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# SOME RESULTS ON AN EQUIVALENCE RELATION ON THE SET OF CLOSED AND BOUNDED VALUED MULTIFUNCTIONS

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Abstract. By using the notion of the fixed point set of multi-valued mappings, we introduce an equivalence relation on the set of all closed and bounded valued multifunction on a metric space. By using the notion we provide some related results.

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## 1. Introduction

In 1966, Sam Bernard Jr. Nadler finished his Ph.D. thesis on differential analysis in university of Georgia ([2]). Later, he published some works about results of his thesis ([3], [4] and [6]). He interested fixed point theory by starting basic notions of fixed points and contractive mappings ([5], [7] and [8]). In 1969, he started study of fixed points of multivalued contractive mappings ([9]). In 1970, he published his most famous work in this area ([10]). Hereafter, many researchers reviewed common fixed points of different types of multivalued contractions (see for example, [11], [12] and [13]). In this paper, we introduce an equivalence relation on the set of all closed and bounded valued multifunction on a metric space. Also by using the notion, we provide some related results.

Let $X$ be a nonempty set, $\mathcal{P}(X)$ the set of all nonempty subsets of $X$, $T$ a multi-valued mapping on $X$ into $\mathcal{P}(X)$ and $\mathfrak{F}_T$ the fixed point set of $T$, that is, $\mathfrak{F}_T = \{x \in X : x \in Tx\}$. For a topological space $(Y, \tau)$, we denote the set of all nonempty closed subsets of $Y$ by $P_{cl}(Y)$ and the set of all nonempty closed and bounded subsets of $Y$ by $P_{b,cl}(Y)$ whenever $Y$ is a metric space.

Let $(X, d)$ be a metric space, $x \in X$ and $A, B \subseteq X$. It is well-known that $D(x, A) = \inf_{y\in A} d(x, y)$, $H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}$ and $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. Then, $H$ is a metric on closed bounded subsets of $X$ which is called the Hausdorff metric.

## 2. Main results

Let $(X, d)$ be a metric space. Denote by $\mathcal{F}$ the set of all multi-valued mappings on $X$ into $P_{b,cl}(X)$. Define the relation $\sim$ on $\mathcal{F}$ by $F \sim G$ whenever $\mathfrak{F}_F = \mathfrak{F}_G$ for all $F, G \in \mathcal{F}$. One can check that $\sim$ is an equivalence relation on $\mathcal{F}$. Denote by $\tilde{\mathcal{F}}$ the equivalence classes of $\mathcal{F}$, that is, $\tilde{\mathcal{F}} = \frac{\mathcal{F}}{\sim} = \{\tilde{F} : F \in \mathcal{F}\}$. Also, define $\tilde{d} : \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \to [0, \infty)$ by $\tilde{d}(\tilde{F}, \tilde{G}) = H(\mathfrak{F}_F, \mathfrak{F}_G)$. It is easy to see that $(\tilde{\mathcal{F}}, \tilde{d})$ is metric space. Note that, there is a connection between common fixed points of two multivalued mappings $S$ and $T$ whenever $S \in \tilde{T}$.

**Lemma 2.1.** Let $(X, d)$ be a metric space, $m \geq 1$ $c > 1$ and $S, T : X \to P_{b,cl}(X)$ two multi-valued mappings such that $\mathfrak{F}_S \neq \emptyset$. Suppose that for each $x \in X$ and $y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) such that

$$d^{3m}(x, y) - \frac{3}{4\sqrt[3]{4}} c^2 d^{2m}(y, z) d(x, y) - \frac{c^3}{8} d^{3m}(y, z) \geq 0. \qquad (1)$$

Then $\mathfrak{F}_T \neq \emptyset$ and $\tilde{S} = \tilde{T}$.

