric spaces.

Quite recently, in [2], [3], [11] some fixed point theorems under various contractive conditions are proved. In [12] and [13] some fixed point theorems for pairs of mappings in partial metric spaces are proved. Other results are obtained in [1].

In 1994, Pant [17] introduced the notion of pointwise R - weak commutativity which is equivalent to commutativity in coincidence points. Jungck [10] defined f and g to be weak compatible if $fx = gx$ implies $fgx = gfx$.

Then, f and g are weakly compatible if and only if f and g are pointwise R - weakly commuting.

In [12] some fixed point theorems for weakly compatible mappings in partial metric spaces are proved.

The notion of pointwise absorbing mappings is introduced and studied in [4] - [6].

Some results for two pairs of absorbing mappings are obtained in [14], [16].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [18], [19]. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, convex metric spaces, Hilbert spaces, compact metric spaces, in two and three metric spaces, for single valued mappings, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points of mappings satisfying contractive / extensive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, G - metric spaces, G_p - metric spaces. With this method the proofs of some fixed points theorems are more simple.

Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for mappings in complete partial metric spaces satisfying implicit relations is initiated in [8], [9], [20] - [22].

The purpose of this paper is to prove some fixed point theorems for two pairs of pointwise absorbing mappings satisfying an implicit relation in complete partial metric spaces, generalizing Theorem 2.2 [13]. As application, a general fixed point theorem for a sequence of mappings in partial metric spaces is obtained.

2. PRELIMINARIES

Definition 1 ([15]). *Let X be a nonempty set. A function $p : X \times X \rightarrow \mathbb{R}_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:*

(P_1) : $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$,

(P_2) : $p(x, x) \leq p(x, y)$,

(P_3) : $p(x, y) = p(y, x)$,

(P_4) : $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a partial metric space.

If $p(x, y) = 0$, then $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as a base the family of open p - balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$

with respect to τ_p if and only if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x).$$

If p is a partial metric on X , then the function $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ defines a metric on X .

Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{m, n \rightarrow \infty} p(x_n, x_m) = \lim_{n \rightarrow \infty} p(x_n, x) = p(x, x). \quad (1)$$

Definition 2. Let (X, p) be a partial metric space.

a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and is finite.

b) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$.

Lemma 1 ([3], [15]). Let (X, p) be a partial metric space. Then:

a) A sequence $\{x_n\}$ in X is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in (X, p^s) .

b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

Lemma 2 ([3], [1]). Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X . If $\lim_{n \rightarrow \infty} x_n = x$ and $p(x, x) = 0$, then

$$\lim_{n \rightarrow \infty} p(x_n, y) = p(x, y), \forall y \in X.$$

Theorem 3 (Theorem 2.2 [13]). Suppose that (X, p) is a complete partial metric space and T, S are self mappings on X . If there exists $r \in [0, 1)$ such that

$$p(Tx, Sy) \leq rM(x, y),$$

for any $x, y \in X$, where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Sy), \frac{1}{2}[p(x, Sy) + p(y, Tx)]\},$$

then there exists $z \in X$ such that $z = Tz = Sz$.

Definition 3 ([4], [5]). 1) Let (X, d) be a metric space and f, g be two self mappings on X . f is said to be g absorbing if there exists $R > 0$ such that

$$d(gx, gfx) \leq Rd(gx, fx)$$

for all $x \in X$.

2) f is said to be pointwise g absorbing if for given $x \in X$, there exists $R > 0$ such that

$$d(gx, gfx) \leq Rd(gx, fx).$$

Similarly we can define f absorbing and pointwise f absorbing map.

Remark 1. If (X, p) is a partial metric space we have similar definition of Definition 3 with p instead d .

Remark 2. If g is I - identity map in X , then f is trivial absorbing. Similarly, I is f absorbing in respect of any f .

3. IMPLICIT RELATIONS

Definition 4. Let \mathcal{F}_5 be the set of all continuous functions $F : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ satisfying:

- (F_1) : F is nonincreasing in variable t_5 ,
- (F_2) : There exists $h \in [0, 1)$ such that for all $u, v \geq 0$,
- (F_{2a}) : $F(u, v, v, u, u + v) \leq 0$
- and
- (F_{2b}) : $F(u, v, u, v, u + v) \leq 0$
- implies $u \leq hv$,
- (F_3) : $F(t, t, 0, 0, 2t) > 0, \forall t > 0$.

In the following examples, property (F_1) is obviously.

