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FEKETE-SZEGÖ THEOREM FOR A CLASS OF FUNCTIONS DEFINED BY A DERIVATIVE OPERATOR

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Abstract. In this paper, a new subclass $S^{\alpha,n}(m,l,q,\lambda,\phi)$ defined by the generalised derivative operator $D^{\alpha,n}(m,l,q,\lambda)$ is introduced. The Fekete-Szegö functional $|a_3-\mu a_2^2|$ of the subclass $S^{\alpha,n}(m,l,q,\lambda,\phi)$ is obtained. Then by convolution, we state another subclass $S^{\alpha,n,g}(m,l,q,\lambda,\phi)$ defined by fractional derivatives. Another set of Fekete-Szegö result is determined.

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1. Introduction

Denote by $\mathcal{A}$ the class of normalised analytic univalent functions f of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1}$$

where $z\in\mathbb{U}=\{z:|z|<1\}$.
For the function $f\in\mathcal{A}$ given by (1), we state a generalised derivative operator given by Eghbiq and Darus [2] as follows:

$$D^{\alpha,n}(m,l,q,\lambda)(f)(z)=z+\sum_{k=2}^{\infty}k^{\alpha}\left(\frac{q+\lambda(k-1)+l}{q+l}\right)^m c(n,k)a_kz^k, \tag{2}$$

where $n,\alpha\in\mathbb{N}_0=\{0,1,2...\},m\in\mathbb{Z},\ \lambda,l,q\geq 0,l+q\neq 0$ and $c(n,k)=\frac{\prod_{j=1}^{k-1}(j+n)}{(k-1)!}$.
In terms of convolution, $D^{\alpha,n}(m,l,q,\lambda)f(z)$ can also be written as

$$\phi(z):=\left(\frac{l+q-\lambda}{l+q}\right)\frac{z}{1-z}+\left(\frac{\lambda}{l+q}\right)\frac{z}{(1-z)^2},\quad (z\in\mathbb{U}).$$

If $m = 0, 1, 2, ...,$ then

$$\begin{aligned} D^{\alpha,n}(m,l,q,\lambda)f(z) &= \underbrace{\phi(z) * ... * \phi(z)}_{(m)-times} * \left[\frac{z}{(1-z)^{n+1}}\right] * \sum_{k=1}^{\infty} k^{\alpha} z^{k} * f(z) \\ &= R^{n} * D^{\alpha}(m,l,q,\lambda)f(z), \end{aligned}$$

where $R^n = z + \sum_{k=2}^{\infty} c(n,k) z^k$, the Ruscheweyh derivative operator[18] and $D^{\alpha}(m,l,q,\lambda) = z + \sum_{k=2}^{\infty} k^{\alpha}(1 + \frac{k-1}{l+q}\lambda)^m$. If $m = -1, -2, ...,$ then

$$\begin{aligned} D^{\alpha,n}(m,l,q,\lambda)f(z) &= \underbrace{\phi(z) * ... * \phi(z)}_{(m)-times} * \left[\frac{z}{(1-z)^{n+1}}\right] * \sum_{k=1}^{\infty} k^{\alpha} z^{k} * f(z) \\ &= R^{n} * D^{\alpha}(m,l,q,\lambda)f(z). \end{aligned}$$

Note that:

$$D^{0,0}(0,l,q,\lambda)f(z) = f(z), \quad and$$
$$D^{1,0}(0,l,q,\lambda)f(z) = zf'(z).$$

By specialising the parameters of $D^{\alpha,n}(m,l,q,\lambda)f(z)$, we get the following derivative and integral operators.

