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UNIVALENCY OF SOME OPERATORS FOR ANALYTIC FUNCTIONS

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Abstract. For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, univalency of some integral operators concerning with Alexander type integrals is considered. Also some subordinations for analytic functions $f(z)$ in $\mathbb{U}$ are discussed with the Schwarzian derivative of $f(z)$.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ which are univalent in $\mathbb{U}$. If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\alpha\,(0 \leqq \alpha < 1)$, then $f(z)$ is said to be starlike of order α in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{S}^*(\alpha)$. For $\alpha = 0$, we say that $f(z) \in \mathcal{S}^*$ is starlike with respect to the origin. Further, if a fnction $f(z) \in \mathcal{A}$ satisfies $zf'(z) \in \mathcal{S}^*(\alpha)$ $(0 \leqq \alpha < 1)$, then $f(z)$ is said to be convex of order α in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{K}(\alpha)$. A function $f(z) \in \mathcal{K}(\alpha)$ satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}). \tag{1.3}$$

For $\alpha = 0$, we write that $\mathcal{K}(0) \equiv \mathcal{K}$. We note that

$$\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S} \subset \mathcal{A} \subset \mathcal{H}.$$

If there exists a function $g(z) \in \mathcal{K}$ such that

$$\text{Re}\left(e^{-i\beta}\frac{f'(z)}{g(z)}\right) > 0 \qquad (z \in \mathbb{U}) \tag{1.4}$$

for $\beta \in (-\pi/2, \pi/2)$ and $f(z) \in \mathcal{A}$, then $f(z)$ is said to be close-to-convex in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{C}$. It is known that $\mathcal{C} \subset \mathcal{S}$.

For $f(z) \in \mathcal{H}$, the Schwarzian derivative of $f(z)$ is given by

$$\{f; z\} = 6\left(\frac{\partial^2}{\partial z \partial \zeta}\log\left(\frac{f(z) - f(\zeta)}{z - \zeta}\right)\right)_{z=\zeta} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{1.5}$$

For the Schwarzian derivative $\{f; z\}$ for $f(z) \in \mathcal{H}$, it is well-known that if $f(z) \in \mathcal{H}$ is univalent in $\mathbb{U}$, then

$$|\{f; z\}| \leqq \frac{6}{(1 - |z|^2)^2} \qquad (z \in \mathbb{U}) \tag{1.6}$$

and the equality holds true for the Koebe function $f(z) = z/(1 - z)^2$. Further, we know that the Nehari's condition (see Nehari [10])

$$|\{f; z\}| \leqq \frac{2}{(1 - |z|^2)^2} \qquad (z \in \mathbb{U}) \tag{1.7}$$

implies that $f(z) \in \mathcal{H}$ is univalent in $\mathbb{U}$.
Note that $f(z) \in \mathcal{A}$ is uniformly locally univalent if and only if the pre-Schwarzian derivative

$$T_f(z) = \frac{f''(z)}{f'(z)} \tag{1.8}$$

is hyperbolically bounded, that is, that the norm

$$\| f \| = \sup_{|z|<1}(1 - |z|^2)|T_f(z)| \tag{1.9}$$

is finite. This quantity can be regarded as the Bloch norm of function $(\log f(z))'$. Both of the pre-Schwarzian derivative and the norm $\| f \|$ play a central role in the theory of Teichmüller spaces, inner radius of univalence, quasiconformal extension, etc.. If $f(z) \in \mathcal{A}$ is univalent in $\mathbb{U}$, then $\| f \| < 6$ and the bound 6 is sharp for the

