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CONVEXITY PROPERTIES FOR A NEW INTEGRAL OPERATOR

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Abstract. For some classes of analytic functions f, g and h in the open unit disk U, we define a new integral operator $H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{h_i(t)} g_i'(t)\right)^{\alpha_i} dt$ and we study convexity properties of this general integral operator.

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1. Introduction

Let $U = \{z : |z| < 1\}$ be the unit disk and $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in U \tag{1}$$

which are analytic in U and satisfy the conditions $f(0) = f'(0) - 1 = 0$.

We denote by S the subclass of $\mathcal{A}$ consisting of univalent functions on U.

A function $f \in \mathcal{A}$ is a convex function of complex order b, ($b \in \mathbb{C} \setminus \{0\}$) and type λ ($0 \leq \lambda < 1$), if it verifies one of these condtions

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > \lambda, \ \left|\frac{1}{b}\frac{zf''(z)}{f'(z)}\right| < 1 - \lambda, \ z \in U. \tag{2}$$

We denote by $C_\lambda^*(b)$ the class of these functions.

A function $f \in \mathcal{A}$ is a starlike function of order β, $0 \leq \beta < 1$ if it satisfies one of the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \ \left|\frac{zf'(z)}{f(z)}\right| < \beta, \ z \in U. \tag{3}$$

We denote this class by $S^*(\beta)$.

We done by $K(\beta)$ the class of convex functions of oder β, $0 \le \beta < 1$ that satisfies the inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \beta,\ z \in U. \tag{4}$$

A function $f \in \mathcal{A}$ belongs to class $R(\beta)$, $0 \le \beta < 1$, if

$$\operatorname{Re}\left(f'(z)\right) > \beta,\ z \in U. \tag{5}$$

A function $f \in \mathcal{A}$ is a starlike function of the complex order b, $b \in \mathbb{C} \setminus \{0\}$ and type λ, $(0 \le \lambda < 1)$, if and only if

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\} > \lambda \ or \left|\frac{1}{b}\frac{zf'(z)}{f(z)}\right| \le 1-\lambda,\ z \in U. \tag{6}$$

We denote by $S^*_\lambda(b)$ the class of these functions.

F. Ronning introduced in [6] the class of univalent functions $\mathcal{SP}(\alpha,\beta)$, $\alpha > 0$, $\beta \in [0,1)$. So, we denote by $\mathcal{SP}(\alpha,\beta)$ the class of all functions $f \in S$ which satisfies the inequality:

$$\left|\frac{zf'(z)}{f(z)}-(\alpha+\beta)\right| \le \operatorname{Re}\frac{zf'(z)}{f(z)}+\alpha-\beta, \quad z \in U. \tag{7}$$

Silverman defined in [7] the class G_b. So, a function $f \in \mathcal{A}$ is in the class G_b, $0 < b \le 1$ if and only if

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| < b\left|\frac{zf'(z)}{f(z)}\right|, \quad z \in U. \tag{8}$$

Uralegaddi in [8], Owa and Srivastava in [3] defined the class $\mathcal{N}(\beta)$. So, a function $f \in \mathcal{A}$ is in the class $\mathcal{N}(\beta)$ if it verifies the inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) < \beta, \quad z \in U,\ \beta > 1. \tag{9}$$

2. Main results

In this paper, we study new properties for a general integral operator defined by

$$H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{h_i(t)}g_i'(t)\right)^{\alpha_i} dt \tag{10}$$

Remark 1. *If we consider $h_i(z) = z$, for $i = 1, 2, ..., n$, in relation* (10)*, we obtain the integral operator:*

$$G_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} g_i^{'}(t) \right)^{\alpha_i} dt \tag{11}$$

introduced and studied by Adriana Oprea and Daniel Breaz in [2].

Remark 2. *If $f_i(z) = z$, $h_i(z) = z$, for $i = 1, 2, ...n$ from* (10)*, we obtain the integral operator:*

$$F_{\alpha_1,\alpha_2,...,\alpha_n}(z) = \int_0^z (g_1(t))^{\alpha_1}.(g_2(t))^{\alpha_2}...(g_n(t))^{\alpha_n} dt, \tag{12}$$

introduced and studied by D.Breaz et all in [1].