- The derivative operator introduced by Ruscheweyh [18];

$$D^{0,n}(0,l,q,\lambda); (n \in \mathbb{N}_0) \equiv R^n = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k.$$

- The derivative operator introduced by Sălăgean [6];

$$D^{n,0}(0,l,q,\lambda); (n \in \mathbb{N}_0) \equiv D^n = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

- The generalised Sălăgean derivative operator given by Al-Oboudi (see [5]);

$$D^{0,0}(n,1,0,\lambda); (n \in \mathbb{N}_0) \equiv D_{\lambda}^n = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k.$$

• The generalised Ruscheweyh derivative operator given by Al-Shaqsi and Darus in [9];

$$D^{0,n}(1,1,0,\lambda);(n\in\mathbb{N}_0)\equiv R^n_\lambda = z+\sum_{k=2}^{\infty}(1+\lambda(k-1))c(n,k)a_k z^k.$$

• The generalised Ruscheweyh and Sălăgean derivative operator introduced by Darus and Al-shaqsi (see [11]);

$$D^{0,\beta}(m,1,0,\lambda);(m\in\mathbb{N}_0)\equiv D^m_{\lambda,\beta} = z+\sum_{k=2}^{\infty}(1+\lambda(k-1))^m c(\beta,k)a_k z^k.$$

• The derivative operator introduced by Catas (see [1]);

$$D^{0,\beta}(m,l,1,\lambda);(m\in\mathbb{N}_0)\equiv D^m(\lambda,\beta,l) = z+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m c(\beta,k)a_k z^k.$$

• The operator introduced by Uralegaddi and Somanatha (see [4]);

$$D^{0,0}(n,1,1,1)\equiv I^n = z+\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^n a_k z^k.$$

• The multiplier transformations studied by Flett (see [19]);

$$D^{0,0}(n,1,\lambda,1)\equiv I^n_\lambda = z+\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^k.$$

• The integral operator introduced by Cho and Kim (see [12]);

$$D^{0,0}(-n,1,\lambda,1)\equiv I^\lambda_n = z+\sum_{k=2}^{\infty}k\left(\frac{1+\lambda}{k+\lambda}\right)^n a_k z^k.$$

• The derivative operator introduced by Mustafa and Darus ((see [13]);

$$D^{\alpha,n}(m,1,q,\lambda)(f)(z)\equiv D^{\alpha,n}(m,q,\lambda)(f)(z) = z+\sum_{k=2}^{\infty}k^\alpha\left(1+\frac{k-1}{1+q}\lambda\right)^m c(n,k)a_k z^k,$$

The Fekete-Szegö problem has attracted many researchers to solve similar problems for different classes.

In this paper, we obtain the Fekete-Szegö inequality for functions f of the class $S^{\alpha,n}(m,l,q,\lambda,\phi)$ which we will define below. Also, we give applications of our results to certain functions defined through convolution (or the Hadamard product). In particular, we consider a class $S^{\alpha,n,\eta}(m,l,q,\lambda,\phi)$, of functions defined by fractional derivatives. The work here is much motivated by Srivastava and Mishra (see [7]), that leads to our new results.

By using the following definition and the operator $D^{\alpha,n}(m,l,q,\lambda)$, we define the class $S^{\alpha,n}(m,l,q,\lambda,\phi)$ as follows:

Definition 1. Let $\phi(z)$ be a univalent starlike function with respect to 1 that maps the unit disc $\mathbb{U}$ onto the right half plane which is symmetric with respect to the real axis,$\phi(0)=1,\ \phi'(0)>0$. The function $f\in\mathcal{A}$ belongs to the class $S^{\alpha,n}(m,l,q,\lambda,\phi)$, if

$$\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)}\prec\phi(z).$$

It is clear that:

(1) $S^{0,0}(0,0,q,\lambda,\phi)\equiv S^*(\phi)$, introduced by Ma and Minda (see [21]).
(2) $S^{0,0}(0,0,q,\lambda,(1+Az)/(1+Bz))=S^*[A,B],(-1\le B<A\le 1)$, introduced by Janowski (see [20]).
(3)$S^{0,0}(0,0,q,\lambda,\beta)=S^*(\beta)$, introduced by Robertson in 1936 (see [15]).

