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CLASSES OF AN UNIVALENT INTEGRAL OPERATOR

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Abstract. In this paper we consider an integral operator for analytic functions in the open unit disk and we obtain sufficient conditions for univalence of this integral operator.

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1. Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{S}$ the subclass of A consisting of functions $f \in A$, which are univalent in U.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. We note

$$\mathcal{A}_n = \left\{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots ,\ n \in \mathbb{N} - \{0\} \right\}$$

with $\mathcal{A}_1 = A$.

In this paper we consider the integral operator

$$G_h : \mathcal{A}_n \to \mathcal{H}(U), \quad \mathcal{A}_n \subset \mathcal{H}(U),$$
$$G_h(f)(z) = \left[\beta \int_0^z f^{\beta}(t) h^{-1}(t) h'(t) dt \right]^{\frac{1}{\beta}}, \tag{1}$$

$\beta \in \mathbb{C}$, $\beta \neq 0$, $f,\ h \in \mathcal{A}_n$.

For $n=1$, $\beta\in\mathbb{C}$, $\beta\neq 0$, $f,h\in A$, $h(z)=z$, from (1) we obtain the integral operator Pascu-Pescar [9],

$$I(z)=\left[\beta\int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\beta}dt\right]^{\frac{1}{\beta}},\quad z\in U. \tag{2}$$

For $n=1$, $\beta=1$, $f,h\in A$, $h(z)=z$, from (1) we obtain the integral operator Alexander [1],

$$T(z)=\int_0^z\frac{f(t)}{t}dt,\quad z\in U. \tag{3}$$

Properties of certain integral operators were studied by different authors in the following papers [2, 16, 17, 18, 19, 20].

In this paper we obtain sufficient conditions for univalence of integral operator G_h.

2. Preliminaries

We need the following lemmas.

Lemma 1 (Pascu, [8])**.** *Let α be a complex number, $Re\,\alpha>0$ and $f\in A$. If*

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq 1, \tag{4}$$

for all $z\in U$, then the function

$$F_\alpha(z)=\left[\alpha\int_0^z t^{\alpha-1}f'(t)dt\right]^{\frac{1}{\alpha}} \tag{5}$$

is regular and univalent in U.

Lemma 2 (General Schwarz Lemma, [4])**.** *Let f the function regular in the disk $U_R=\{z\in\mathbb{C}:|z|<R\}$ with $|f(z)|<M$, M fixed. If the function f has in $z=0$ one zero with multiply $\geq m$, then*

$$|f(z)|\leq\frac{M}{R^m}|z|^m,\ z\in U_R, \tag{6}$$

the equality (in the inequality (6) for $z\neq 0$) can hold only if

$$f(z)=e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

Lemma 3 (Mocanu and Şerb, [6]). *Let $M_0 = 1.5936\ldots$ be the positive solution of equation*

$$(2-M)e^M = 2.$$

If $f \in A$ and

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0,\ z \in U, \tag{7}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, z \in U. \tag{8}$$

The edge M_0 is sharp.

3. Main results

Theorem 4. *Let β be a complex number, $a = Re\beta > 0$, the functions $f, h \in \mathcal{A}_n$, $f(z) = z + a_{n+1}z^{n+1} + \cdots$, $h(z) = z + b_{n+1}z^{n+1} + \cdots$, M, L, K positive real numbers.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \quad z \in U, \tag{9}$$

$$\left|\frac{zf'(z)}{f(z)} - \frac{zh'(z)}{h(z)}\right| < L, \quad z \in U, \tag{10}$$

$$\left|\frac{zh''(z)}{h'(z)}\right| < K, \quad z \in U \tag{11}$$

and

$$|\beta - 1|M + L + K \le \frac{(2a+n)^{\frac{n+2a}{2a}}}{2n^{\frac{n}{2a}}}, \quad n \in \mathbb{N} - \{0\}, \tag{12}$$

then the function $G_h(f)(z)$ belongs to the class $\mathcal{S}$.

Proof. From (1) we have

$$G_h(f)(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\beta-1} \frac{f(t)}{h(t)} h'(t)dt\right]^{\frac{1}{\beta}}, \tag{13}$$

for all $z \in U$.

We consider the function

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\beta-1} \frac{f(t)}{h(t)} h'(t)dt, \quad z \in U, \tag{14}$$

which is regular in U and $g(0) = g'(0) - 1 = 0$.

We have

$$\frac{zg''(z)}{g'(z)} = (\beta - 1)\left(\frac{zf'(z)}{f(z)} - 1\right) + \frac{zf'(z)}{f(z)} - \frac{zh'(z)}{h(z)} + \frac{zh''(z)}{h'(z)}, \tag{15}$$

for all $z \in U$.

Using (15) we obtain

$$\begin{aligned} \frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \quad &\leq \quad \frac{1-|z|^{2a}}{a}\left[|\beta - 1|\left|\frac{zf'(z)}{f(z)} - 1\right| + \left|\frac{zf'(z)}{f(z)} - \frac{zh'(z)}{h(z)}\right| + \right. \\ &+ \quad \left.\left|\frac{zh''(z)}{h'(z)}\right|\right], \end{aligned} \tag{16}$$

for all $z \in U$.