Remark 3. *For $n = 1$, $f(z) = z$, $h(z) = z$, $g_1 = g$, $\alpha_1 = \gamma_1 = \gamma$ from* (10)*, we obtain the integral operator:*

$$G(z) = \int_0^z (g^{'}(t))^{\gamma} dt \tag{13}$$

studied in [4] *and* [5].

Theorem 1. *Let $f_i, g_i, h_i \in \mathcal{A}$, where $g_i \in G_{b_i}, 0 < b_i \leq 1$, for $i = 1, 2, ..., n$. For any $M_i, N_i \geq 1$, which verify*

$$\left| \frac{z f_i^{'}(z)}{f_i(z)} \right| \leq M_i, \left| \frac{z h_i^{'}(z)}{h_i(z)} \right| \leq N_i \text{ and } \left| \frac{z g_i^{'}(z)}{g_i(z)} - 1 \right| < 1, \tag{14}$$

for all $z \in U$, there are α_i real numbers, with $\alpha_i > 0$, $i = 1, 2, ...n$, so that

$$\lambda = 1 - \sum_{i=1}^n \alpha_i (M_i + N_i + 2b_i + 1) > 0. \tag{15}$$

In these conditions, the integral operator

$$H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{h_i(t)} g_i'(t) \right)^{\alpha_i} dt$$

is in the class $K(\lambda)$.

Proof. We calculate the first and second order derivatives for $H_{n,\alpha}$ and we obtain:

$$H'_{n,\alpha}(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{h_i(z)} g'_i(z) \right)^{\alpha_i}$$

and

$$H''_{n,\alpha}(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{h_i(z)} g'_i(z) \right)^{\alpha_i - 1} \left[\frac{f'_i(z)h_i(z) - f_i(z)h'_i(z)}{h_i^2(z)} g'_i(z) + \frac{f_i(z)}{h_i(z)} g''_i(z) \right] \times \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(\frac{f_k(z)}{h_k(z)} g'_k(z) \right)^{\alpha_k}.$$

Further, we have:

$$\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^{n} \alpha_i \left[\frac{zf'_i(z)}{f_i(z)} - \frac{zh'_i(z)}{h_i(z)} \right] + \sum_{i=1}^{n} \alpha_i \frac{zg''_i(z)}{g'_i(z)}$$
$$= \sum_{i=1}^{n} \alpha_i \left[\frac{zf'_i(z)}{f_i(z)} - \frac{zh'_i(z)}{h_i(z)} \right] + \sum_{i=1}^{n} \alpha_i \left(\frac{zg''_i(z)}{g'_i(z)} - \frac{zg'_i(z)}{g_i(z)} + 1 \right) + \sum_{i=1}^{n} \alpha_i \left(\frac{zg'_i(z)}{g_i(z)} - 1 \right). \tag{16}$$

So, we have:

$$\left| \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \right| \leq \sum_{i=1}^{n} \alpha_i \left| \frac{zf'_i(z)}{f_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zh'_i(z)}{h_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zg''_i(z)}{g_i(z)} - \frac{zg'_i(z)}{g_i(z)} + 1 \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right|. \tag{17}$$

Since functions $g_i \in G_{b_i}$, $0 < b_i \leq 1$, for $i = 1, 2, ..., n$, using inequality (8), we get:

$$\left| \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \right| \leq \sum_{i=1}^{n} \alpha_i (M_i + N_i) + \sum_{i=1}^{n} \alpha_i b_i \left| \frac{zg'_i(z)}{g_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right|$$
$$\leq \sum_{i=1}^{n} \alpha_i (M_i + N_i) + \sum_{i=1}^{n} \alpha_i b_i \left(\left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| + 1 \right) + \sum_{i=1}^{n} \alpha_i \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right|$$

$$\leq \sum_{i=1}^{n} \alpha_i (M_i + N_i + 2b_i + 1) = 1 - \lambda. \tag{18}$$

So, the integral operator $H_{n,\alpha}$ is in the class $K(\lambda)$.

If we consider $n = 1$ in Theorem 1, we get the following corollary:

Corollary 2. *Let $f, g, h \in \mathcal{A}$, where $g \in G_b, 0 < b \leq 1$. For any $M, N \geq 1$, which verify the conditions*

$$\left| \frac{zf'(z)}{f(z)} \right| \leq M, \left| \frac{zh'(z)}{h(z)} \right| \leq N, \left| \frac{zg'(z)}{g(z)} - 1 \right| < 1, \tag{19}$$

for all $z \in U$, with $\alpha > 0$ is a real number, so that $\lambda = 1 - \alpha(M + N + 2b + 1) > 0$. In these conditions, the integral operator $H_{1,\alpha}(z) = \int_0^z \left(\frac{f(t)}{h(t)} g'(t) \right)^{\alpha} dt$ is in the class $K(\lambda)$.