(4) $S^{0,n}(1,1,0,\lambda,(1+z)/(1-z))\equiv R_n$ was studied by Singh and Singh (see [16]), and also Owa et al (see [17]).

2. Main results

In order to prove our result we have to recall the following lemmas:

Lemma 1. (Ma and Minda [21]) If $p(z)=1+c_1z+c_2z^2+...$ is an analytic function with positive real part in $\mathbb{U}$, then

$$|c_2-\nu c_1^2|\le\begin{cases}-4\nu+2 & if \quad \nu\le 0,\\ 2 & if \quad 0\le\nu\le 1,\\ 4\nu+2 & if \quad \nu\ge 1.\end{cases}$$

When $\nu<0$ or $\nu>1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$, or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$, or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}a\right)\frac{1-z}{1+z}, \quad (0 \leq a < 1),$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu|c_1| \leq 2, \qquad (0 < \nu \leq \frac{1}{2}),$$

and

$$|c_2 - \nu c_1^2| + (1-\nu)|c_1| \leq 2, \qquad (\frac{1}{2} < \nu \leq 1).$$

Lemma 2. (Ma and Minda [21]) If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function with positive real part in $\mathbb{U}$, then for any complex number μ

$$|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}.$$

The result is sharp for the function

$$p(z) = \frac{1+z}{1-z} \quad or \quad p(z) = \frac{1+z^2}{1-z^2}.$$

Next, we state and prove the following theorem.

Theorem 1. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_k are real with $B_1 > 0$. Let the function f be given by (1) and belongs to the class $S^{\alpha,n}(m, l, q, \lambda, \phi)$. Then

$|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \frac{B_1^2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} + \frac{B_2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} - \frac{\mu B_1^2(l+q)^m}{2^{2\alpha}(n+1)^2(l+q+\lambda)^m} & if \quad \mu \leq \sigma_1, \\ \frac{B_1(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} & if \sigma_1 \leq \mu \geq \sigma_2, \\ -\frac{B_1^2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} - \frac{B_2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} + \frac{\mu B_1^2(l+q)^m}{2^{2\alpha}(n+1)^2(l+q+\lambda)^m} & if \quad \mu \leq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{2^\alpha(n+1)(l+q+\lambda)^{2m}[B_1 - B_2 + B_1^2]}{3^\alpha B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)},$$

$$\sigma_2 = \frac{2^\alpha (n+1)(l+q+\lambda)^{2m}[B_1 + B_2 + B_1^2]}{3^\alpha B_1^2 (l+q)^m (l+q+2\lambda)^m (n+2)}.$$

Proof. For $f \in S^{\alpha,n}(m,l,q,\lambda,\phi)$, let

$$p_1(z) = \frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)} = 1 + b_1 z + b_2 z^2 + \cdots . \tag{3}$$

It follows from (3) that

$$2^\alpha (n+1)\frac{(l+q+\lambda)^m}{(l+q)^m} a_2 = b_1,$$

and

$$3^\alpha (n+1)(n+2)\frac{(l+q+2\lambda)^m}{(l+q)^m} a_3 = b_2 + b_1 2^\alpha (n+1)\frac{(l+q+\lambda)^m}{(l+q)^m} a_2.$$

Knowing that $\phi(z)$ is univalent and $p \prec \phi$, we can say

$$p(z) = \frac{1+\phi^{-1}(p_1(z))}{1-\phi^{-1}(p_1(z))} = 1 + c_1 z + c_2 z^2 + \cdots ,$$

is analytic with positive real part in $\mathbb{U}$. Thus we obtain

$$p_1(z) = \phi\left(\frac{p(z)-1}{p(z)+1}\right),$$

and from equation,

$$\begin{aligned} 1 + b_1 z + b_2 z^2 + ... &= \phi\Big(\frac{c_1 z + c_2 z^2 + ...}{2 + c_1 z + c_2 z^2 + ...}\Big) \\ &= \phi\Big[\frac{1}{2}c_1 z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + ...\Big] \\ &= 1 + B_1 \frac{1}{2}c_1 z + B_1 \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + ... + B_2 \frac{1}{4}c_1^2 z^2 + ..., \end{aligned}$$

we get

$$b_1 = \frac{1}{2}B_1 c_1 \text{ and } b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2).$$