Koebe function $k(z) = z/(1-z)^2$.
Conversely, if $f(z) \in \mathcal{A}$ satisfies $\| f \| < 1$, then $f(z)$ is univalent in $\mathbb{U}$ by Becker [1]. Also, it is known that $\| f \| < 4$ for $f(z) \in \mathcal{K}$. For $f(z) \in \mathcal{A}$, the Alexander transformation $J[f](z)$ is defined by

$$J[f](z) = \int_0^z \frac{f(t)}{t} dt. \tag{1.10}$$

If $f(z) \in \mathcal{S}$, then $f(z)$ is locally univalent and $\| J[f] \| < 6$ by Kim, Choi and Sugawa [6]. Also, Yamashita [12] proved that if $f(z) \in \mathcal{S}^*(\alpha)$, then $\| f \| < 6-4\alpha$ and $\| J[f] \| < 4(1-\alpha)$. By means of (1.5) and (1.8), we see that

$$\{f; z\} = (T_f(z))' - \frac{1}{2}(T_f(z))^2. \tag{1.11}$$

The Alexander transformation $J[f](z)$ of $f(z) \in \mathcal{A}$ is also called as Biernacki's integral. It is known that $J[f](\mathcal{S}^*) = \mathcal{K}$ while $J[f](\mathcal{S})$ is not in $\mathcal{S}$. In this paper, we would like to extend the type of functions $f(z)$ to be considered by introducing a parameter α and setting an integral of the form

$$F_\alpha(z) = \int_0^z \left(\frac{tf'(t)}{f(t)} \right)^\alpha dt. \tag{1.12}$$

For more details on this integral, we refer to Goodman [4]. The following lemma due to Fukui and Sakaguchi [3] is a generalization of Jack's lemma by Jack [5] (also by Miller and Mocanu [9]).

Lemma 1.1 *Let $w(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots$ be analytic in $\mathbb{U}$ with $a_p \neq 0$ and $p \geqq 1$. If the maximum value of $|w(z)|$ on the circle $|z| = r < 1$ is attained at $z = z_0$, then $z_0 w'(z_0)/w(z_0)$ is real and*

$$\frac{z_0 w'(z_0)}{w(z_0)} \geqq p. \tag{1.13}$$

2. Univalency of some operators

We first derive

Theorem 2.1 *Let $f(z)$ be analytic in $\mathbb{U}$ with $f(0) = 0$. If $f(z)$ satisfies*

$$|f(z)| \leqq \frac{M}{1-|z|^2} \qquad (z \in \mathbb{U}) \tag{2.1}$$

for a bounded positive constant M, then

$$|f(z)| \leqq \frac{3\sqrt{3}M|z|}{2} \leqq \frac{3\sqrt{3}M|z|}{2(1-|z|^2)} \qquad (|z| \leqq \frac{\sqrt{3}}{3}) \tag{2.2}$$

and

$$|f(z)| \leqq \frac{\sqrt{3}M|z|}{1-|z|^2} \leqq \frac{3\sqrt{3}M|z|}{2(1-|z|^2)} \qquad (\frac{\sqrt{3}}{3} \leqq |z| < 1). \tag{2.3}$$

Proof For the case of $|z| \leqq \sqrt{3}/3$, we have

$$\frac{1}{1-|z|^2} \leqq \frac{3}{2}. \tag{2.4}$$

Thus, the inequality (2.1) gives

$$|f(z)| \leqq \frac{3M}{2} \qquad (|z| \leqq \frac{\sqrt{3}}{3}). \tag{2.5}$$

Therefore, applying the Schwarz lemma for $f(z)$ with $|z| \leqq \sqrt{3}/3$, we obtain that

$$|f(z)| \leqq \sqrt{3}|z|\frac{3M}{2} \qquad (|z| \leqq \frac{\sqrt{3}}{3}) \tag{2.6}$$

which shows (2.2). If $\sqrt{3}/3 \leqq |z| < 1$, we know that $\sqrt{3}|z| \geqq 1$. This gives us that

$$|f(z)| \leqq \frac{\sqrt{3}M|z|}{1-|z|^2} \qquad (\frac{\sqrt{3}}{3} \leqq |z| < 1) \tag{2.7}$$

which implies the inequality (2.3).

Corollary 2.1 *If $f(z)$ is analytic in $\mathbb{U}$ with $f(0) = 0$, then there exists some $z \in \mathbb{U}$ such that*

$$|f(z)| \leqq \frac{M}{1-|z|^2} \tag{2.8}$$

satisfies

$$|f(z)| \leqq \frac{3\sqrt{3}M|z|}{2(1-|z|^2)} \tag{2.9}$$

for a positive constant M.