Theorem 3. *Let $f_i \in S^*(\beta_i)$ and $h_i \in S^*(\delta_i)$ with $0 \leq \beta_i, \delta_i < 1$ and $g_i \in \mathcal{K}(\lambda_i), 0 \leq \lambda_i < 1$, for $i = 1, 2, ..., n$. If α_i are real numbers with $\alpha_i > 0$, for $i = 1, 2, ...n$ so that*

$$\sum_{i=1}^{n} \alpha_i (\beta_i + \delta_i - \lambda_i + 3) < 1, \tag{20}$$

then the integral operator

$$H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{h_i(t)} g_i'(t) \right)^{\alpha_i} dt$$

is convex of order $\rho = 1 + \sum_{i=1}^{n} \alpha_i (\lambda_i - \beta_i - \delta_i - 3)$, for all $i = 1, 2, ..., n$.

Proof. After the same steps as in the proof of Theorem 1, we get

$$\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i \frac{zh_i'(z)}{h_i(z)} + \sum_{i=1}^{n} \alpha_i \frac{zg_i''(z)}{g_i'(z)}.$$

Further, we obtain

$$\left| \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \right| = \left| \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i \frac{h_i'(z)}{h_i(z)} + \sum_{i=1}^{n} \alpha_i \frac{zg_i''(z)}{g_i'(z)} \right|$$
$$\leq \sum_{i=1}^{n} \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zh_i'(z)}{h_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zg_i''(z)}{g_i'(z)} \right|.$$

$$\leq \sum_{i=1}^{n} \alpha_i \left(\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + 1 \right) + \sum_{i=1}^{n} \alpha_i \left(\left| \frac{zh_i(z)}{h_i(z)} - 1 \right| + 1 \right) + \sum_{i=1}^{n} \alpha_i \left| \frac{zg_i^{''}(z)}{g_i^{'}(z)} \right|$$

$$\leq \sum_{i=1}^{n} \alpha_i \left[\beta_i + 1 + \delta_i + 1 + 1 - \lambda_i \right] = \sum_{i=1}^{n} \alpha_i (\beta_i + \delta_i - \lambda_i + 3). \tag{21}$$

From (21), we get:

$$\left| \frac{zH_{n,\alpha}^{''}(z)}{H_{n,\alpha}^{'}(z)} \right| \leq \sum_{i=1}^{n} \alpha_i (\beta_i + \delta_i - \lambda_i + 3) = 1 - \rho. \tag{22}$$

So, the integral operator $H_{n,\alpha}$ is convex of order
$\rho = 1 + \sum_{i=1}^{n} \alpha_i (\lambda_i - \beta_i - \delta_i - 3)$, for $i = 1, 2, ..., n$.

If we consider $n = 1$ in Theorem 3, we get the following corollary:

Corollary 4. *Let $f \in S^*(\beta)$, $h \in S^*(\delta)$, $0 \leq \beta < 1, 0 \leq \delta < 1$ and $g \in K(\lambda)$, $0 \leq \lambda < 1$. If α is a real number so that $\alpha > 0$ and $\alpha(\beta + \delta - \lambda + 3) < 1$, then the integral operator*

$$H_{1,\alpha}(z) = \int_0^z \left(\frac{f(t)}{h(t)} g^{'}(t) \right)^{\alpha} dt$$

is convex of order $1 + \alpha(\lambda - \beta - \delta - 3)$.

Theorem 5. *Let functions $f_i \in \mathcal{SP}(\alpha, \beta), h_i \in \mathcal{SP}(\delta, \eta)$, with $\alpha > 0$ and $\delta > 0, \beta \in [0, 1), \eta \in [0, 1)$ and $g_i \in \mathcal{N}(\lambda_i)$, $\lambda_i > 1$ for $i = 1, 2, ..., n$. For any $M_i, N_i \geq 1, i = 1, 2, ..., n$, which verify*

$$\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq M_i, \left| \frac{zh_i'(z)}{h_i(z)} \right| \leq N_i \quad for \quad all \quad z \in U, \tag{23}$$

there are $\alpha_i > 0$ real numbers with $\alpha_i > 0$, $i = 1, 2, ..., n$, so that

$$\rho = 1 + \sum_{i=1}^{n} \alpha_i (M_i + N_i + 2\alpha + 4\delta - 2\eta + \lambda_i - 1) > 1. \tag{24}$$

In these conditions, the integral operator

$$H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{h_i(t)} g_i^{'}(t) \right)^{\alpha_i} dt$$

is in the class $\mathcal{N}(\rho)$.