Example 1. $F(t_1, \dots, t_5) = t_1 - k \max \left\{ t_2, t_3, t_4, \frac{t_5}{2} \right\}$, where $k \in [0, 1)$.

(F_2) : Let $u, v \geq 0$ be such that

$$F(u, v, v, u, u + v) = u - k \max \left\{ u, v, \frac{u + v}{2} \right\} \leq 0.$$

If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k < 1$.

Similarly, $F(u, v, u, v, u + v) \leq 0$ implies $u \leq hv$.

(F_3) : $F(t, t, 0, 0, 2t) = t(1 - k) > 0, \forall t > 0$.

Example 2. $F(t_1, \dots, t_5) = t_1 - k \max \{t_2, t_3, t_4, t_5\}$, where $k \in \left[0, \frac{1}{2}\right)$.

The proof is similar to the proof of Example 1.

Example 3. $F(t_1, \dots, t_5) = t_1 - \frac{at_3t_1}{1+t_2} - bt_2 - c(t_3 + t_4) - dt_5$, where $a, b, c, d \geq 0$ and $a + b + 2c + 2d < 1$.

(F₂): Let $u, v \geq 0$ be such that

$$F(u, v, v, u, u + v) = u - \frac{auv}{u + v} - c(u + v) - d(u + v) \leq 0.$$

If $u > v$, then $u[1 - (a + b + 2c + 2d)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + b + 2c + 2d < 1$.

Similarly, $F(u, v, u, v, u + v) \leq 0$ implies $u \leq hv$.

(F₃): $F(t, t, 0, 0, 2t) = t[1 - (b + 2d)] > 0, \forall t > 0$.

Example 4. $F(t_1, \dots, t_5) = t_1 - a \max\{t_2, t_3, t_4\} - bt_5$, where $a, b \geq 0$ and $a + 2b < 1$.

Example 5. $F(t_1, \dots, t_5) = t_1^2 - at_2t_3 - bt_3t_4 - ct_5^2$, where $a, b, c \geq 0$ and $a + b + 4c < 1$.

Example 6. $F(t_1, \dots, t_5) = t_1^2 + \frac{t_1}{1 + t_5} - (at_2^2 + bt_3^2 + ct_4^2)$, where $a, b, c \geq 0$ and $a + b + c < 1$.

Example 7. $F(t_1, \dots, t_5) = t_1 - at_2 - bt_3 - c \max\{2t_4, t_5\}$, where $a, b, c \geq 0$ and $a + b + 2c < 1$.

Example 8. $F(t_1, \dots, t_5) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5^2$, where $a, b, c, d \geq 0$ and $a + b + c + 4d < 1$.

4. MAIN RESULTS

Theorem 4. Let (X, p) be a partial metric space and A, B, S, T be self mappings on X such that for all $x, y \in X$

$$F \left(\begin{array}{l} p(Ax, By), p(Sx, Ty), p(Sx, Ax), \\ p(Ty, By), p(Sx, By) + p(Ax, Ty) \end{array} \right) \leq 0, \quad (2)$$

where $F \in \mathcal{F}_5$. If z is a common fixed point of A, B, S and T , then $p(z, z) = 0$.

Proof. Suppose that $x = y = z$. Then by (2) we obtain

$$F(p(z, z), p(z, z), p(z, z), p(z, z), 2p(z, z)) \leq 0,$$

a contradiction of (F₃) if $p(z, z) > 0$. Hence, $p(z, z) = 0$.

Theorem 5. Let A, B, S and T be self mappings of a partial metric space (X, p) as in Theorem 4. Then A, B, S and T have at most one common fixed point.

Proof. Suppose that u, v are common fixed points of A, B, S and T . Then by (2) we have

$$F(p(u, v), p(u, v), p(u, u), p(v, v), p(u, v) + p(u, v)) \leq 0.$$

By Theorem 4, $p(u, u) = p(v, v) = 0$. Hence,

$$F(p(u, v), p(u, v), 0, 0, 2p(u, v)) \leq 0,$$

a contradiction of (F_3) if $p(u, v) > 0$. Hence, $p(u, v) = 0$, which implies $u = v$.

Theorem 6. *Let (X, p) be a complete partial metric space and A, B, S and T be self mappings on X such that*

- 1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
- 2) the inequality (2) holds for all $x, y \in X$ and some $F \in \mathcal{F}_5$.

If one of $A(X), B(X), S(X)$ or $T(X)$ is closed, then:

- 1) A and S have a coincidence point,
- 2) B and T have a coincidence point.