Applying Lemma 2, from (9), (10), (11) we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M|z|^n, \quad z \in U, \tag{17}$$

$$\left|\frac{zf'(z)}{f(z)} - \frac{zh'(z)}{h(z)}\right| \leq L|z|^n, \quad z \in U, \tag{18}$$

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq K|z|^n, \quad z \in U. \tag{19}$$

From (17), (18), (19) and (16) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}|z|^n\left[|\beta - 1|M + L + K\right], \quad z \in U. \tag{20}$$

We consider the function $J : [0,1] \to \mathbb{R}$, $J(x) = \frac{(1-x^{2a})x^n}{a}$ where $x = |z|$, $x \in [0,1]$.

We have

$$\max_{x\in[0,1]} J(x) = \frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{n+2a}{2a}}},\ n \in \mathbb{N} - \{0\}. \tag{21}$$

By (12), (21) and (20) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \le 1, \tag{22}$$

for all $z \in U$.

Now, from (22) and Lemma 1, it results that the function $G_h(f)(z)$ belongs to the class S,

$$G_h(f)(z) = z + c_2z^2 + c_3z^3 + \cdots \tag{23}$$

We note by $\mathcal{K}_1$ the class of univalent integral operator $G_h(f)$, obtained by the conditions of Theorem 4.

Corollary 5. *Let β be a complex number, $a = Re\beta > 0$, the function $f \in A$, $f(z) = z + a_2z + \cdots$, M a positive real number.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \quad z \in U, \tag{24}$$

and

$$|\beta| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2M}, \tag{25}$$

then the function $I(z)$ defined by (2) is in the class $\mathcal{S}$.

Proof. For $n = 1$, $h(z) = z$ and using (15) from Theorem 4, we obtain Corollary 5.

Corollary 6. *Let the function $f \in A$, $f(z) = z + a_2z^2 + \cdots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{3\sqrt{3}}{2}, \quad z \in U, \tag{26}$$

then the function $T(z)$ defined by (3) is in the class $\mathcal{S}$.

Proof. For $n=1$, $\beta=1$, $h(z)=z$, from Corollary 5, we obtain Corollary 6.

Theorem 7. *Let β be a complex number, $a=Re\beta>0$, the functions $f,h\in A$, $f(z)=z+a_2z^2+\dots$, $h(z)=z+b_2z^2+\dots$, $M_0=1.5936\dots$, the positive solution of equation $(2-M)e^M=2$.*

If

$$\left|\frac{f''(z)}{f'(z)}\right|\le M_0,\ z\in U, \tag{27}$$

$$\left|\frac{h''(z)}{h'(z)}\right|\le M_0, z\in U, \tag{28}$$

and

$$\frac{|\beta-1|+2}{a}+\frac{2M_0}{(2a+1)^{\frac{2a+1}{2a}}}\le 1, \tag{29}$$

then the function $G_h(f)(z)$ belongs to the class S.

Proof. We consider the function $G_h(f)(z)$ defined by (13) and the function $g(z)$ defined by (14).

From (15) we obtain:

$$\left|\frac{zg''(z)}{g'(z)}\right|\le|\beta-1|\left|\frac{zf'(z)}{f(z)}-1\right|+\left|\frac{zf'(z)}{f(z)}-1\right|+\left|\frac{zh'(z)}{h(z)}-1\right|+|z|\left|\frac{h''(z)}{h'(z)}\right| \tag{30}$$

for all $z\in U$.

Using (27), (28) and Lemma 3, from (30) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{1-|z|^{2a}}{a}\left[|\beta-1|+2\right]+\frac{1-|z|^{2a}}{a}|z|M_0,\ z\in U \tag{31}$$

We have

$$\max_{|z|\le 1}\frac{1-|z|^{2a}}{a}|z|=\frac{2}{(2a+1)^{\frac{2a+1}{2a}}}. \tag{32}$$

From (31) and (32) we obtain:

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{|\beta-1|+2}{a}+\frac{2M_0}{(2a+1)^{\frac{2a+1}{2a}}}, \tag{33}$$

for all $z\in U$.

Using (29), from (33) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le 1,\ z\in U. \tag{34}$$

Now, from (34) and Lemma 1 we obtain that the function $G_h(f)(z)$ belongs to the class S,

$$G_h(f)(z) = z + c_2 z^2 + \dots$$

We note by $\mathcal{K}_2$ the class of univalent integral operator $G_h(f)$, obtained by the conditions of Theorem 7.

Corollary 8. *Let β be a real number, $\beta > 1$, the function $f \in A$, $f(z) = z + a_2 z^2 + \dots$, $M_0 = 1.5936\dots$ the positive solution of the equation $(2-M)e^M = 2$.*

If

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0,\ z \in U, \tag{35}$$

then the function $I(z)$ defined by (2) is in the class S.

Proof. Using (30) and Theorem 7 for $h(z) = z$, we obtain Corollary 8.

Corollary 9. *Let the function $f \in A$, $f(z) = z + a_2 z^2 + \dots$, $M_0 = 1.5936\dots$ the positive solution of equation $(2-M)e^M = 2$.*

If

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0,\ z \in U \tag{36}$$

then the function $T(z)$ defined by (3) is in the class S.

Proof. For $\beta = 1$, $h(z) = z$, using (30) and Theorem 7 we obtain Corollary 9.

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