Proof. From **Theorem** 3, we get:

$$\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)}+1=\sum_{i=1}^{n}\alpha_i\frac{zf'_i(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i\frac{h'_i(z)}{h_i(z)}+\sum_{i=1}^{n}\alpha_i\frac{zg''_i(z)}{g'_i(z)}+1$$
$$=\sum_{i=1}^{n}\alpha_i\left(\frac{zf'_i(z)}{f_i(z)}+\alpha-\beta\right)-\sum_{i=1}^{n}\alpha_i\left(\frac{zh'_i(z)}{h_i(z)}+\delta-\eta\right)$$
$$+\sum_{i=1}^{n}\alpha_i(\delta-\eta-\alpha+\beta)+\sum_{i=1}^{n}\alpha_i\left(\frac{zg''_i(z)}{g'_i(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+1. \tag{25}$$

We calculate the real part of both terms in the above expresion and obtain:

$$\mathrm{Re}\left(\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)}+1\right)=\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zf'_i(z)}{f_i(z)}+(\alpha-\beta)\right)-\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zh'_i(z)}{h_i(z)}+(\delta-\eta)\right)$$
$$+\sum_{i=1}^{n}\alpha_i(\delta-\eta-\alpha+\beta)+\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zg''_i(z)}{g'_i(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+1$$
$$=\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left[\left(\frac{zf'_i(z)}{f_i(z)}+(\alpha-\beta)\right)-\left(\frac{zh'_i(z)}{h_i(z)}+(\delta-\eta)\right)\right]$$
$$+\sum_{i=1}^{n}\alpha_i(\delta-\eta-\alpha+\beta)+\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zg''_i(z)}{g'_i(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+1 \tag{26}$$

Since $\mathrm{Re}\omega\le|\omega|$, we have:

$$\mathrm{Re}\left(\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)}+1\right)\le\sum_{i=1}^{n}\alpha_i\left|\left(\frac{zf'_i(z)}{f_i(z)}+(\alpha-\beta)\right)-\left(\frac{zh'_i(z)}{h_i(z)}+(\delta-\eta)\right)\right|$$
$$+\sum_{i=1}^{n}\alpha_i(\delta-\eta-\alpha+\beta)+\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zg''_i(z)}{g'_i(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+1$$
$$\le\sum_{i=1}^{n}\alpha_i\left|\frac{zf'_i(z)}{f_i(z)}+(\alpha-\beta)\right|+\sum_{i=1}^{n}\alpha_i\left|\frac{zh'_i(z)}{h_i(z)}+(\delta-\eta)\right|$$
$$+\sum_{i=1}^{n}\alpha_i(\delta-\eta-\alpha+\beta)+\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zg''_i(z)}{g'_i(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+1. \tag{27}$$

Since $f_i \in \mathcal{SP}(\alpha,\beta)$, $\alpha > 0, \beta \in [0,1)$ and $h_i \in \mathcal{SP}(\delta,\eta)$, $\delta > 0, \eta \in [0,1)$, for $i = 1,2,...,n$ and $g_i \in \mathcal{N}(\lambda_i), \lambda_i > 1, i = 1,2,...,n$, we have:

$$\left|\frac{zf_i'(z)}{f_i(z)} - (\alpha+\beta)\right| \le \mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) + \alpha - \beta,$$

$$\left|\frac{zh_i'(z)}{h_i(z)} - (\delta+\eta)\right| \le \mathrm{Re}\left(\frac{zh_i'(z)}{h_i(z)}\right) + \delta - \eta,$$

and

$$\mathrm{Re}\left(\frac{zg_i''(z)}{g_i'(z)}\right) \le \lambda_i, \lambda_i > 1, i = 1,2,...n, \quad z \in U.$$