Moreover, if A is pointwise S absorbing and B is pointwise T absorbing, then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X . Since $A(X) \subset T(X)$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$.

Since $B(X) \subset S(X)$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}. \quad (3)$$

We prove that $\{y_n\}$ is a Cauchy sequence in (X, p) .

By (2) for $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$F \left(\begin{array}{l} p(Ax_{2n}, Bx_{2n+1}), p(Sx_{2n}, Tx_{2n+1}), p(Sx_{2n}, Ax_{2n}), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Sx_{2n}, Bx_{2n+1}) + p(Ax_{2n}, Tx_{2n+1}) \end{array} \right) \leq 0.$$

By (3) we obtain

$$F \left(\begin{array}{l} p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n}), \\ p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n+1}) + p(y_{2n}, y_{2n}) \end{array} \right) \leq 0. \quad (4)$$

By (P_4) ,

$$p(y_{2n-1}, y_{2n+1}) \leq p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}) - p(y_{2n}, y_{2n}).$$

By (4) and (F_1) we have

$$F \left(\begin{array}{l} p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n}), \\ p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}) \end{array} \right) \leq 0. \quad (5)$$

By (5) and (F_{2a}) we obtain

$$p(y_{2n}, y_{2n+1}) \leq hp(y_{2n-1}, y_{2n}).$$

Similarly by (2) for $x = x_{2n+2}$ and $y = x_{2n+1}$ we have

$$F \left(\begin{array}{l} p(Ax_{2n+2}, Bx_{2n+1}), p(Sx_{2n+2}, Tx_{2n+1}), p(Sx_{2n+2}, Ax_{2n+2}), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Sx_{2n+2}, Bx_{2n+1}) + p(Ax_{2n+2}, Tx_{2n+1}) \end{array} \right) \leq 0.$$

By (3) we have

$$F \left(\begin{array}{l} p(y_{2n+2}, y_{2n+1}), p(y_{2n+1}, y_{2n}), p(y_{2n+1}, y_{2n+2}), \\ p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+1}) + p(y_{2n+2}, y_{2n}) \end{array} \right) \leq 0. \quad (6)$$

By (P_4) ,

$$p(y_{2n}, y_{2n+2}) \leq p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}) - p(y_{2n+1}, y_{2n+1}).$$

By (6) and (F_1) we have

$$F \left(\begin{array}{l} p(y_{2n+2}, y_{2n+1}), p(y_{2n+1}, y_{2n}), p(y_{2n+1}, y_{2n+2}), \\ p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}) \end{array} \right) \leq 0.$$

By (F_{2b}) we have

$$p(y_{2n+2}, y_{2n+1}) \leq hp(y_{2n}, y_{2n+1}).$$

Hence

$$p(y_n, y_{n+1}) \leq hp(y_{n-1}, y_n) \leq \dots \leq h^n p(y_0, y_1).$$

For $n, m \in \mathbb{N}$ with $m > n$ and (P_4) we obtain

$$\begin{aligned} p(y_n, y_m) &\leq (h^n + h^{n+1} + \dots + h^{m-1}) p(y_0, y_1) \\ &\leq \frac{h^n}{1-h} p(y_0, y_1) \end{aligned}$$

and so

$$p^s(y_n, y_m) \leq 2p(y_n, y_m) \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

This implies that $\{y_n\}$ is a Cauchy sequence in the metric space (X, p^s) . Since (X, p) is complete, then by Lemma 1, (X, p^s) is complete. Therefore, there exists $y \in X$ such that $\lim_{n \rightarrow \infty} p^s(x_n, y) = 0$ and by (1), $p(y, y) = \lim_{n \rightarrow \infty} p(y_n, y_n) = \lim_{n, m \rightarrow \infty} p(y_n, y_m) = 0$. This implies

$$\lim_{n \rightarrow \infty} p(y_{2n+1}, y) = \lim_{n \rightarrow \infty} p(y_{2n-1}, y) \text{ and } \lim_{n \rightarrow \infty} p(Sx_{2n}, y) = 0.$$

Now we suppose without loss of generality that $S(X)$ is a closed subset of partial metric space (X, p) .

Then, there exists $u \in X$ such that $y = Su$.