*Proof.* Let $u \in \mathfrak{F}_S$ and $z \in Tu$. By using the relation (1), we get

$$d^3(u, u) - \frac{3}{4\sqrt[3]{4}} c^2 d^2(u, z) d(u, u) - \frac{c^3}{8} d^3(u, z) \geq 0.$$

Hence, $-\frac{c^3}{8} d^3(u, z) \geq 0$ and so $d(u, z) = 0$. This implies that $z = u$ and so $u \in Tu$. Thus, $\mathfrak{F}_T \neq \emptyset$ and $\mathfrak{F}_S \subset \mathfrak{F}_T$. A similar proof shows that $\mathfrak{F}_T \subset \mathfrak{F}_S$. Therefore, $\tilde{S} = \tilde{T}$.

Let $(X, d)$ be a metric space and $V : X \to P_{b,cl}(X)$ a multi-valued map. We say that $T$ has the property $(M)$ whenever for each convergent sequence $\{x_n\}_{n \geq 0}$ with $x_n \to x$ and $x_{2n-1} \in Tx_{2n-2}$ for all $n$ (or $x_{2n} \in TVx_{2n-1}$ for all $n$) we have $x \in Tx$.

**Theorem 2.2.** Let $(X, d)$ be a complete metric space, $S, T : X \to P_{b,cl}(X)$ two multi-valued mappings, $m \geq 1$ and $c > 1$. Suppose that for each $x \in X$ and $y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) such that

$$d^{3m}(x, y) - \frac{3}{4\sqrt[3]{4}} c^2 d^{2m}(y, z) d(x, y) - \frac{c^3}{8} d^{3m}(y, z) \geq 0.$$

If one of the multi-valued mappings $S$ and $T$ have the property $(M)$, then $\tilde{S} = \tilde{T}$.

*Proof.* Let $x_0 \in X$ be an arbitrary element and $x_1 \in Sx_0$. Choose $x_2 \in Tx_1$ such that $d^{3m}(x_0, x_1) - \frac{3}{4\sqrt[3]{4}} c^2 d^{2m}(x_1, x_2) d(x_0, x_1) - \frac{c^3}{8} d^{3m}(x_1, x_2) \geq 0$. There exists

$x_3 \in Sx_2$ such that $d^{3m}d(x_1,x_2) - \frac{3}{4\sqrt[3]{4}}c^2 d^{2m}(x_2,x_3)d(x_1,x_2) - \frac{c^3}{8}d^{3m}(x_2,x_3) \geq 0$. By continuing this process we obtain a sequence $\{x_n\}_{n\geq 0}$ in $X$ such that $x_{2n-1} \in Sx_{2n-1}$ and $x_{2n} \in Tx_{2n-1}$ for all $n$ and

$$d^{3m}(x_n,x_{n-1}) - \frac{3}{4\sqrt[3]{4}}c^2 d^{2m}(x_n,x_{n+1})d(x_n,x_{n-1}) - \frac{c^3}{8}d^{3m}(x_n,x_{n+1}) \geq 0 \qquad (2)$$

for all $n$. Note that, the inequality (2) is a third degree polynomial in the variable $d^m(x_n,x_{n-1})$ with the discriminant

$$\Delta = 4(\frac{-3}{4\sqrt[3]{4}}c^2 d^{2m}(x_n,x_{n+1}))^3 + 27(\frac{-c^3}{8}d^{3m}(x_n,x_{n+1}))^2.$$

Thus, $d^m(x_n,x_{n-1}) \geq -2\sqrt[3]{\frac{c^3}{8}d^{3m}(x_n,x_{n+1})} = cd^m(x_n,x_{n+1})$. If $k^m = \frac{1}{c}$, then we obtain $k < 1$ and $0 \leq d^m(x_n,x_{n+1}) < k^m d^m(x_n,x_{n-1})$. This implies that $d(x_n,x_{n+1}) \leq kd(x_{n-1},x_n)$ for all $n$. Hence, $d(x_n,x_{n+1}) \leq k^n d(x_0,x_1)$ for all $n$. It is easy to see that $d(x_n,x_{n+p}) \leq \frac{k^n}{1-k}d(x_0,x_1)$ for all $n$ and $p$. Thus, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in $X$. Choose $u \in X$ such that $x_n \to u$. Since $x_{2n-1} \in Sx_{2n-1}$ and $x_{2n} \in Tx_{2n-1}$ for all $n$ and one of the multi-valued mappings $S$ and $T$ have the property $(M)$, we conclude that $u \in Su$ or $u \in Tu$. By using Lemma 2.1, we get $\tilde{S} = \tilde{T}$.