Remark 2.1 Noting that $3\sqrt{3}/2 = 2.598\ldots$, we conjecture that $3\sqrt{3}/2$ in Corollary 2.1 can be replaced by 1.

Next, we derive

Theorem 2.2 *For a function $f(z) \in \mathcal{S}$, we assume that the function $(zf'(z)/f(z))^\alpha$ is analytic in $\mathbb{U}$ for $\alpha > 0$ with*

$$\left.\left(\frac{zf'(z)}{f(z)}\right)^\alpha\right|_{z=0} = 1. \tag{2.10}$$

Then, the integral transformation $F_\alpha(z)$ defined by (1.12) *is univalent in $\mathbb{U}$ for*

$$0 < \alpha \leqq \alpha_0 = \frac{2\sqrt{5}-4}{15\sqrt{3}} = 0.0181725\ldots. \tag{2.11}$$

Proof Note that

$$F'_\alpha(z) = \left(\frac{zf'(z)}{f(z)}\right)^\alpha \qquad (z \in \mathbb{U}) \tag{2.12}$$

by $F_\alpha(z)$ in (1.12). This gives us that

$$\frac{F''_\alpha(z)}{F'_\alpha(z)} = \frac{\alpha}{z}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right). \tag{2.13}$$

If we put

$$h(z) = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U}), \tag{2.14}$$

we have that $h(0) = 0$ and

$$|h(z)| \leqq \left|1 + \frac{zf''(z)}{f'(z)}\right| + \left|\frac{zf'(z)}{f(z)}\right|. \tag{2.15}$$

On the other hand, it is well-known that if $f(z) \in \mathcal{S}$, then

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leqq \frac{4|z|}{1-|z|^2} \qquad (z \in \mathbb{U}) \tag{2.16}$$

that is,

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{1+|z|^2}{1-|z|^2}\right| \leqq \frac{4|z|}{1-|z|^2} \qquad (z \in \mathbb{U}). \tag{2.17}$$

This gives that

$$\left|1+\frac{zf''(z)}{f'(z)}\right| \leqq \frac{4|z|}{1-|z|^2}+\frac{1+|z|^2}{1-|z|^2}<\frac{6}{1-|z|^2} \qquad (z\in\mathbb{U}). \tag{2.18}$$

Further, we know that

$$\left|\frac{zf'(z)}{f(z)}\right| \leqq \frac{1+|z|}{1-|z|}=\frac{(1+|z|)^2}{1-|z|^2}<\frac{4}{1-|z|^2} \qquad (z\in\mathbb{U}). \tag{2.19}$$

Therefore, the inequality (2.15) implies that

$$|h(z)|<\frac{10}{1-|z|^2} \qquad (z\in\mathbb{U}). \tag{2.20}$$

Considering $M=10$ in (2.1) of Theorem 2.1, we say that

$$|h(z)|<\frac{15\sqrt{3}|z|}{1-|z|^2} \qquad (z\in\mathbb{U}). \tag{2.21}$$

Therefore, we have that

$$\left|\frac{F''_\alpha(z)}{F'_\alpha(z)}\right| \leqq \frac{\alpha}{|z|}|h(z)|<\frac{15\sqrt{3}\alpha}{1-|z|^2} \qquad (z\in\mathbb{U}). \tag{2.22}$$

By using of the result in [11], we know that there exists a point $z\in\mathbb{U}$ that if

$$|h(z)|<\frac{1}{1-|z|^2} \qquad (z\in\mathbb{U}), \tag{2.23}$$

then

$$|h'(z)|<\frac{4}{(1-|z|^2)^2} \qquad (z\in\mathbb{U}). \tag{2.24}$$

It follows from the above that

$$\left|\left(\frac{F''_\alpha(z)}{F'_\alpha(z)}\right)'\right|<\frac{60\sqrt{3}\alpha}{(1-|z|^2)^2} \qquad (z\in\mathbb{U}). \tag{2.25}$$