By (2) for $x = u$ and $y = x_{2n+1}$ we obtain

$$F \left(\begin{array}{l} p(Au, Bx_{2n+1}), p(Su, Tx_{2n+1}), p(Su, Au), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Su, Bx_{2n+1}) + p(Au, Tx_{2n+1}) \end{array} \right) \leq 0,$$

$$F \left(\begin{array}{l} p(Au, y_{2n+1}), p(Su, y_{2n}), p(Su, Au), \\ p(y_{2n}, y_{2n+1}), p(Su, y_{2n+1}) + p(Au, y_{2n}) \end{array} \right) \leq 0. \quad (7)$$

Letting n tend to infinity by (7) and Lemma 2 we obtain

$$F(p(Au, y), 0, p(y, Au), 0, 0 + p(Au, y)) \leq 0.$$

By (F_{2b}) we have $p(y, Au) = 0$ which implies $y = Au = Su$.

Since $A(X) \subset T(X)$, $y \in T(X)$, hence, there exists $v \in X$ such that $y = Tv$.

By (2) for $x = u$ and $y = v$ we obtain

$$F \left(\begin{array}{l} p(Au, Bv), p(Su, Tv), p(Su, Au), \\ p(Tv, Bv), p(Su, Bv) + p(Au, Tv) \end{array} \right) \leq 0,$$

$$F(p(y, Bv), 0, 0, p(y, Bv), p(y, Bv) + 0) \leq 0.$$

By (F_{2a}) , $p(y, Bv) = 0$ which implies $y = Bv = Tv$. Hence, $y = Au = Su = Tv = Bv$ with $p(y, y) = 0$.

If A is pointwise S absorbing, there exists $R > 0$ such that

$$p(Su, SAu) \leq Rp(Su, Au) = Rp(y, y) = 0.$$

Hence,

$$Su = SAu = SSu = y$$

which implies

$$y = Sy.$$

Now we prove that $y = Ay$. By (2) we have

$$F \left(\begin{array}{l} p(Ay, Bx_{2n+1}), p(Sy, Tx_{2n+1}), p(Sy, Ay), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Sy, Bx_{2n+1}) + p(Ay, Tx_{2n+1}) \end{array} \right) \leq 0,$$

$$F \left(\begin{array}{l} p(Ay, y_{2n+1}), p(x, y_{2n}), p(y, Ay), \\ p(y_{2n}, y_{2n+1}), p(y, y_{2n+1}) + p(Ay, y_{2n}) \end{array} \right) \leq 0.$$

Letting n tend to infinity by Lemma 2 we obtain

$$F(p(Ay, y), 0, p(y, Ay), 0, 0 + p(Ay, y)) \leq 0.$$

Hence, by (F_{2b}) we obtain $p(y, Ay) = 0$, which implies $y = Ay$ and y is a common fixed point of A and S .

If B is pointwise T absorbing, there exists $R_1 > 0$ such that

$$p(Tv, TBv) \leq R_1 p(Tv, Bv) = R_1 p(y, y) = 0.$$

Hence,

$$Tv = TBv = TTv = y$$

which implies

$$y = Ty.$$

By (2) we have

$$F\left(\begin{array}{l} p(Au, By), p(Su, Ty), p(Su, Au), \\ p(Ty, By), p(Su, By) + p(Au, Ty) \end{array}\right) \leq 0,$$

$$F(p(y, By), 0, 0, p(y, By), p(y, By) + 0) \leq 0.$$

By (F_{2a}) we obtain $p(y, By) = 0$, which implies $y = By$ and y is a common fixed point of B and T .

Hence, y is a common fixed point of A, B, S and T . By Theorem 5, y is the unique common fixed point of A, B, S and T .

Example 9. Let $X = [0, 1]$ be a partial metric space with $p(x, y) = \max\{x, y\}$. Then (X, p) is a complete partial metric space. We define the following mappings on X .

$Ax = 0, Sx = \frac{x}{x+1}, Bx = \frac{x}{3}, Tx = x$. Then $A(X) = \{0\}, S(X) = \left[0, \frac{1}{2}\right], B(X) = \left[0, \frac{1}{3}\right], T(X) = [0, 1]$. Hence, $A(X) \subset T(X), B(X) \subset S(X)$ and $B(X), S(X), T(X)$ are closed subsets of (X, p) . Moreover,

$$\begin{aligned} p(Sx, SAx) &= p\left(\frac{x}{x+1}, 0\right) = \frac{x}{x+1}, \\ p(Sx, Ax) &= \frac{x}{x+1}. \end{aligned}$$