Using above inequalities, we get:

$$\mathrm{Re}\left(\frac{zH_{n,\alpha}''(z)}{H_{n,\alpha}'(z)} + 1\right) \le \sum_{i=1}^{n} \alpha_i \left(\mathrm{Re}\frac{zf_i'(z)}{f_i(z)} + \alpha - \beta\right) - \sum_{i=1}^{n} \alpha_i \left(\mathrm{Re}\frac{zh_i'(z)}{h_i(z)} + \delta - \eta\right)$$

$$+ \sum_{i=1}^{n} \alpha_i(2\alpha + 2\delta) + \sum_{i=1}^{n} \alpha_i(\delta - \eta - \alpha - \beta) + \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i + 1. \tag{28}$$

From (28), we obtain:

$$\mathrm{Re}\left(\frac{zH_{n,\alpha}''(z)}{H_{n,\alpha}'(z)} + 1\right) \le \sum_{i=1}^{n} \alpha_i \left|\frac{zf_i'(z)}{f_i(z)}\right| + \sum_{i=1}^{n} \alpha_i \left|\frac{zh_i'(z)}{h_i(z)}\right| + \sum_{i=1}^{n} \alpha_i(\alpha - \beta + 2\alpha)$$

$$+ \sum_{i=1}^{n} \alpha_i(\delta - \eta + 2\delta) + \sum_{i=1}^{n} \alpha_i(\delta - \eta - \alpha + \beta) + \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i + 1$$

$$= \sum_{i=1}^{n} \alpha_i(M_i + N_i + 2\alpha + 4\delta - 2\eta + \lambda_i - 1) + 1 = \rho \tag{29}$$

So, the integral operator $H_{n,\alpha}$ is in the class $\mathcal{N}(\rho)$.

If we consider $n = 1$ in the Theorem 5, we obtain the following corallary:

Corollary 6. *Let functions $f \in \mathcal{SP}(\alpha,\beta), h \in \mathcal{SP}(\delta,\eta)$ with $\alpha > 0, \delta > 0, \beta \in [0,1), \eta \in [0,1)$ and $g \in \mathcal{N}(\lambda)$, $\lambda > 1$. For any $M, N \ge 1$, which verify*

$$\left|\frac{zf'(z)}{f(z)}\right| \le M, \left|\frac{zh'(z)}{h(z)}\right| \le N, \quad for \quad all \quad z \in U,$$

there is α *real number with* $\alpha > 0$*, so that*

$$\rho = 1 + \alpha(M + N + 2\alpha + 4\delta - 2\eta + \lambda - 1) > 1.$$

In these conditions, the integral operator

$$H_{1,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f(t)}{h(t)} g'(t) \right)^{\alpha} dt$$

is in the class $\mathcal{N}(\rho)$.

Theorem 7. *Let* $f_i \in S^*_{\lambda_i}(b)$, $h_i \in S^*_{\delta_i}(b)$ $g_i \in C_{\lambda_i}(b)$, *with* $0 \leq \lambda_i < 1$, $0 \leq \delta_i < 1$ *for* $i = 1, 2, ..., n$ *and* $b \in \mathbb{C} - \{0\}$. *Also, let* α_i *be real numbers, with* $\alpha_i > 0$ *for* $i = 1, 2, ..., n$. *If*

$$0 \leq 1 + \sum_{i=1}^{n} \alpha_i (2\lambda_i + \delta_i - 5) < 1,$$

then the integral operator

$$H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{h_i(t)} g_i'(t) \right)^{\alpha_i} dt$$

is in the class the $C_\mu(b)$, *with* $\mu = 1 + \sum_{i=1}^{n} \alpha_i (2\lambda_i + \delta_i - 5)$, *for* $i = 1, 2, ..., n$.

Proof. After the same steps from previous Theorems, we obtain:

$$\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^{n} \alpha_i \left[\frac{zf_i'(z)}{f_i(z)} - \frac{zh_i'(z)}{h_i(z)} + \frac{zg_i''(z)}{g_i'(z)} \right].$$

Multiplying relation with $1/b$, we get:

$$\frac{1}{b} \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^{n} \alpha_i \left[\frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - \frac{zh_i'(z)}{h_i(z)} \right) + \frac{1}{b} \frac{zg_i''(z)}{g_i'(z)} \right].$$

