

INVERSE PROBLEM OF FINDING A PARAMETER IN A SEMI-LINEAR HEAT EQUATION BASED ON THE PICARD'S ITERATION METHOD

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ABSTRACT. In this paper, The Picard's iteration method (PIM) is successfully applied to obtain the exact solution for an inverse problem of finding a parameter in a semi-linear heat equation. By employing some transformation, the parabolic inverse problem is transferred into a direct one given by a non classical parabolic equation with one unknown parameter which has a suitable form to apply the Picard's iteration method. By applying the PIM method, the solution of the resulting problem is provided in the form of an infinite series, which can be written in a closed form. Then, the obtained solution is used to deduce the solution of the original problem. Two numerical examples are used to explain and illustrate the application of the proposed approach. The obtained results reveal that the Picard's iteration method is efficient and straightforward

Keywords: Picard's iteration method, inverse problem, parabolic equation, heat equations, control function.

1. INTRODUCTION

Inverse problem of determining a source parameter in a parabolic partial differential equation can be used to model several engineering applications and physical phenomena, such as the heat transfer process, chemical diffusion, thermoelasticity and control theory [3, 4, 5, 25, 6, 7, 8, 13, 9, 14, 17]. The parabolic inverse problem that describes the heat transfer process in one and two dimensions can be formulated by the following parabolic partial differential equation subject to appropriate initial and boundary conditions given by :

$$\frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) + p(t)u(x, t) + \phi(x, t), \quad x \in \Omega, 0 < t \leq T, \quad (1)$$

$$u(x, 0) = f(x), \quad x \in \Omega \quad (2)$$

$$u(x, t) = \psi(t), \quad x \in \partial\Omega, 0 < t \leq T, \quad (3)$$

and the additional condition which describes the specification of the solution at an internal point in the spatial domain :

$$u(x_0, t) = E(t), \quad x_0 \in \Omega, \quad 0 < t \leq T. \quad (4)$$

where Δ is the Laplace operator, $\Omega \subset \mathbb{R}^d$ is the spatial domain, $d = 1, 2$, $x = (x_1, \dots, x_d)$, T is the time horizon, Φ , f , ψ and E are known functions, and x_0 is a given interior point. $u(x, t)$ and $p(t)$ are unknown, where u represents the temperature distribution and p is a control parameter that produces at each time t a desired temperature $E(t)$ at a given interior point $x_0 \in \Omega$ [10]. Hence, The problem (1) – (4) can be viewed as a control problem [5, 25].

In the last decades, much attention have been devoted by many researchers to study and develop accurate methods to obtain numerical or analytical solution for the parabolic inverse problems, i.e., the determination of the control parameter $p(t)$ and the solution $u(x, t)$. One can cite, finite difference method [5, 7, 8, 9, 19, 18], Radial Basis Functions [12], Method of Lines coupled with the method of Runge-Kutta [11], Tau-method [10], Sinc collocation method [28], Variational Iteration Method [24, 27].

The aim of this paper, is to employ the Picard's iteration method (PIM) [21, 23] to obtain the exact solution of the above parabolic inverse problem (1) – (4), that is the determination of the pair of functions (u, p) . The Picard's iteration method is a semi-analytical method, that provides the solution in the form of an infinite series, by transforming the differential equation into an integral equation. The resulting integral equation serves to construct an iterative process to generate a sequences of functions, that converges to the exact solution if it exists. The method has been proved by many authors to be effective, and powerful mathematical tool for solving a wide class of scientific and engineering applications [1, 2, 16, 21, 22, 23, 15].

The rest of this paper is organised as follows. In section 2, the Picard's iteration method is briefly explained. In section 3, the parabolic inverse problem is transferred into a direct problem, to be able to employ the PIM method as illustrated in the subsection 3.2. Two numerical examples are given in section 4 to demonstrate and to valid the efficiency of the proposed approach, and the section 5 is reserved to the conclusion.