Hence,

$$p(Sx, SAx) \leq Rp(Sx, Ax) \text{ for some } R \geq 1.$$

$$p(Tx, TBx) = \max \left\{ x, \frac{x}{3} \right\} = x.$$

Hence,

$$p(Tx, TBx) \leq R_1 p(Tx, Bx) \text{ for some } R_1 \geq 1.$$

Hence, A is pointwise S absorbing and B is pointwise T absorbing. On the other hand,

$$\begin{aligned} p(Ax, By) &= p\left(0, \frac{y}{3}\right) = \frac{y}{3}, \\ p(Ty, By) &= p\left(y, \frac{y}{3}\right) = y. \end{aligned}$$

Hence,

$$p(Ax, By) \leq kp(Ty, By),$$

where $k \in \left[\frac{1}{3}, \frac{1}{2}\right)$, which implies

$$p(Ax, By) \leq k \max \{p(Sx, Ty), p(Sx, Ax), p(Ty, By), p(Sx, By) + p(Ty, Ax)\}.$$

By Theorem 6, A, B, S and T have a unique common fixed point.

If $S = T = I$, by Theorem 6 and Remark 2 we obtain

Theorem 7. Let (X, p) be a complete partial metric space and A, B be self mappings on X such that the inequality

$$F \left(\begin{array}{l} p(Ax, By), p(x, y), p(x, Ax), \\ p(y, By), p(x, By) + p(y, Ax) \end{array} \right) \leq 0$$

holds for all $x, y \in X$ and some F satisfying properties (F_1) , (F_{2a}) and (F_3) . Then A and B have a unique common fixed point.

Remark 3. By Theorem 7 and Example 1 we obtain Theorem 3.

For a function $f : (X, p) \rightarrow (X, p)$ we denote

$$Fix(f) = \{x \in X : x = fx\}.$$

If instead of (F_1) we have property

(F'_1) : F is nonincreasing in variables t_2, t_3, t_4, t_5 , we have

Theorem 8. *Let A, B, S and T be self mappings of a partial metric space (X, p) . If the inequality (2) holds for all $x, y \in X$ and some F satisfying properties (F'_1) , (F_2) and (F_3) , then we have:*

$$[Fix(S) \cap Fix(T)] \cap Fix(A) = [Fix(S) \cap Fix(T)] \cap Fix(B).$$

Proof. Let $x \in [Fix(S) \cap Fix(T)] \cap Fix(A)$. Then by (2) we have

$$F \left(\begin{array}{l} p(Ax, Bx), p(Sx, Tx), p(Sx, Ax), \\ p(Tx, Bx), p(Sx, Bx) + p(Ax, Tx) \end{array} \right) \leq 0,$$

$$F(p(x, Bx), p(x, x), p(x, x), p(x, Bx), p(x, Bx) + p(x, x)) \leq 0.$$

By (F_2) ,

$$p(x, x) \leq p(x, Bx)$$

and by (F'_1) we obtain

$$F(p(x, Bx), p(x, Bx), p(x, Bx), p(x, Bx), 2p(x, Bx)) \leq 0,$$

which implies by (F_2) that

$$p(x, Bx) \leq hp(x, Bx).$$

Hence, $p(x, Bx) = 0$, so $x = Bx$ and $x \in Fix(B)$. Therefore,

$$[Fix(S) \cap Fix(T)] \cap Fix(A) \subset [Fix(S) \cap Fix(T)] \cap Fix(B).$$

Similarly,

$$[Fix(S) \cap Fix(T)] \cap Fix(B) \subset [Fix(S) \cap Fix(T)] \cap Fix(A).$$

By Theorems 6 and 6 we obtain

Theorem 9. *Let S, T and $\{A_i\}_{i \in \mathbb{N}^*}$ be self mappings of a complete partial metric space such that:*

- a) $A_2(X) \subset S(X)$ and $A_1(X) \subset T(X)$,
- b) one of $A_1(X)$, $A_2(X)$, $S(X)$, $T(X)$ is a closed subset of X ,
- c) the inequality

$$F \left(\begin{array}{l} p(A_i x, A_{i+1} y), p(Sx, Ty), p(Sx, A_i x), \\ p(Ty, A_{i+1} y), p(Sx, A_{i+1} y) + p(Ty, A_i x) \end{array} \right) \leq 0,$$

holds for all $x, y \in X$, $i \in \mathbb{N}^$ and F satisfies conditions (F'_1) , (F_2) and (F_3) .*

If A_1 is S pointwise absorbing and A_2 is T pointwise absorbing, then S, T and $\{A_i\}_{i \in \mathbb{N}^}$ have a unique common fixed point.*

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