Therefore

$$a_3 - \mu a_2^2 = \frac{\left[(B_1^2 c_1^2 + 2B_1[c_2 - \frac{1}{2}c_1^2] + B_2 c_1^2)(l+q)^m\right]}{4(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}$$

$$-\frac{\mu B_1^2 c_1^2 (l+q^{2m})}{2^{2(\alpha+1)}(n+1)^2(l+q+\lambda)^2 m},$$

$$a_3 - \mu a_2^2 = \frac{B_1(l+q)^m}{(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}$$

$$\left[c_2 - c_1^2\left(\frac{1}{2}\left\{1 - B_1 - \frac{B_2}{B_1} + \mu\frac{3^\alpha(l+q)^m(n+2)(l+q+2\lambda)^m}{2^{2\alpha}(n+1)(l+q+\lambda)^m}\right]\right)\right],$$

$$a_3 - \mu a_2^2 = \frac{B_1(l+q)^m}{2(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}[c_2 - \nu c_1^2], \tag{4}$$

where

$$\nu = \frac{1}{2}[1 - B_1 - \frac{B_2}{B_1} + \mu\frac{3^\alpha(l+q)^\mu(n+2)(l+q+2\lambda)^m}{2^{2\alpha}(n+1)(l+q+\lambda)^m}].$$

If $\mu \leq \sigma_1$, then by applying Lemma 1 we have

$$|a_3 - \mu a_2^2| = \frac{B_1^2(1+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} + \frac{B_2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} -$$

$$\frac{\mu B_1^2(l+q)^m}{2^{2\alpha}(n+1)^2(l+q+\lambda)^m}.$$

If $\sigma_1 \leq \mu \leq \sigma_2$, we get

$$|a_3 - \mu a_2^2| = \frac{B_1(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m}.$$

Similarly, if $\mu \leq \sigma_2$, we get

$$|a_3 - \mu a_2^2| = -\frac{B_1^2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} - \frac{B_2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} +$$

$$\frac{\mu B_1^2(l+q)^m}{2^{2\alpha}(n+1)^2(l+q+\lambda)^m}.$$

If $\mu = \sigma_1$, then equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}a\right)\frac{1-z}{1+z}, \quad (0 \leq a < 1, z \in \mathbb{U}),$$

or one of its rotations. Also, if $\mu = \sigma_2$, then

$$\frac{1}{2}\left[1 - B_1 - \frac{B_2}{B_1} + \mu\frac{3^{\alpha}(l+q)^{\mu}(n+2)(l+q+2\lambda)^m}{2^{2\alpha}(n+1)(l+q+\lambda)^m}\right] = 0.$$

Therefore

$$\frac{1}{p(z)} = \left(\frac{1}{2} + \frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}a\right)\frac{1-z}{1+z}, \quad (0 \le a < 1, z \in \mathbb{U}).$$

We obtain an interesting result contained in the following remark.

Remark 1.

- For $\alpha = q = 0$, $m = l = 1$ in Theorem 1, we get the results obtained by Al-Shaqsi and Darus (see [10]).
- For $l = 1$ in Theorem 1, we get the results of Mustafa and Darus (see [14])

