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COEFFICIENT ESTIMATES ASSOCIATED WITH A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS

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Abstract. In our present investigation, we aim at introducing a new subclass of the function class Σ of bi-univalent functions defined in the open unit disc $\mathbb{U}$. Furthermore, we establish bounds for the coefficients for this subclass and several related classes are also considered and connections to earlier known results are made.

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1. Introduction and Definitions

Let $\mathcal{A}$ indicate the class of functions f which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of A consisting of the form (1) which are univalent in U. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z,\ (z \in \mathbb{U})$ and $f(f^{-1}(w)) = w,\ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w\ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disc $\mathbb{U}$. For a brief history and interesting examples of functions in the class Σ, see the pioneering work on this area by Srivastava *et al.* [12], which has apparently revived the study of bi-univalent functions in recent years.

The research into Σ was started by Lewin [10]. It focused on problems connected with coefficients and obtained the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [6] conjectured that $|a_2| \leqq \sqrt{2}$ for

$f \in \Sigma$. Later on, Netanyahu [11] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [5] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^{\star}(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β $(0 \leqq \beta < 1)$ in $\mathbb{U}$, respectively (see [11]). The classes $\mathcal{S}^{\star}_{\Sigma}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$ of bi-starlike functions of order β in $\mathbb{U}$ and bi-convex functions of order β in $\mathbb{U}$, corresponding to the function classes $\mathcal{S}^{\star}(\beta)$ and $\mathcal{K}(\beta)$, were also introduced analogously. For each of the function classes $\mathcal{S}^{\star}_{\Sigma}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$, they found non-sharp estimates for the initial coefficients. Recently, motivated substantially by the aforementioned work on this area Srivastava *et al.* [12], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see, for example, [2], [7], [13]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geqq 4$. In the literature, there are only a few works determining the general coefficient bounds for $|a_n|$ for the analytic bi-univalent functions (see, for example, [4], [8], [9]). The coefficient estimate problem for each of the coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\};\ \mathbb{N} = \{1, 2, 3, \cdots\})$ is still an open problem.

In our present investigation, we aim at introducing a new subclass of the function class Σ of bi-univalent functions defined in the open unit disc $\mathbb{U}$. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass of the function class Σ employing the techniques used earlier by Altınkaya and Yalçın [2] (see also [1]).

We note the following definition required for obtaining our results.

Definition 1. *Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that*

$$\min \{\Re(h(z)), \Re(p(z))\} > 0$$

and

$$h(0) = p(0) = 1.$$

2. Coefficient Estimates for the Function Class $S^{h,p}_{\Sigma}(\alpha)$

We begin this section by introducing the function class $S^{h,p}_{\Sigma}(\alpha)$ and finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this class.

Definition 2. *A function $f \in \Sigma$ is said to be in the class $S^{h,p}_{\Sigma}(\alpha)$, $0 < \alpha \leq 1$, if the following conditions are satisfied:*

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right) \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \tag{2}$$

and

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right)\in p\left(\mathbb{U}\right)\quad\left(w\in\mathbb{U}\right) \tag{3}$$

where $g\left(w\right)=f^{-1}\left(w\right)$.

Remark 1. *There are many choices of* h,p *and* α *which would provide interesting subclasses of class* $S_{\Sigma}^{h,p}\left(\alpha\right)$. *For example,*

1. For $0<\alpha\leq 1$ and $h\left(z\right)=p\left(z\right)=\left(\frac{1+z}{1-z}\right)^{\lambda}$ where $(0<\lambda\leq 1)$ it can be directly verified that the functions $h\left(z\right)$ and $p\left(z\right)$ satisfy the hypotheses of Definition 1. Now if $f\in S_{\Sigma}(\lambda,\alpha)$ then

$$f\in\Sigma,\quad\left|\arg\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right)\right|<\frac{\lambda\pi}{2}\quad\left(0<\lambda\leq 1,\quad z\in U\right)$$

   and

$$\left|\arg\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right)\right|<\frac{\lambda\pi}{2}\quad\left(0<\lambda\leq 1,\quad w\in U\right).$$

   Therefore in this case, the class $S_{\Sigma}^{h,p}\left(\alpha\right)$ reduces to class $S_{\Sigma}(\lambda,\alpha)$ which is defined by Altınkaya and Yalçın [3].