We need the followings for our last result.

**Lemma 2.3.** [13] Let $(X,d)$ be a metric space, $A$ and $B$ two bounded subsets of $X$ and $k > 1$. Then for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq kH(A,B)$.

This implies easily next Lemma.

**Lemma 2.4.** [13] Let $(X,d)$ be a metric space, $k > 1$ and $S,T : X \to P_{cl,b}(X)$ two multi-valued mappings. Then for each $y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) such that $d(y,z) \leq kH(Sx,Ty)$.

**Theorem 2.5.** Let $(X,d)$ be a complete metric space, $T_1,T_2 : X \to P_{b,cl}(X)$ two multi-valued mappings, $m \geq 1$ and $c > 1$. Suppose that for each $x,y \in X$ with $c^2\delta^{2m}(y,T_2y) + 6c\delta^m(y,T_2y)\delta^m(x,T_1x) + 8\delta^{2m}(x,T_1x) \neq 0$ we have

$$H^m(T_1x,T_2y) \leq \frac{8d^{3m}(x,T_1x)}{c^2\delta^{2m}(y,T_2y) + 6c\delta^m(y,T_2y)\delta^m(x,T_1x) + 8\delta^{2m}(x,T_1x)}. \qquad (3)$$

Then $\tilde{T_1} = \tilde{T_2}$.

*Proof.* By using the inequality (3), we obtain

$$H^m(T_1x,T_2y)(c^2\delta^{2m}(y,T_2y)+6c\delta^m(y,T_2y)\delta^m(x,T_1x)+8\delta^{2m}(x,T_1x)) \leq 8d^{3m}(x,T_1x)$$

for all $x \in X$ and $y \in T_1x$. Let $1 < c < k^m$, $x \in X$ and $y \in T_1x$. By using Lemma 2.4, there exists $z \in T_2y$ such that $d(y,z) \leq kH(T_1x,T_2y)$. Hence,

$$cd^m(y,z)(c^2d^{2m}(y,z) + \frac{6cd^m}{\sqrt[3]{4}}d^m(y,z)d(x,y) \leq 8d^{3m}(x,y).$$

Thus for each $x \in X$ and $y \in T_1x$ there exists $z \in T_2y$ such that

$$d^{3m}(x,y) - \frac{3}{4\sqrt[3]{4}}cd^m(y,z)d^m(x,y) - \frac{c^3}{8}d^3(y,z) \geq 0.$$

Now, we show that $T_1$ has the property $(M)$. Let $(x_n)_{n\geq 0}$ be a convergent sequence in $X$ with $x_n \to x$, $x_{2n-1} \in T_1x_{2n-2}$ and $x_{2n} \in T_2x_{2n-1}$ for all $n$. Then, we have

$$d(T_1x, x_{2n}) \leq H(T_1x, T_2x_{2n-1})$$

for all $n$. Hence,

$$cd^m(T_1x,x_{2n})(c^2d^{2m}(x_{2n-1},x_{2n}) + 6cd^m(x_{2n-1},x_{2n})d^m(x_{2n},T_1x) + 8d^(x_{2n},T_1x))$$
$$\leq 8d^{3m}(x_{2n},T_1x)$$

for all $n$ and so $d(x,T_1x) \leq \frac{1}{c}d(x,T_1x)$, that is, $d(T_1x,x) = 0$. Since $T_1x$ is a closed subset of $X$, we conclude that $x \in T_1x$. Now by using Lemma 2.1 and Theorem 2.2, we get $\mathfrak{F}_{T_1} = \mathfrak{F}_{T_2}$ and $\tilde{T_1} = \tilde{T_2}$.

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