Theorem 2. If $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 1, Theorem 1 can be improved. Let σ_3 be given by

$$\sigma_3 = \frac{2^{\alpha}(n+1)(l+q+\lambda)^{2m}[B_2 + B_1^2]}{3^{\alpha}B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)}.$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{2^{\alpha}(n+1)(l+q+\lambda)^{2m}}{3^{\alpha}B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)}$$

$$\left[B_1 - B_2 + \frac{(3^{\alpha}B_1^2(l+q)^m(l+q+2\lambda)^m(n+2) + (n+1)(l+q+\lambda)^{2m}B_1^2)}{2^{\alpha}(n+1)(l+q+\lambda)^{2m}}\right]|a_2|^2$$

$$\le \frac{B_1(l+q)^m}{3^{\alpha}(n+1)(n+2)(l+q+2\lambda)^m}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{2^{\alpha}(n+1)(l+q+\lambda)^{2m}}{3^{\alpha}B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)}$$

$$\left[B_1 + B_2 - \frac{(3^{\alpha}B_1^2(l+q)^m(l+q+2\lambda)^m(n+2) + (n+1)(l+q+\lambda)^{2m}B_1^2)}{2^{\alpha}(n+1)(l+q+\lambda)^{2m}}\right]|a_2|^2$$

$$\le \frac{B_1(l+q)^m}{3^{\alpha}(n+1)(n+2)(l+q+2\lambda)^m}.$$

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\begin{aligned}
|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2 &= \frac{B_1(l+q)^m}{2(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}[c_2-\nu c_1^2]\\
&+ (\mu-\sigma_1)\frac{B_1^2(l+q)^{2m}}{2^{2(\alpha+1)}(n+1)^2(l+q+\lambda)^{2m}}|c_1|^2\\
&= \frac{B_1(l+q)^m}{2(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}[c_2-\nu c_1^2]\\
&+ \left(\mu-\frac{2^\alpha(n+1)(l+q+\lambda)^{2m}[B_1-B_2+B_1^2]}{3^\alpha B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)}\right)\\
&\times \left(\frac{B_1^2(l+q)^{2m}}{2^{2(\alpha+1)}(n+1)^2(l+q+\lambda)^{2m}}\right)|c_1|^2\\
&= \frac{B_1(l+q)^m}{(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}\left[\frac{1}{2}[|c_2-\nu c_1^2|+\nu|c_1|^2]\right]\\
&\le \frac{B_1(l+q)^m}{(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}.
\end{aligned}$$

Similarly, if $\sigma_3 \le \mu \le \sigma_2$, we can write

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2 = \frac{B_1(l+q)^m}{2(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}[c_2-\nu_1^2]$$

$$\begin{aligned}
&+ (\sigma_2-\mu)\frac{B_1^2(l+q)^{2m}}{2^{2(\alpha+1)}(n+1)^2(l+q+\lambda)^2m}|c_1|^2\\
&= \frac{B_1(l+q)^m}{2(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}[c_2-\nu c_1^2]\\
&+ \left(\frac{2^\alpha(n+1)(l+q+\lambda)^{2m}[B_1+B_2+B_1^2]}{3^\alpha B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)}-\mu\right)\\
&\times \left(\frac{B_1^2(l+q)^{2m}}{2^{2(\alpha+1)}(n+1)^2(l+q+\lambda)^2m}\right)|c_1|^2\\
&= \frac{B_1(l+q)^m}{(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}\left[\frac{1}{2}[|c_2-\nu c_1^2|+(1-\nu)|c_1|^2]\right]\\
&\le \frac{B_1(1+q)^m}{(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m}.
\end{aligned}$$

Using the above method for the class $S^{\alpha,n}(m,l,q,\lambda,(1+Az)/(1+Bz))$, we give the following corollary.

Corollary 1. Let $-1 \le B < A \le 1$. Let the function f be given by (1) and belongs to the class $S^{\alpha,n}(m,l,q,\lambda,(1+Az)/(1+Bz))$. Then