Therefore, we have that

$$|\{F_\alpha(z);z\}| \leqq \left|\left(\frac{F''_\alpha(z)}{F'_\alpha(z)}\right)'\right|+\frac{1}{2}\left|\frac{F''_\alpha(z)}{F'_\alpha(z)}\right|^2 \tag{2.26}$$

$$\leqq \frac{60\sqrt{3}\alpha}{(1-|z|^2)^2} + \frac{1}{2}\left(\frac{15\sqrt{3}\alpha}{1-|z|^2}\right)^2 = \frac{15(45\alpha+8\sqrt{3})\alpha}{2(1-|z|^2)^2} \qquad (z \in \mathbb{U}).$$

Applying the Nehari's condition (1.7) for $F_\alpha(z)$, we need that

$$\frac{15(45\alpha+8\sqrt{3})\alpha}{2} \leqq 2, \tag{2.27}$$

that is, that

$$0 < \alpha \leqq \alpha_0 = \frac{2\sqrt{5}-4}{15\sqrt{3}} = 0.0181725\ldots. \tag{2.28}$$

This completes the proof of the theorem.

Next, we recall here a result by Chichra and Singh [2] that if

$$z + z^2 \log\frac{g(z)}{z} \in \mathcal{S}^*, \tag{2.29}$$

then there exist some t $(0 \leqq t \leqq 1)$ and α $(0 \leqq \alpha \leqq 1/2)$ such that

$$tz + (1-t)\int_0^z \left(\frac{tg'(t)}{g(t)}\right)^\alpha dt \in \mathcal{S}^*. \tag{2.30}$$

Letting

$$\frac{g(z)}{z} = \frac{zf'(z)}{f(z)} \tag{2.31}$$

for $f(z) \in \mathcal{A}$, Theorem 2.2 becomes

Theorem 2.3 *Assume that* $g(z) \in \mathcal{A}$ *satisfies*

$$z\exp\left(\int_0^z \frac{\frac{g(t)}{t}-1}{t}dt\right) \in \mathcal{S}, \tag{2.32}$$

the function $(g(z)/z)^\alpha$ *is analytic in* $\mathbb{U}$ *with* $0 < \alpha < 1$ *and*

$$\left.\left(\frac{g(z)}{z}\right)^\alpha\right|_{z=0} = 1. \tag{2.33}$$

If $0 < \alpha \leqq \alpha_0 = (2\sqrt{5}-4)/15\sqrt{3} = 0.0181725\cdots$ *, then the integration* $\int_0^z (g(t)/t)^\alpha dt$ *is univalent in* $\mathbb{U}$.

By means of the result due to Krzyż [7], we know that $g(z) \in \mathcal{S}$ is not implies that $\int_0^z (g(t)/t)dt \in \mathcal{S}$. The counterexample for the above is given by

$$g(z) = \frac{z}{(1-iz)^{1-i}}. \tag{2.34}$$

On the other hand, Merkes and Wright [8] showed that if $g(z) \in \mathcal{S}^*$, then

$$\int_0^z \left(\frac{g(t)}{t}\right)^\alpha dt \in \mathcal{C} \tag{2.35}$$

for $-1/2 \leqq \alpha \leqq 3/2$. Theorem 2.3 says that if

$$z\exp\left(\int_0^z \frac{\frac{g(t)}{t}-1}{t}dt\right) \in \mathcal{S}, \tag{2.36}$$

then

$$\int_0^z \left(\frac{g(t)}{t}\right)^\alpha dt \in \mathcal{S} \tag{2.37}$$

for $0 < \alpha \leqq \alpha_0 = (2\sqrt{5}-4)/15\sqrt{3}$.