2. PICARD'S ITERATION METHOD

To illustrate the basic concept of the Picard's iteration method [20, 21], consider the following partial differential equation :

$$\frac{\partial}{\partial t} w(x, t) = N w(x, t) + g(x, t), \quad x \in \Omega, t > 0, \quad (5)$$

with initial condition

$$w(x, 0) = h(x). \quad (6)$$

where N is a nonlinear operator, g and h are given analytical functions. By integrating the both sides of equation (5) with respect to t , yields the following integral equation :

$$w(x, t) = w(x, 0) + \int_0^t (N w(x, \tau) + g(x, \tau)) d\tau. \quad (7)$$

The solution $w(x, t)$ of the equation (7) is given as the limit of the sequence of functions $\{w_n\}$ generated by the following iterative formula :

$$w_{n+1}(x, t) = w(x, 0) + \int_0^t (N w_n(x, \tau) + g(x, \tau)) d\tau. \quad (8)$$

By selecting the initial approximation $w_0(x, t)$ of the solution as $w_0(x, t) = w(x, 0) = h(x)$, the other successive approximations $w_n(x, t), k \geq 1$ of the solution $w(x, t)$ will be readily obtained upon using the iteration formula (8).

3. REFORMULATION AND SOLVING THE INVERSE PROBLEM BY PICARD'S ITERATION METHOD

3.1. Reformulation of the inverse problem

The equation (1) of the inverse problem (1)-(4) has two unknown functions $p(t)$ and $u(x, t)$, which is difficult to solve, hence to overcome the difficulty of the presence of the control function $p(t)$, we make the following transformations [3]. Set,

$$v(x, t) = u(x, t) \exp \left\{ \int_0^t -p(s) ds \right\}, \quad (9)$$

$$r(t) = \exp \left\{ \int_0^t -p(s) ds \right\}, \quad (10)$$

so we have :

$$u(x, t) = \frac{v(x, t)}{r(t)}, \quad (11)$$

$$p(t) = -\frac{\dot{r}(t)}{r(t)}, \quad (12)$$

where $\dot{r}(t) = \frac{dr(t)}{dt}$.

By employing the transformations (9) and (10) allow us to eliminate the unknown function $p(t)$ from equation (1) and the problem (1)-(4) becomes :

$$\frac{\partial v(x, t)}{\partial t} = \Delta v(x, t) + r(t) \phi(x, t), \quad x \in \Omega \subset \mathbb{R}^d, \quad 0 < t \leq T, \quad (13)$$

subject to the following initial and boundary conditions :

$$v(x, 0) = f(x), \quad x \in \Omega \quad (14)$$

$$v(x, t) = r(t) \psi(t), \quad x \in \partial\Omega, \quad 0 < t \leq T, \quad (15)$$

and

$$v(x_0, t) = r(t) E(t), \quad x \in \Omega, \quad 0 < t \leq T. \quad (16)$$

Assume that $E(t) \neq 0$, then we get :

$$r(t) = \frac{v(x_0, t)}{E(t)}. \quad (17)$$

Equation (13) has only one unknown function $v(x, t)$ which has a suitable form to apply the Picard's iteration method.

It's worth mentioning that the problem (13)-(16) is equivalent to the main problem (1)-(4) as shown by the following lemma.

Lemma 1. [3]

If the problem (1)-(4) has a solution (u, p) , then the problem (13)-(16) has a solution (v, r) defined by the transformation (9)-(12) and vice versa.

3.2. Application of the Picard's iteration method

To solve the obtained problem (13)-(16) by means of the Picard's iteration method (PIM), we transform the partial differential equation (13) into an integral equation as follow :

$$v(x, t) = v(x, 0) + \int_0^t \left(\Delta v(x, \tau) + \frac{v(x_0, \tau)}{E(\tau)} \phi(x, \tau) \right) d\tau, \quad (18)$$

by choosing a suitable initial approximation $v_0(x, t)$, the solution of the above equation (18) is given as the limit of the sequence of functions $v_n(x, t)$ generated by the following iteration formula :

$$v_{n+1}(x, t) = v(x, 0) + \int_0^t \left(\Delta v_n(x, \tau) + \frac{v_n(x_0, \tau)}{E(\tau)} \phi(x, \tau) \right) d\tau. \quad (19)$$

Once the solution (v, r) of the problem (13)-(16) is determined, the solution (u, p) of the main problem (1)-(4) can be easily deduced using the inverse transformations (11) and (12).