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{(A-B)^2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} + \frac{B(B-A)(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} - \frac{\mu(A-B)^2(l+q)^m}{2^{2\alpha}(n+1)^2(l+q+\lambda)^m} & if \quad \mu \le \sigma_1, \\ \frac{(A-B)(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} & if \sigma_1 \le \mu \le \sigma_2, \\ -\frac{(A-B)^2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} - \frac{B(B-A)(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} + \frac{\mu(A-B)^2(l+q)^m}{2^{2\alpha}(n+1)^2(l+q+\lambda)^m} & if \quad \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{2^\alpha(n+1)(l+q+\lambda)^{2m}[(A-B)(1-2B+A]}{3^\alpha(A-B)^2(l+q)^m(l+q+2\lambda)^m(n+2)},$$

$$\sigma_2 = \frac{2^\alpha(n+1)(l+q+\lambda)^{2m}[(A-B)(1+A)]}{3^\alpha(A-B)^2(l+q)^m(l+q+2\lambda)^m(n+2)}.$$

Making use of Lemma 2, and equation (4), we immediately obtain the following result.

Corollary 2. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ and the function f be given by (1) and belongs to the class $S^{\alpha,n}(m,l,q,\lambda,\phi)$. For a complex number μ

$$|a_3-\mu a_2^2| \le \frac{B_1(l+q)^m}{(3^\alpha)(n+1)(n+2)(l+q+2\lambda)^m} \max\left\{1, \left[-B_1 - \frac{B_2}{B_1} + \mu\frac{3^\alpha(l+q)^m(n+2)(l+q+2\lambda)^m}{2^{2\alpha}(n+1)(l+q+\lambda)^m}\right]\right\}$$

The next result is motivated by Srivastava and Mishra (see [7]). The methods of proving are similar [7]. Here we define the class $S^{\alpha,n,\eta}(m,l,q,\lambda,\phi)$. This will require the following definition.

Definition 2. (Srivastava and Owa [8]) Let the function f be analytic in a simply connected region of the z-plane $\mathbb{C}$ containing the origin and $0 \le \alpha < 1$, then the fractional derivative of order α is defined by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{f(\alpha)}{(z-t)^\alpha}dt; \quad 0 \le \alpha < 1,$$

where the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Srivastava and Owa[8] introduced and studied the operator $\Omega^\alpha : \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^\alpha f(z) = \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z), \qquad (\alpha \ne 2,3,4,...),$$

where D_z^α is the fractional derivative of f of order α.

The class $S^{\alpha,n,\eta}(m,l,q,\lambda,\phi)$ consists of the functions $f \in \mathcal{A}$ for which $\Omega^{\eta} f \in S^{\alpha,n}(m,l,q,\lambda,\phi)$. Note that $S^{\alpha,n,\eta}(m,l,q,\lambda,\phi)$ is a special case of the class $S^{\alpha,n,g}(m,l,q,\lambda,\phi)$ when

$$g(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma((2-\eta)}{\Gamma((k+1-\eta)} z^k.$$

Let

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k \qquad (g_k > 0).$$

Since

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S^{\alpha,n,g}(m,l,q,\lambda)$$

if and only if

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k g_k z^k \in S^{\alpha,n}(m,l,q,\lambda,\phi).$$

The coefficient estimate for functions in the class $S^{\alpha,n,g}(m,l,q,\lambda,\phi)$ is obtained from the corresponding estimate of functions in the class $S^{\alpha,n}(m,l,q,\lambda)$. Applying Theorem 1 for the function $(f * g)(z) = z + a_2 g_2 z^2 + a_3 g_3 z^3 + ...$, we have Theorem 2 below after some changes of the parameter μ.

Theorem 2 Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_k are real with $B_1 > 0$. Let the function f be given by (1) and belongs to the class $S^{\alpha,n,g}(m,l,q,\lambda,\phi)$. Then