Further, we have

$$\left|\frac{1}{b}\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)}\right| = \left|\sum_{i=1}^{n}\alpha_i\frac{1}{b}\left(\frac{zf'_i(z)}{f_i(z)} - \sum_{i=1}^{n}\alpha_i\frac{1}{b}\frac{zh'_i(z)}{h_i(z)}\right) + \sum_{i=1}^{n}\alpha_i\frac{1}{b}\frac{g''_i(z)}{g'_i(z)}\right|$$

$$\leq \sum_{i=1}^{n}\alpha_i\left|\frac{1}{b}\frac{zf'_i(z)}{f_i(z)}\right| + \sum_{i=1}^{n}\alpha_i\left|\frac{1}{b}\frac{zh'_i(z)}{h_i(z)}\right| + \sum_{i=1}^{n}\alpha_i\left|\frac{1}{b}\frac{g''_i(z)}{g'_i(z)}\right|$$

$$\leq \sum_{i=1}^{n}\alpha_i\left|\frac{1}{b}\right|\left(\left|\frac{zf'_i(z)}{f_i(z)} - 1\right| + 1\right) + \sum_{i=1}^{n}\alpha_i\left|\frac{1}{b}\right|\left(\left|\frac{zh'_i(z)}{h_i(z)} - 1\right| + 1\right)$$

$$+ \sum_{i=1}^{n}\alpha_i\left|\frac{1}{b}\right|\left|\frac{g''_i(z)}{g'_i(z)}\right|$$

$$\leq \sum_{i=1}^{n}\alpha_i\left(\left|\frac{1}{b}\left(\frac{zf'_i(z)}{f_i(z)} - 1\right)\right| + 1\right) + \sum_{i=1}^{n}\alpha_i\left(\left|\frac{1}{b}\left(\frac{zh'_i(z)}{h_i(z)} - 1\right)\right| + 1\right)$$

$$+ \sum_{i=1}^{n}\alpha_i\left|\frac{1}{b}\frac{zg''_i(z)}{g'_i(z)}\right|. \tag{30}$$

Since $f_i \in S^*_{\lambda_i}(b)$, $h_i \in S^*_{\delta_i}(b)$ and $g_i \in C_{\lambda_i}(b)$ for $i = 1, 2, ..., n$, we have
$\left|\frac{1}{b}\left(\frac{zf'_i(z)}{f'_i(z)} - 1\right)\right| \leq 1 - \lambda_i$, $\left|\frac{1}{b}\left(\frac{zh'_i(z)}{h'_i(z)} - 1\right)\right| \leq 1 - \delta_i$ and $\left|\frac{1}{b}\frac{zg''_i(z)}{g'_i(z)}\right| \leq 1 - \lambda_i$.
So, we get:

$$\left|\frac{1}{b}\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)}\right| \leq \sum_{i=1}^{n}\alpha_i\left((1-\lambda_i)+1\right) + \sum_{i=1}^{n}\alpha_i((1-\delta_i)+1) + \sum_{i=1}^{n}\alpha_i(1-\lambda_i)$$

$$= \sum_{i=1}^{n}(2-\lambda_i) + \sum_{i=1}^{n}\alpha_i(2-\delta_i) + \sum_{i=1}^{n}\alpha_i(1-\lambda_i) = \sum_{i=1}^{n}\alpha_i(5-2\lambda_i-\delta_i).$$

Since $0 \leq 1 + \sum_{i=1}^{n}\alpha_i(2\lambda_i + \delta_i - 5) < 1$, we get, $H_{n,\alpha}$ is in the class $C_\mu(b)$, with $\mu = 1 + \sum_{i=1}^{n}\alpha_i(2\lambda_i + \delta_i - 5)$.

If we consider $n = 1$ in Theorem 7, we get the following corollary:

Corollary 8. *Let $f \in S^*_\lambda$ and $h \in S^*_\delta$, $g \in C_\lambda(b)$ with $0 \leq \lambda < 1$, $0 \leq \delta < 1$ and $b \in \mathbb{C} - \{0\}$. Also, let α be a real number, with $\alpha > 0$. If $0 \leq 1 + \alpha(2\lambda + \delta - 5) < 1$, then the integral operator*

$$H_{1,\alpha}(z) = \int_0^z \left(\frac{f(t)}{h(t)}g'(t)\right)^\alpha dt$$

is in the class $C_\mu(b)$, *with* $\mu = 1 + \alpha(2\lambda + \delta - 5)$.