Remark 1.

The n th approximation $v_n(x, t)$ can be used as an approximate solution of the exact solution by considering the truncated series of the first n -terms approximations that satisfy the following criterion :

$$\|v_n(x, t) - v_{n-1}(x, t)\|_{L^2(Q)} = \left(\int_Q |v_n(x, t) - v_{n-1}(x, t)|^2 dQ \right)^{1/2} \leq \epsilon. \quad (20)$$

where $Q = \Omega \times [0, T]$ and $\epsilon > 0$ is a desired threshold.

In what follows, we will apply the PIM method to illustrate the strength of the method and to establish exact solution for two parabolic inverse problems in one and two dimensions.

4. NUMERICAL APPLICATION

In this section, we present two numerical examples to show the efficiency and the accuracy of the Picard's iteration method.

4.1. Example 1 : One dimensional inverse problem

Consider the following one dimensional inverse problem

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + p(t) u(x, t) + (\pi^2 + 2t) \exp(t) \cos(\pi x) + 2xt \exp(t), \quad (21)$$

subject to

$$u(x, 0) = x + \cos(\pi x), \quad (22)$$

$$u(0, t) = \exp(t), \quad (23)$$

$$u(1, t) = 0, \quad (24)$$

$$u(0.5, t) = E(t) = \frac{1}{2} \exp(t), \quad (25)$$

where the exact solution is :

$$u(x, t) = \exp(t) (x + \cos(\pi x)), \quad (26)$$

and

$$p(t) = 1 - 2t. \quad (27)$$

By making the transformations (9) and (10), yields the following initial boundary value problem :

$$\frac{\partial v(x, t)}{\partial t} = \frac{\partial^2 v(x, t)}{\partial x^2} + \frac{v(x_0, t)}{E(t)} (\pi^2 + 2t) \exp(t) \cos(\pi x) + 2xt \exp(t), \quad (28)$$

subject to

$$v(x, 0) = x + \cos(\pi x), \quad (29)$$

$$v(0, t) = r(t) \exp(t), \quad (30)$$

$$v(1, t) = 0, \quad (31)$$

applying the Picard's iteration method, we construct the following iteration formula :

$$v_{n+1}(x, t) = v(x, 0) + \int_0^t \left[\frac{\partial^2 v_n(x, \tau)}{\partial x^2} + \frac{v_n(x_0, \tau)}{E(\tau)} ((\pi^2 + 2\tau) \exp(\tau) \cos(\pi x) + 2x\tau \exp(\tau)) \right] d\tau, \quad (32)$$

By selecting the initial approximation $v_0(x, t) = v(x, 0) = x + \cos(\pi x)$, we get following successive approximations :

$$v_0(x, t) = x + \cos(\pi x), \quad (33)$$

$$v_1(x, t) = (x + \cos(\pi x)) (1 + t^2) + \text{noise terms}, \quad (34)$$

$$v_2(x, t) = (x + \cos(\pi x)) \left(1 + t^2 + \frac{t^4}{2} \right) + \text{noise terms}, \quad (35)$$

$$v_3(x, t) = (x + \cos(\pi x)) \left(1 + t^2 + \frac{t^4}{2} + \frac{t^6}{3!} \right) + \text{noise terms}, \quad (36)$$

$$\vdots \quad (37)$$

$$v_n(x, t) = (x + \cos(\pi x)) \left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \dots + \frac{t^{2n}}{n!} \right) + \text{noise terms}, \quad (38)$$

By removing the noise terms from v_1, v_2, \dots, v_n [26], we find :

$$v_n(x, t) = (x + \cos(\pi x)) \left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \dots + \frac{t^{2n}}{n!} \right), \quad (39)$$

recall that

$$v(x, t) = \lim_{n \rightarrow \infty} = (x + \cos(\pi x)) \exp(t^2), \quad (40)$$

obtained upon using the Taylor expansion. Using the inverse transformations (11) and (12), yields

$$u(x, t) = (x + \cos(\pi x)) \exp(t), \quad (41)$$

$$p(t) = 1 - 2t. \quad (42)$$

which correspond to the exact solution of the main problem.

