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Coefficient Estimates for Initial Taylor-Maclaurin Coefficients for a Subclass of Analytic and Bi-univalent Functions Associated with q-Derivative Operator

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Abstract: In the present paper, we introduce and investigate a new subclass of analytic and bi-univalent functions $\Sigma_q(\varphi)$ in the open unit disk with respect to q-derivative operator. For functions belonging to this class, we obtain estimates on the first two Taylor- Maclaurin coefficients $|a_2|$ and $|a_3|$. Various other results, which presented in this paper, would generalize and improve those in related works of several earlier authors

Keywords: Analytic functions, Bi-univalent functions, q-derivative, Coefficient estimates.

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1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.
An analytic function f is subordinate to an analytic function g, written as $f \prec g$, provided there is an analytic (Schwarz) function w with $w(0) = 0, |w(z)| < 1$, for all $z \in \mathbb{U}$ satisfying $f(z) = g(w(z))$ for all $z \in \mathbb{U}$.

The well-known Koebe one-quarter theorem [1] ensure that the image of $\mathbb{U}$ under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Hence, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$ and

$$f^{-1}(f(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4})$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi univalent functions in $\mathbb{U}$ given by (1).

In 1986, Brannan and Taha [2] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses of starlike and convex functions of order α. In 2012, Ali et al. [3] widen the result of Brannan and Taha using subordination. Since then, various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two oefficients a_2 and a_3 of the Taylor-Maclaurin series expansion (1) were found in several recent studies. For interesting study on this topic can be found in ([5]-[6]-[7]-[8]).

In [11], [12], Jackson defined the q-derivative operator D_q of a function as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0, q \neq 0) \tag{2}$$

and $D_q f(z) = f'(0)$. In case $f(z) = z^k$ for k is a positive integer, the q-derivative of $f(z)$ is given by

$$D_q z^k = \frac{z^k - (zq)^k}{z(1-q)} = [k]_q z^{k-1}.$$

As $q \to 1^-$ and $k \in \mathbb{N}$, we have

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + ... + q^k \to k. \tag{3}$$

Quite a number of great mathematicians studied the concepts of q-derivative, for example by Gasper and Rahman [10], Aral et.al [13] and many others (see [15]-[20]).

Let φ be an analytic function with positive real part in $\mathbb{U}$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathbb{U})$ is symmetric with respect to real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{4}$$

We now introduce the following subclass of analytic and bi-univalent functions using the q-operator.

Definition 1.1 *A function $f \in \Sigma$ is said to be in the class $\Sigma_q(\varphi)$ if each of the following subordination condition holds true:*

$$D_q(f(z)) \prec \varphi(z), \quad z \in \mathbb{U}. \tag{5}$$

and

$$D_q(g(w)) \prec \varphi(w), \quad w \in \mathbb{U}. \tag{6}$$

where $g(w) = f^{-1}(w)$.

The subclass $\Sigma_q(\varphi)$ in Definition 1.1 can be reduced to many subclasses introduced before as seen in the following Remarks.

Remark 1.2 *Setting $q \to 1^-$, the class $\Sigma_q(\varphi)$ reduces to the class $\mathcal{H}_\sigma(\varphi)$ introduced by Ali et al.[3] which is a subclass of the functions $f \in \Sigma$ satisfying*

$$f'(z) \prec \varphi(z), \quad g'(w) \prec \varphi(z)$$

Remark 1.3 *Setting $q \to 1^-$ and*

$$\varphi(z) = \frac{1 + (1-2\beta)}{1-z} \quad (0 \leq \beta < 1), \qquad \varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad (0 < \alpha \leq 1),$$

the class $\Sigma_q(\varphi)$ *reduces to the classes* $\mathcal{H}_\Sigma^\alpha$ *and* $\mathcal{H}_\Sigma(\beta)$ *introduced by Srivastava et al.[4] which are subclasses of the functions* $f \in \Sigma$ *satisfying*

$$\left|\arg(f'(z))\right| < \frac{\alpha\pi}{2}, \quad \left|\arg(g'(w))\right| < \frac{\alpha\pi}{2}$$

and

$$Re(f'(z)) > \beta, \quad Re(g'(w)) > \beta$$

respectively.

Remark 1.4 *Setting*

$$\varphi(z) = \frac{1+(1-2\beta)}{1-z} \quad (0 \le \beta < 1) \quad \text{and} \quad \varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad (0 < \alpha \le 1)$$

the class $\Sigma_q(\varphi)$ *reduces to the classes* $\mathcal{H}_\Sigma^{q;\alpha}$ *and* $\mathcal{H}_\Sigma^q(\beta)$ *introduced by Bulut[9] which are subclasses of the functions* $f \in \Sigma$ *satisfying*

$$|\arg(D_q f(z))| < \frac{\alpha\pi}{2}, \quad |\arg(D_q g(w))| < \frac{\alpha\pi}{2}$$

and

$$Re(D_q f(z)) > \beta, \quad Re(D_q g(w)) > \beta$$

respectively.