2. For $0<\alpha\leq 1$ and $h\left(z\right)=p\left(z\right)=\frac{1+(1-2\beta)z}{1-z}$ where $(0\leq\beta<1)$ it can be directly verified that the functions $h\left(z\right)$ and $p\left(z\right)$ satisfy the hypotheses of Definition 1. Now if $f\in S_{\Sigma}(\lambda,\beta)$ then

$$f\in\Sigma,\quad\Re\left(\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right)\right)>\beta\quad\left(0\leq\beta<1,\ 0<\alpha\leq 1,\ z\in U\right)$$

   and

$$\Re\left(\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right)\right)>\beta\quad\left(0\leq\beta<1,\ 0<\alpha\leq 1,\ w\in U\right).$$

   Therefore in this case, the class $S_{\Sigma}^{h,p}\left(\alpha\right)$ reduces to class $S_{\Sigma}(\lambda,\beta)$ which is defined by Altınkaya and Yalçın [3].

3. For $\alpha=1$ and $h\left(z\right)=p\left(z\right)=\left(\frac{1+z}{1-z}\right)^{\lambda}$ where $(0<\lambda\leq 1)$ it can be directly verified that the functions $h\left(z\right)$ and $p\left(z\right)$ satisfy the hypotheses of Definition 1. Now if $f\in S_{\Sigma}(\alpha)$ then

$$f\in\Sigma,\quad\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\lambda\pi}{2}\quad\left(0<\lambda\leq 1,\quad z\in U\right)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\lambda\pi}{2} \quad (0 < \lambda \leq 1, \quad w \in U)\ .$$

Therefore in this case, the class $S_{\Sigma}^{h,p}(\alpha)$ reduces to class $S_{\Sigma}(\alpha)$ which is defined by Brannan and Taha [5] (see also [14]).

4. For $\alpha = 1$ and $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ where $(0 \leq \beta < 1)$ it can be directly verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 1. Now if $f \in S_{\Sigma}(\beta)$ then

$$f \in \Sigma, \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (0 \leq \beta < 1,\ 0 < \alpha \leq 1,\ z \in U)$$

and

$$\Re\left(\frac{wg'(w)}{g(w)}\right) > \beta \quad (0 \leq \beta < 1,\ 0 < \alpha \leq 1,\ w \in U)\ .$$

Therefore in this case, the class $S_{\Sigma}^{h,p}(\alpha)$ reduces to class $S_{\Sigma}(\beta)$ which is defined by Brannan and Taha [5] (see also [14]).

Theorem 1. *Let f given by (1) be in the class $S_{\Sigma}^{h,p}(\alpha)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2\left(|h'(0)|^2+|p'(0)|^2\right)\alpha^2}{(1+\alpha)^2}}, \sqrt{\frac{(|h''(0)|+|p''(0)|)\alpha^2}{2\alpha^2+\alpha+1}}\right\} \tag{4}$$

and

$$|a_3| \leq \min\left\{\begin{array}{l} \frac{2\left(|h'(0)|^2+|p'(0)|^2\right)\alpha}{(1+\alpha)^2} + \frac{(|h''(0)|+|p''(0)|)\alpha}{4(1+\alpha)}, \\ \\ \frac{(6\alpha^3+5\alpha^2+\alpha)|h''(0)|}{4(1+\alpha)(2\alpha^2+\alpha+1)} + \frac{(2\alpha^3+3\alpha^2-\alpha)|p''(0)|}{4(1+\alpha)(2\alpha^2+\alpha+1)} \end{array}\right. . \tag{5}$$

Proof. Let $f \in S_{\Sigma}^{h,p}(\alpha)$. It follows from (2) and (3) that

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right) = h(z) \tag{6}$$

and

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right) = p(w), \tag{7}$$

where $h(z)$ and $p(w)$ satisfy the conditions of Definition 1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots ,$$

respectively. Thus, upon comparing the corresponding coefficients in (6) and (7), we get

$$\frac{\alpha+1}{2\alpha} a_2 = h_1, \tag{8}$$

$$\frac{\alpha+1}{2\alpha}\left(2a_3 - a_2^2\right) + \frac{1-\alpha}{4\alpha^2} a_2^2 = h_2, \tag{9}$$

and

$$-\frac{\alpha+1}{2\alpha} a_2 = p_1, \tag{10}$$

$$\frac{\alpha+1}{2\alpha}\left(3a_2^2 - 2a_3\right) + \frac{1-\alpha}{4\alpha^2} a_2^2 = p_2. \tag{11}$$

From (8) and (10) we obtain

$$h_1 = -p_1,$$

and

$$\frac{(\alpha+1)^2}{2\alpha^2} a_2^2 = h_1^2 + p_1^2. \tag{12}$$

Now, by adding (9) to (11), we find that

$$\frac{2\alpha^2+\alpha+1}{2\alpha^2} a_2^2 = h_2 + p_2, \tag{13}$$

which gives us the desired estimate on $|a_2|$ as asserted in (4).