Corollary 2.2 *If $g(z) \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(\frac{g(z)}{z}\right) > 0 \qquad (z \in \mathbb{U}), \tag{2.38}$$

then

$$\int_0^z \left(\frac{g(t)}{t}\right)^\alpha dt \tag{2.39}$$

is univalent in $\mathbb{U}$, where $0 < \alpha \leqq \alpha_0 = (2\sqrt{5}-4)/15\sqrt{3}$.

3. An application of Schwarzian derivative

Next, we would like to consider an application of Schwarzian derivative concerning with the subordinations. Let $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$. Then the function $f(z)$ is said to subordinate to $g(z)$ if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for $z \in \mathbb{U}$. We write that

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) \tag{3.1}$$

if $f(z)$ subordinates to $g(z)$ in $\mathbb{U}$. Also, if $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see Miller and Mocanu [9]).

Now, we derive

Theorem 3.1 *Let $f(z) \in \mathcal{A}$ satisfy*

$$|z^2\{f; z\}| < \alpha(1-\beta) \qquad (z \in \mathbb{U}), \tag{3.2}$$

where $0 < \alpha < 1$ and

$$\left| \frac{zh''(z)}{h'(z)} - \frac{2zh'(z)}{h(z)+1} \right| \leqq \beta \qquad (z \in \mathbb{U}) \tag{3.3}$$

with

$$h(z) = (f'(z))^{1/\alpha} \neq \pm 1. \tag{3.4}$$

Then we have that

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (z \in \mathbb{U}) \tag{3.5}$$

or

$$|\arg f'(z)| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U}). \tag{3.6}$$

Therefore, $f(z)$ is univalent in $\mathbb{U}$.

Proof For $h(z) = (f'(z))^{1/\alpha}$ $(0 < \alpha < 1)$, we define the function $w(z)$ by

$$w(z) = \frac{h(z)-1}{h(z)+1} = \frac{c_n}{2}z + \cdots \tag{3.7}$$

with $w(0) = 0$. This implies that

$$f'(z) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}. \tag{3.8}$$

It follows from (3.8) that

$$f''(z) = \frac{2\alpha w'(z)}{1-w(z)^2}\left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha} = \frac{2\alpha w'(z)}{1-w(z)^2} f'(z), \tag{3.9}$$

that is, that

$$\frac{f''(z)}{f'(z)} = \frac{2\alpha w'(z)}{1 - w(z)^2}. \tag{3.10}$$

Thus, we obtain that

$$\left(\frac{f''(z)}{f'(z)}\right)^2 = \left(\frac{zf''(z)}{f'(z)}\right)^2 \frac{1}{z^2} = \left(\frac{2\alpha z w'(z)}{1 - w(z)^2}\right)^2 \frac{1}{z^2}. \tag{3.11}$$

We suppose that there exists a point $z_0 \in \mathbb{U}$ such that $|w(z)| < 1$ $(|z| < |z_0| < 1)$ and $|w(z_0)| = 1$. Then Lemma 1.1 gives us that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geqq 1. \tag{3.12}$$

Further, by the result due to Miller and Mocanu [9], we have that

$$\text{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) \geqq 0. \tag{3.13}$$

Therefore, we have that

$$\left(\frac{f''(z_0)}{f'(z_0)}\right)^2 = \left(\frac{2\alpha k w(z_0)}{1 - w(z_0)^2}\right)^2 \frac{1}{z_0^2} \tag{3.14}$$

$$= \left(\frac{i\alpha k}{\sin\theta}\right)^2 \frac{1}{z_0^2} = -\left(\frac{\alpha k}{\sin\theta}\right)^2 \frac{1}{z_0^2},$$

where $w(z_0) = e^{i\theta}$ $(0 \leqq \theta < 2\pi)$.
Also, we see that

$$\left.\left(\frac{f''(z)}{f'(z)}\right)'\right|_{z=z_0} = \left.\left(\frac{2\alpha w'(z)}{1 - w(z)^2}\right)'\right|_{z=z_0} \tag{3.15}$$

$$= 2\alpha\left(\frac{w''(z_0)}{1 - w(z_0)^2}\right) + \left.\frac{4\alpha w(z)(w'(z))^2}{(1 - w(z)^2)^2}\right|_{z=z_0}$$

$$= \frac{ik\alpha}{\sin\theta}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right)\frac{1}{z_0} + \left(\frac{ik}{\sin\theta}\right)^2 \frac{\alpha w(z_0)}{z_0^2}$$

$$= \frac{k\alpha}{\sin\theta}\left\{i\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) - \frac{k w(z_0)}{\sin\theta}\right\}\frac{1}{z_0^2}.$$