Theorem 9. *Let* $f_i, g_i, h_i \in \mathcal{A}$, *where* $g_i \in \mathcal{N}(\lambda_i)$, *with* $\lambda_i > 1$ *for* $i = 1, 2, ..., n$. *For any* $\lambda_i > 1$, *and* f_i, h_i *verifying conditions*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)} - 1\right| \le 1, \left|\frac{zh_i^{'}(z)}{h_i(z)} - 1\right| \le 1, \, z \in U$$

there are numbers $\alpha_i \in \mathbb{R}$ *with* $\alpha_i > 0$ *so that* $\mu = \sum_{i=1}^{n} \alpha_i(\lambda_i{+}1){+}1$ *for* $i = 1, 2, ..., n$. *In these conditions, the integral operator*

$$H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{h_i(t)} g_i^{'}(t)\right)^{\alpha_i} dt$$

is in the class $\mathcal{N}(\mu)$.

Proof. From the previous Theorems, we obtain

$$\frac{zH_{n,\alpha}^{''}(z)}{H_{n,\alpha}^{'}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i^{'}(z)}{f_i(z)} - \frac{zh_i^{'}(z)}{h_i(z)} + \frac{zg_i^{''}(z)}{g_i^{'}(z)}\right).$$

Further, we get:

$$\frac{zH_{n,\alpha}^{''}(z)}{H_{n,\alpha}^{'}(z)} + 1 = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i^{'}(z)}{f_i(z)} - 1\right) - \sum_{i=1}^{n} \alpha_i \left(\frac{zh_i^{'}(z)}{h_i(z)} - 1\right)$$

$$+ \sum_{i=1}^{n} \alpha_i \left(\frac{zg_i^{''}(z)}{g_i^{'}(z)} + 1\right) - \sum_{i=1}^{n} \alpha_i + 1 \tag{31}$$

and

$$\mathrm{Re}\left(\frac{zH_n^{''}(z)}{H_n^{'}(z)} + 1\right) = \sum_{i=1}^{n} \alpha_i \mathrm{Re}\left(\frac{zf_i^{'}(z)}{f_i(z)} - 1\right) - \sum_{i=1}^{n} \alpha_i \mathrm{Re}\left(\frac{zh_i^{'}(z)}{h_i(z)} - 1\right)$$

$$+ \sum_{i=1}^{n} \alpha_i \mathrm{Re}\left(\frac{zg_i^{''}(z)}{g_i^{'}(z)} + 1\right) - \sum_{i=1}^{n} \alpha_i + 1. \tag{32}$$

Since $g_i \in \mathcal{N}(\lambda_i), i = 1, 2, ..., n$ and $\mathrm{Re}(\omega) \le |\omega|$ and applying the conditions from the hypothesis of Theorem 9, (31) and (32), we get:

$$\mathrm{Re}\left(\frac{zH_{n,\alpha}^{''}(z)}{H_{n,\alpha}^{'}(z)} + 1\right) \le \sum_{i=1}^{n} \alpha_i \left|\frac{zf_i^{'}(z)}{f_i(z)} - 1\right| + \sum_{i=1}^{n} \alpha_i \left|\frac{zh_i^{'}(z)}{h_i(z)} - 1\right|$$

$$+ \sum_{i=1}^{n} \alpha_i \lambda_i - \sum_{i=1}^{n} \alpha_i + 1$$

$$\leq 2\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\alpha_i\lambda_i - \sum_{i=1}^{n}\alpha_i + 1 = \sum_{i=1}^{n}\alpha_i(\lambda_i+1)+1. \tag{33}$$

So, $H_{n,\alpha}$ is in the class $\mathcal{N}(\mu)$, where $\mu = 1+\sum_{i=1}^{n}\alpha_i(\lambda_i+1), i=1,2,...,n$.

If consider $n=1$ and in Theorem 9, we get the following corollary:

Corollary 10. *Let $f,h\in\mathcal{A}$, where $g\in\mathcal{N}(\lambda)$, $\lambda>1$ and f,h verify conditions*

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1, \left|\frac{zh'(z)}{h(z)}-1\right|\leq 1, z\in U,$$

there is number $\alpha\in\mathbb{R}$ with $\alpha>0$ so that $\mu=\alpha(\lambda+1)$.
In these conditions, the integral operator

$$H_{1,\alpha}(z) = \int_o^z\left(\frac{f(t)}{h(t)}g'(t)\right)^{\alpha}dt$$

is in the class $\mathcal{N}(\mu)$.

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