In our investigation, we shall need the following Lemma

Lemma 1.5 *[14] Let the function* $p \in \mathcal{P}$ *be given by the following series:*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (z \in \mathbb{U}).$$

The sharp estimate given by

$$|p_n| \le 2 \quad (n \in \mathbb{N}),$$

holds true.

The object of the present paper is to find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass $\Sigma_q(\varphi)$ of the function class Σ.

2 A set of main results

For functions in the class $\Sigma_q(\varphi)$, the following result is obtained.

Theorem 2.1 *Let* $f \in \Sigma_q(\varphi)$ *be of the form (1). Then*

$$|a_2| \le \min\left\{\frac{B_1}{[2]_q}, \frac{B_1^{\frac{3}{2}}}{\sqrt{[3]_q B_1^2 + [2]_q^2(B_1 - B_2)}}\right\} \tag{7}$$

and

$$|a_3| \le \min\left\{\frac{B_2}{[3]_q}, \frac{B_1}{[3]_q} + \frac{B_1^2}{[2]_q^2}\right\} \tag{8}$$

where the coefficients B_1 *and* B_2 *are given as in (4).*

Proof. Let $f \in \Sigma_q(\varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = v(0) = 0$, satisfying the following conditions:

$$D_q(f(z)) = \varphi(u(z)), \quad z \in \mathbb{U} \tag{9}$$

and

$$D_q(g(w)) = \varphi(v(w)), \quad w \in \mathbb{U} \tag{10}$$

Define the functions p and q by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots \tag{11}$$

and

$$q(z) = \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots . \tag{12}$$

Then p and q are analytic in $\mathbb{U}$ with $p(0) = q(0) = 1$. Since $u, v : \mathbb{U} \to \mathbb{U}$, each of the functions p and q has a positive real part in $\mathbb{U}$. Therefore, in view of the above Lemma, we have

$$|p_n| \leq 2 \quad and \quad |q_n| \leq 2 \quad (n \in \mathbb{N}). \tag{13}$$

Solving for $u(z)$ and $v(z)$, we get

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2\right] + \cdots \quad (z \in \mathbb{U}) \tag{14}$$

and

$$v(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2\right] + \cdots \quad (z \in \mathbb{U}). \tag{15}$$

Upon substituting from (14) and (15) into (9) and (10), respectively, and making use of (4), we obtain

$$D_q(f(z)) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + B_1 p_1 z + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right]]z^2 + \cdots \tag{16}$$

and

$$D_q(g(w)) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + B_1 q_1 w + \left[\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right]]w^2 + \cdots \tag{17}$$

Equating the coefficients in (9) and (10),we find that

$$[2]_q a_2 = \frac{1}{2}B_1 p_1 \tag{18}$$

$$[3]_q a_3 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2 \tag{19}$$

$$-[2]_q a_2 = \frac{1}{2}B_1 p_1 \tag{20}$$

$$[3]_q(2a_2^2 - a_3) = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2 \tag{21}$$

From (18) and (20), we get

$$p_1 = -q_1 \tag{22}$$

and

$$2[2]_q^2 a_2^2 = \frac{1}{4} B_1^2 (p_1^2 + q_1^2) \tag{23}$$

Also from (19) and equation (21), we get

$$2[3]_q a_2^2 = \frac{1}{2} B_1 \left[p_2 + q_2 - \left(\frac{p_1^2 + q_1^2}{2} \right) \right] + \frac{1}{4} B_2 [p_1^2 + q_1^2], \tag{24}$$

by using (23), we get

$$a_2^2 = \frac{B_1^3 (p_2 + q_2)}{4 \left[[3]_q B_1^2 + [2]_q^2 (B_1 - B_2) \right]} \tag{25}$$

Applying Lemma 1.5 for the coefficients p_1, p_2, q_1, q_2 in the equalities (23) and (25), we obtain

$$|a_2| \leq \frac{B_1^{\frac{3}{2}}}{\sqrt{[3]_q B_1^2 + [2]_q^2 (B_1 - B_2)}} \tag{26}$$

$$|a_2| \leq \frac{B_1}{[2]_q} \tag{27}$$

Hence equations (26) and (27) gives the estimates of $|a_2|$.