Next, in order to find the bound on $|a_3|$, by subtracting (11) from (9), we obtain

$$\frac{2(\alpha+1)}{\alpha}\left(a_3 - a_2^2\right) = h_2 - p_2. \tag{14}$$

Therefore, in view of (12) and (13), we have

$$a_3 = \frac{2\left(h_1^2+p_1^2\right)\alpha}{(\alpha+1)^2} + \frac{(h_2-p_2)\,\alpha}{2(\alpha+1)}$$

and

$$a_3 = \frac{2(h_2+p_2)\,\alpha^2}{2\alpha^2+\alpha+1} + \frac{(h_2-p_2)\,\alpha}{2(\alpha+1)}.$$

which completes the proof of Theorem 1.

3. Corollaries and Consequences

Corollary 2. *If we let*

$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\lambda}=1+2\lambda z+2\lambda^2z^2+...\qquad(0<\lambda\leq 1),$$

then inequalities (4) and (5) become

$$|a_2|\leq\min\left\{\tfrac{4\alpha\lambda}{1+\alpha},2\lambda\alpha\sqrt{\tfrac{2}{2\alpha^2+\alpha+1}}\right\}=2\lambda\alpha\sqrt{\tfrac{2}{2\alpha^2+\alpha+1}}$$

and

$$|a_3|\leq\min\left\{\frac{16\lambda^2\alpha^2}{(1+\alpha)^2}+\frac{2\lambda^2\alpha}{1+\alpha},\frac{8\alpha^2\lambda^2}{2\alpha^2+\alpha+1}\right\}.$$

Corollary 3. *If we let*

$$h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots\qquad(0\leq\beta<1),$$

then inequalities (4) and (5) become

$$|a_2|\leq\min\left\{\frac{4\alpha(1-\beta)}{1+\alpha},2\alpha\sqrt{\frac{2(1-\beta)}{2\alpha^2+\alpha+1}}\right\}.$$

and

$$|a_3|\leq\min\left\{\frac{16(1-\beta)^2\alpha^2}{(1+\alpha)^2}+\frac{2(1-\beta)\alpha}{1+\alpha},\frac{8\alpha^2(1-\beta)}{2\alpha^2+\alpha+1}\right\}.$$

Taking $\alpha=1$ in Theorem 1, we get

Corollary 4. *If $f\in S_{\Sigma}^{h,p}$ then*

$$|a_2|\leq\min\left\{\sqrt{\tfrac{|h'(0)|^2+|p'(0)|^2}{2}},\sqrt{\tfrac{|h''(0)|+|p''(0)|}{4}}\right\}\tag{15}$$

and

$$|a_3|\leq\min\left\{\tfrac{|h'(0)|^2+|p'(0)|^2}{2}+\tfrac{|h''(0)|+|p''(0)|}{8},\tfrac{3|h''(0)|}{8}+\tfrac{|p''(0)|}{8}\right\}\tag{16}$$

Corollary 5. *If we let*

$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\lambda}=1+2\lambda z+2\lambda^2z^2+...\qquad(0<\lambda\leq 1),$$

then inequalities (15) and (16) become

$$|a_2| \leq \min\left\{2\lambda, \sqrt{2}\lambda\right\} = \sqrt{2}\lambda$$

and

$$|a_3| \leq \min\left\{5\lambda^2, 2\lambda^2\right\} = 2\lambda^2.$$

Remark 2. *Corollary 8 provides an improvement estimates obtained by Altınkaya and Yalçın [3].*

Corollary 6. *(see [3]) If we let*

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \qquad (0 \leq \beta < 1),$$

then inequalities (15) and (16) become

$$|a_2| \leq \min\left\{2(1-\beta), \sqrt{2(1-\beta)}\right\} = \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq \min\left\{4(1-\beta)^2 + (1-\beta), 2(1-\beta)\right\} = 4(1-\beta)^2 + (1-\beta).$$

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