

Down Continuity of Real Functions

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Abstract: In this paper, we introduce and investigate the concepts of down continuity and down compactness. A real valued function f on a subset E of \mathbb{R} , the set of real numbers is down continuous if it preserves downward half Cauchy sequences, i.e. the sequence $(f(\alpha_n))$ is downward half Cauchy whenever (α_n) is a downward half Cauchy sequence of points in E , where a sequence (α_k) of points in \mathbb{R} is called downward half Cauchy if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\alpha_m - \alpha_n < \varepsilon$ for $m \geq n \geq n_0$. It turns out that the set of down continuous functions is a proper subset of the set of continuous functions.

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1 Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics, but also in other branches of sciences involving mathematics especially in computer science, dynamical systems, economics, information theory, biological science.

Using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([16], [58]), quasi-slowly oscillating continuity ([29]), ward continuity ([5], [21],[13]), p -ward continuity ([23]), δ -ward continuity ([17]), δ^2 -ward continuity ([2], statistical ward continuity ([19], [20]), λ -statistical ward continuity ([36], [49]), ρ -statistical ward continuity ([7]), arithmetic continuity ([8]) strongly lacunary ward continuity ([14], [31], [44], [43], and [44]), lacunary statistical ward continuity ([27], [32], and [60]), downward statistical continuity ([24]), lacunary statistical downward continuity ([10]) which enabled some authors to obtain conditions on the domain of a function to be uniformly continuous in terms of sequences in the sense that a function preserves a certain kind of sequences (see for example [58, Theorem 6],[13, Theorem 1 and Theorem 2],[29, Theorem 2.3].

The purpose of this paper is to introduce the concepts of down continuity of a real function, and the concept of down compactness of a subset of R which cannot be given by means of a sequential method G and prove interesting theorems.

2 Down compact sets

Throughout this paper, \mathbb{N} and \mathbb{R} will denote the set of positive integers, and the set of real numbers, respectively. Modifying the definition of a *downward Cauchy sequence* introduced in [12] and [3], we state a definition of downward half Cauchyness of a real sequence given in [51] in the following.

Definition 1 A sequence (α_k) of points in \mathbb{R} is called downward half Cauchy if for each $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ so that $\alpha_m - \alpha_n < \varepsilon$ for $m \geq n \geq n_0$.

We note that a sequence (α_k) is downward half Cauchy if and only if $(-\alpha_k)$ is upward half Cauchy. Trivially, any Cauchy sequence is downward half Cauchy, so is a convergent sequence, but there are downward half Cauchy sequences which are not Cauchy. For example, the sequence $(a_k) = (-k)$ is downward half Cauchy, but not Cauchy. Thus the set of downward half Cauchy sequences is a proper subset of the set of Cauchy sequences. Any subsequence of a downward half Cauchy sequence is downward half Cauchy. We note that the sum of two downward half Cauchy sequences is downward half Cauchy, the product of two bounded downward half Cauchy sequences is downward half Cauchy. For any positive real number c and for any downward half Cauchy sequence (α_k) , the sequence $(c\alpha_k)$ is downward half Cauchy.

Definition 2 A subset A of \mathbb{R} is called down compact if any sequence of points in E has a downward half Cauchy subsequence.

First, we note that any finite subset of \mathbb{R} is down compact, the union of two down compact subsets of \mathbb{R} is down compact, the intersection of any family of down compact subsets of \mathbb{R} is down compact, and any subset of a down compact set is down compact. Any statistically ward compact subset of \mathbb{R} is down compact. An I -sequentially compact subset of \mathbb{R} is down compact for a nontrivial admissible ideal I (see [30], [55], and [22]). Furthermore any bounded subset of \mathbb{R} is down compact, any slowly oscillating compact subset of \mathbb{R} is down compact (see [16] for the definition of slowly oscillating compactness), any above bounded subset of \mathbb{R} is down compact. These observations suggest to us the following.

Theorem 1 A subset of \mathbb{R} is down compact if and only if it is bounded above.

Proof. The proof can be obtained by using a contradiction method. ■

3 Down continuous functions

A real valued function f defined on a subset of \mathbb{R} is continuous if and only if, for each point ℓ in the domain, $\lim_{n \rightarrow \infty} f(\alpha_n) = f(\ell)$ whenever $\lim_{n \rightarrow \infty} \alpha_n = \ell$. This is equivalent to the statement that $(f(\alpha_n))$ is a convergent sequence whenever (α_n) is. This is also equivalent to the statement that $(f(\alpha_n))$ is a Cauchy sequence whenever (α_n) is Cauchy provided that the domain of the function is closed. These known results for continuity for real functions in terms of sequences might suggest to us introducing a new type of continuity, namely, down continuity.

Definition 3 A function $f : E \rightarrow \mathbb{R}$ is called down continuous on a subset of \mathbb{R} , if it preserves downward half Cauchy sequences, i.e. the sequence $(f(\alpha_k))$ is downward half Cauchy whenever (α_k) is a downward half Cauchy sequence of points in E .

It should be noted that down continuity cannot be given by any G -continuity in the manner of [4]. We see that the composition of two down continuous functions is down continuous, and for every positive real number c , cf is down continuous, whenever f is down continuous.