$|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \frac{1}{g_3}\left[\frac{B_1^2(l+q)^m}{3^{\alpha}(n+1)(n+2)(l+q+2\lambda)^m} + \frac{B_2(l+q)^m}{3^{\alpha}(n+1)(n+2)(l+q+2\lambda)^m} - \frac{\mu g_3 B_1^2(l+q)^m}{g_2^2 2^{2\alpha}(n+1)^2(l+q+\lambda)^m}\right] & if \quad \mu \leq \sigma_1, \\ \frac{1}{g_3}\left[\frac{B_1(l+q)^m}{3^{\alpha}(n+1)(n+2)(l+q+2\lambda)^m}\right] & if \sigma_1 \leq \mu \geq \sigma_2, \\ \frac{1}{g_3}\left[-\frac{B_1^2(l+q)^m}{3^{\alpha}(n+1)(n+2)(l+q+2\lambda)^m} - \frac{B_2(l+q)^m}{3^{\alpha}(n+1)(n+2)(l+q+2\lambda)^m} + \frac{g_3 \mu B_1^2(l+q)^m}{g_2^2 2^{2\alpha}(n+1)^2(l+q+\lambda)^m}\right] & if \quad \mu \leq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2 2^{\alpha}(n+1)(l+q+\lambda)^{2m}[B_1 - B_2 + B_1^2]}{g_3 3^{\alpha} B_1^2 (l+q)^m (l+q+2\lambda)^m (n+2)},$$

$$\sigma_2 = \frac{g_2^2 2^{\alpha}(n+1)(l+q+\lambda)^{2m}[B_1 + B_2 + B_1^2]}{g_3 3^{\alpha} B_1^2 (l+q)^m (l+q+2\lambda)^m (n+2)}.$$

Since

$$\Omega^\eta D^{\alpha,n}(m,l,q,\lambda) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\eta)}{\Gamma(k+1-\eta)} \left[k^\alpha \left(1 + \frac{k-1}{l+q}\lambda \right)^m c(n,k) \right] a_k z^k.$$

We have

$$g_2 = \frac{\Gamma(3)\Gamma(2-\eta)}{\Gamma(3-\eta)} = \frac{2}{(2-\eta)},$$

$$g_3 = \frac{\Gamma(4)\Gamma(2-\eta)}{\Gamma(4-\eta)} = \frac{6}{(2-\eta)(3-\eta)},$$

for g_2 and g_3 given by the above equalities, Theorem 2 gives to the following.

Theorem 3 Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$, where B_k are real with $B_1 > 0$. Let the function f be given by (1) and belongs to the class $S^{\alpha,n,\eta}(m,l,q,\lambda,\phi)$. Then

$|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \frac{(2-\eta)(3-\eta)}{6}[\frac{B_1^2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} + \frac{B_2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} - \\ \frac{\mu 3(2-\eta)B_1^2(l+q)^m}{2(3-\eta)2^{2\alpha}(n+1)^2(l+q+\lambda)^m}] & if \quad \mu \leq \sigma_1, \\ \frac{(2-\eta)(3-\eta)}{6}\left[\frac{B_1(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m}\right] & if \sigma_1 \leq \mu \geq \sigma_2, \\ \frac{(2-\eta)(3-\eta)}{6}[-\frac{B_1^2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} - \frac{B_2(l+q)^m}{3^\alpha(n+1)(n+2)(l+q+2\lambda)^m} + \\ \frac{3(2-\eta)\mu B_1^2(l+q)^m}{2(3-\eta)2^{2\alpha}(n+1)^2(l+q+\lambda)^m}] & if \quad \mu \leq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{2(3-\eta)2^\alpha(n+1)(l+q+\lambda)^{2m}[B_1 - B_2 + B_1^2]}{3(2-\eta)3^\alpha B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)},$$

$$\sigma_2 = \frac{2(3-\eta)2^\alpha(n+1)(l+q+\lambda)^{2m}[B_1 + B_2 + B_1^2]}{3(2-\eta)3^\alpha B_1^2(l+q)^m(l+q+2\lambda)^m(n+2)}.$$

Remark 2. When $\lambda = 0, \quad n = 0, \quad m = 0, \quad B_1 = \frac{8}{\pi^2}, \quad B_2 = \frac{16}{3\pi^2}$, the above Theorem 2 reduces to the work of Srivastava and Mishra (see [7]) for a class of functions for which $\Omega^\eta f(z)$ is a parabolic starlike function (see [3]).

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