Consequetly, we obtain that

$$z_0^2\{f;z\} = \frac{k\alpha}{\sin\theta}\left\{i\left(\frac{z_0w''(z_0)}{w'(z_0)}\right) - \frac{kw(z_0)}{\text{sim}\,\theta} + \frac{\alpha k}{2\sin\theta}\right\} \tag{3.16}$$

$$= \frac{k\alpha}{2\sin\theta}\left\{2i\left(\frac{z_0w''(z_0)}{w'(z_0)}\right) + \frac{k}{\sin\theta}(\alpha - 2\cos\theta - 2i\sin\theta)\right\}$$

and so

$$|z_0^2\{f;z_0\}| \geqq \frac{\alpha}{2}\left|\left|\frac{k}{\sin\theta}\right||\alpha - 2\cos\theta - 2i\sin\theta| - 2\left|\frac{z_0w''(z_0)}{w'(z_0)}\right|\right| \tag{3.17}$$

$$\geqq \frac{\alpha}{2}\left|k\sqrt{\frac{\alpha^2 - 4\alpha\cos\theta + 4}{1-\cos^2\theta}} - 2\left|\frac{z_0w''(z_0)}{w'(z_0)}\right|\right|.$$

If we define a function $p(x)$ by

$$p(x) = \frac{\alpha^2 - 4\alpha x + 4}{1-x^2} \qquad (x = \cos\theta), \tag{3.18}$$

then

$$p'(x) = \frac{-2(2x-\alpha)(\alpha x - 2)}{(1-x^2)^2} \tag{3.19}$$

gives that $p(x)$ takes its minimum value at $x = \alpha/2 < 1/2$, because $0 < \alpha < 1$ and $-1 \leqq x \leqq 1$. This shows us that $p(x) \geqq 4$ and so

$$|z_0^2\{f;z_0\}| \geqq \alpha\left|1 - \left|\frac{z_0w''(z_0)}{w'(z_0)}\right|\right| \tag{3.20}$$

$$= \alpha\left|1 - \left|\frac{z_0h''(z_0)}{h'(z_0)} - \frac{2z_0h'(z_0)}{h(z_0)+1}\right|\right| \geqq \alpha(1-\beta).$$

This contradicts the condition (3.2) of the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that there exists $w(z)$ such that

$$f'(z) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha} \qquad (z \in \mathbb{U}) \tag{3.21}$$

with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$. Consequently, we prove the subordination (3.5).

Further, since

$$\left|\arg\left(\frac{1+z}{1-z}\right)\right| < \frac{\pi}{2} \qquad (z \in \mathbb{U}), \tag{3.22}$$

we obtain (3.6) for $\arg f'(z)$.

Making $\alpha = 1/2$ in Theorem 3.1, we derive
Corollary 3.1 *Let $f(z) \in \mathcal{A}$ satisfy*

$$|z^2\{f;z\}| < \frac{1-\beta}{2} \qquad (z \in \mathbb{U}) \tag{3.23}$$

with

$$\left|\frac{zh''(z)}{h'(z)} - \frac{2zh'(z)}{h(z)+1}\right| \leqq \beta \qquad (z \in \mathbb{U}) \tag{3.24}$$

and $h(z) = \sqrt{f'(z)} \neq \pm 1$. Then we have

$$f'(z) \prec \sqrt{\frac{1+z}{1-z}} \qquad (z \in \mathbb{U}) \tag{3.25}$$

or

$$|\arg f'(z)| < \frac{\pi}{4} \qquad (z \in \mathbb{U}). \tag{3.26}$$

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