Next, in order to find the bound on $|a_3|$, we subtract (21) from (19) and also from (22), we get $p_1^2 = q_1^2$, hence

$$2[3]_q a_3 - 2[3]_q a_2^2 = \frac{1}{2} B_1 (p_2 - q_2), \tag{28}$$

which, upon substitution of the value of a_2^2 from (23) into (28), yields

$$a_3 = \frac{B_1}{[3]_q} (p_2 - q_2) + \frac{B_1^2}{[2]_q^2} (p_1^2 + q_1^2). \tag{29}$$

So we get

$$|a_3| \leq \frac{B_1}{[3]_q} + \frac{B_1^2}{[2]_q^2}. \tag{30}$$

On the other hand, upon substituting the value of a_2^2 from (24) into (28),it follows that

$$a_3 = \frac{4B_1 p_2 + (B_2 - B_1)(p_1^2 + q_1^2)}{8[3]_q}. \tag{31}$$

And we get

$$|a_3| \leq \frac{B_2}{[3]_q}. \tag{32}$$

Thus, we get the desired estimate on the coefficient $|a_3|$ as asserted in (40).

3 Corollaries and Consequensec

Taking $q \to 1^-$ in Theorem 2.1, we obtain the following corollary.

Corollary 3.1 *Let the function f given by (1) be in the class $\Sigma(\varphi)$. Then*

$$|a_2| \leq \min\left\{\frac{B_1}{2}, \frac{B_1\sqrt{B_1}}{\sqrt{3B_1^2+4(B_1-B_2)}}\right\} \tag{33}$$

and

$$|a_3| \leq \min\left\{\frac{B_2}{3}, \left(\frac{1}{3}+\frac{B_1}{4}\right)B_1\right\} \tag{34}$$

Remark 3.2 *Corollary 3.1 is an improvement of the following estimates obtained by Ali et al.[3].*

Corollary 3.3 *(see [3]) Let the function f given by (1) be in the function class $\mathcal{H}_\sigma(\varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{3B_1^2-4B_2+4B_1)}} \quad \textit{and} \quad |a_3| \leq \left(\frac{1}{3}+\frac{B_1}{4}\right)B_1. \tag{35}$$

Taking

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\cdots \quad (0 \leq \beta < 1) \tag{36}$$

in Theorem 2.1, we have the following corollary.

Corollary 3.4 *[9] Let the function f given by (1) be in the function class $\Sigma_q(\beta)$ $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{[2]_q}, \sqrt{\frac{2(1-\beta)}{[3]_q}}\right\} \tag{37}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{[3]_q} \tag{38}$$

Taking $q \to 1^-$ in Corollary 3.4, we have the following corollary

Corollary 3.5 *Let the function f given by (1) be in the function class $\Sigma(\beta)$ $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \min\left\{(1-\beta), \sqrt{\frac{2(1-\beta)}{3}}\right\} \tag{39}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3} \tag{40}$$

Remark 3.6 *Corollary 3.5 is an improvement of the following estimates obtained by Srivastave et al. [4].*

Corollary 3.7 *[4] Let the function f given by (1) be in the function class $\mathcal{H}_\Sigma(\alpha)$ $(0 \leq \alpha < 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{3}} \tag{41}$$

and

$$|a_3| \leq \frac{(1-\alpha)(5-3\alpha)}{3}. \tag{42}$$

Taking

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1) \tag{43}$$

in Theorem 2.1, we have the following corollary.

Corollary 3.8 *Let the function f given by (1) be in the function class $\Sigma_q(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \min\left\{\frac{2\alpha}{[2]_q} + \frac{2\alpha}{\sqrt{2[3]_q\alpha + (1-\alpha)[2]_q^2}}\right\} \tag{44}$$

and

$$|a_3| \leq \min\left\{\frac{2\alpha^2}{[3]_q}, \frac{2\alpha}{[3]_q} + \frac{4\alpha^2}{[2]_q^2}\right\}. \tag{45}$$

Remark 3.9 *Corollary 3.8 is an improvement of the following estimates obtained by Bulut [9].*

Corollary 3.10 *[9] Let the function f given by (1) be in the function class $\Sigma_q(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2[3]_q\alpha + (1-\alpha)[2]_q^2}} \tag{46}$$

and

$$|a_3| \leq \frac{2\alpha}{[3]_q} + \frac{4\alpha^2}{[2]_q^2} \tag{47}$$

Taking $q \to 1^-$ in Corollary 3.8, we have the following corollary.

Corollary 3.11 *Let the function f given by (1) be in the function class $\Sigma(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \min\left\{\alpha, \alpha\sqrt{\frac{2}{\alpha+2}}\right\} \tag{48}$$

and

$$|a_3| \leq \min\left\{\frac{2\alpha^2}{3}, \frac{\alpha(3\alpha+2)}{3}\right\}. \tag{49}$$

Remark 3.12 *Corollary 3.11 is an improvement of the following estimates obtained by Srivastave et al. [4].*

Corollary 3.13 *[4] Let the function f given by (1) be in the function class $\mathcal{H}_{\Sigma}^{\alpha}$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}} \tag{50}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}. \tag{51}$$

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