4.2. Example 2 : Two-dimensional case

Consider the following two-dimensional inverse problem [24]

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + p(t) u(x, y, t) + \phi(x, t), \quad (43)$$

subject to

$$u(x, y, 0) = \sin\left(\frac{\pi}{4}(x + 2y)\right), \quad (44)$$

$$u(0, y, t) = \exp(t) \sin\left(\frac{\pi y}{2}\right), \quad (45)$$

$$u(1, y, t) = \exp(t) \sin\left(\frac{\pi}{4}(1 + 2y)\right), \quad (46)$$

$$u(x, 0, t) = \exp(t) \sin\left(\frac{\pi x}{4}\right), \quad (47)$$

$$u(x, 1, t) = \exp(t) \sin\left(\frac{\pi}{4}(x + 2)\right), \quad (48)$$

$$u(0.4, 0.2, t) = E(t) = \exp(t) \sin(0.2\pi), \quad (49)$$

where

$$\phi(x, t) = \left(\frac{5\pi^2}{16} - 5t\right) \exp(t) \sin\left(\frac{\pi}{4}(x + 2y)\right), \quad (50)$$

and the exact solution is :

$$u(x, y, t) = \exp(t) \sin\left(\frac{\pi}{4}(x + 2y)\right), \quad (51)$$

$$p(t) = 1 + 5t. \quad (52)$$

By means of the PIM method, and by selecting the initial approximation $v_0(t) = \sin\left(\frac{\pi}{4}(x+2y)\right)$, the successive approximations are given as follows :

$$v_0(x, y, t) = \sin\left(\frac{\pi}{4}(x+2y)\right), \quad (53)$$

$$v_1(x, y, t) = \sin\left(\frac{\pi}{4}(x+2y)\right) \left(1 - \frac{5t^2}{2}\right), \quad (54)$$

$$v_2(x, y, t) = \sin\left(\frac{\pi}{4}(x+2y)\right) \left(1 - \frac{5t^2}{2} + \frac{\left(\frac{5t^2}{2}\right)^2}{2}\right) + \text{noise terms}, \quad (55)$$

$$v_3(x, y, t) = \sin\left(\frac{\pi}{4}(x+2y)\right) \left(1 - \frac{5t^2}{2} + \frac{\left(\frac{5t^2}{2}\right)^2}{2} - \frac{\left(\frac{5t^2}{2}\right)^3}{6}\right) + \text{noise terms}, \quad (56)$$

⋮

$$v_n(x, y, t) = \sin\left(\frac{\pi}{4}(x+2y)\right) \left(1 - \frac{5t^2}{2} + \frac{\left(\frac{5t^2}{2}\right)^2}{2} + \dots + (-1)^n \frac{\left(\frac{5t^2}{2}\right)^n}{n!}\right) + \text{noise terms}, \quad (57)$$

removing the noise terms from v_2, v_3, \dots, v_n , and taking the limit, we find :

$$v(x, y, t) = \lim_{n \rightarrow \infty} v_n(x, y, t) = \sin\left(\frac{\pi}{4}(x+2y)\right) \exp\left(\frac{-5t^2}{2}\right), \quad (58)$$

obtained upon using the Taylor series. According to (11) and (12) yields the exact values of $u(x, y, t)$ and $p(t)$. The results are the same as with those obtained using the variational iteration method [24].

5. CONCLUSION

In this paper, the Picard's iteration method has been employed successfully to find the exact solution for the semi-linear inverse parabolic equation in one and two-dimensional cases. The method provides the solution in the form of an infinite

series which converges to the exact solution, by using a simple recursive scheme. Furthermore, the proposed method tackles the problem directly without any restrictive assumptions such as discretization, small perturbation as the homotopy perturbation method, and does not require the evaluation of the Lagrange multiplier and the restrictive variation like the variational iteration method, which means that the PIM method reduce the computational size. The proposed method is tested by two numerical examples and the obtained results show that PIM method is a powerful mathematical tool for solving this kind of problems.

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