We see in the following that the sum of two down continuous functions is down continuous.

Theorem 2 If f and g are down continuous functions, then $f + g$ is down continuous.

Proof. The proof can be obtained easily, so is omitted. ■

The case for the product of functions is different. If f and g are bounded positive real valued down continuous functions, then the product of f and g is down continuous.

In connection with downward half Cauchy sequences, and convergent sequences the problem arises to investigate the following types of "continuity" of functions on \mathbb{R} :

$$(\delta^-) (\alpha_n) \in \Delta^- \Rightarrow (f(\alpha_n)) \in \Delta^-$$

$$(\delta^-c) (\alpha_n) \in \Delta^- \Rightarrow (f(\alpha_n)) \in c$$

$$(c\delta^-) (\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in \Delta^-.$$

We see that (c) can be replaced by not only ρ -statistical continuity, but also lacunary statistical continuity, strongly lacunary-sequential continuity, I -sequential continuity, and more generally G -sequential continuity (see [15, 18]). We see that (δ^-) is down continuity of f . It is easy to see that (δ^-c) implies (δ^-) ; (δ^-) does not imply (δ^-c) ; (δ^-) implies $(c\delta^-)$; $(c\delta^-)$ does not imply (δ^-) ; (δ^-c) implies (c) , and (c) does not imply (δ^-c) ; and (c) implies $(c\delta^-)$.

Now we give the implication (δ^-) implies (c) , i.e. any down continuous function is continuous.

Theorem 3 *Any down continuous function is continuous.*

Proof. The proof can be obtained easily, so is omitted. ■ We see in the following example that the converse of the preceding theorem is not always true.

Example 1 *The continuous function f defined by $f(x) = -x$ for every $x \in \mathbb{R}$ is not down continuous.*

Theorem 4 *Down continuous image of any down half compact subset of \mathbb{R} is down half compact.*

Proof. Let E be a subset of \mathbb{R} , $f : E \rightarrow \mathbb{R}$ be a down continuous function, and A be a down half compact subset of E . Take any sequence $\beta = (\beta_n)$ of points in $f(A)$. Write $\beta_n = f(\alpha_n)$, where $\alpha_n \in A$ for each $n \in \mathbb{N}$, $\alpha = (\alpha_n)$. Down half compactness of A implies that there is a downward half Cauchy subsequence ξ of the sequence of α . Write $\eta = (\eta_k) = f(\xi) = (f(\xi_k))$. Then η is a down half Cauchy subsequence of the sequence $f(\beta)$. This completes the proof of the theorem. ■ We note that down continuous image of any N_θ -sequentially compact subset of \mathbb{R} is N_θ -sequentially compact, and down continuous image of any ρ -statistically sequentially compact subset of \mathbb{R} is ρ -statistically sequentially compact. Furthermore down continuous image of any G -statistically sequentially connected subset of \mathbb{R} is G -sequentially connected (see [6], [9], [47], and [48]). On the other hand, uniform continuity does not imply down continuity. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = -x$ for every $x \in \mathbb{R}$ is uniformly continuous, but not down continuous.

Now we have the following result related to uniform convergence, namely, the uniform limit of a sequence of down continuous functions is down continuous.

Theorem 5 *If (f_n) is a sequence of down continuous functions defined on a subset E of \mathbb{R} and (f_n) is uniformly convergent to a function f , then f is down continuous on E .*

Proof. The proof can be obtained directly so is omitted. ■

4 Conclusion

In this paper, mainly a new types of continuity, namely down continuity of a real function, and down compactness of a subset of \mathbb{R} are introduced and investigated. In this investigation we have obtained results related to down continuity, some other kinds of continuities via downward half Cauchy sequences, convergent sequences, statistical convergent sequences, lacunary statistical convergent sequences of points in \mathbb{R} . We note that the set of down continuous functions is a proper subset of the set of ordinary continuous functions. The term downward half Cauchy sequence can be considered to be associated with above boundedness of the underlying space, whereas the term Cauchy sequence is traditionally associated with the completeness of the underlying space. Instead of downward half Cauchy sequence, one might tend to consider an asymptotically non-decreasing sequence in the sense that a sequence (α_n) is asymptotically non-decreasing if $\liminf_k \inf\{\alpha_m - \alpha_n : m > n \geq k, k \in \mathbb{N}\} \geq 0$, and investigate functions that preserve asymptotically non-decreasing sequences. As the set of downward half Cauchy sequences is different from the set of asymptotically non-decreasing sequences, a further new research would be carried out if one allows to find new results. We suggest to investigate downward half Cauchy sequences of fuzzy points in asymmetric fuzzy spaces (see [34], [1], [45], [39] for the definitions and related concepts in fuzzy setting and soft setting). We also suggest to investigate downward half Cauchy double sequences (see for example [33], [52], [53], [38] and [54] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate downward half Cauchy sequences of points in an asymmetric cone metric space since in a cone metric space the notion of a downward half Cauchy sequence coincides with the notion of a Cauchy sequence, and therefore down continuity coincides with ordinary continuity on complete subsets of \mathbb{R} (see [50], [37], [56], [57], [25], [59], and [35]).

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