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NEIMARK-SACKER BIFURCATION AND CHAOS CONTROL IN A MODIFIED NICHOLSON-BAILEY MODEL

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Abstract. This article deals with the study of some qualitative properties of a discrete-time host-parasitoid model. The present model is a modification of classical Nicholson-Bailey model and the modification is based on a particular case of Hassel-Varley model in which the interaction between parasitoids is taken in such a way that the searching area per parasitoid is inversely proportional to the density of parasitoid. Moreover, it is shown that there exists Neimark-Sacker bifurcation for the unique positive steady-state of given system. Confirmation of complexity and chaotic behavior are verified by plotting largest Lyapunov exponents. Furthermore, feedback strategy is introduced in order to stabilize the unstable equilibrium.

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Keywords: Hassel-Varley model, Neimark-Sacker bifurcation, chaos control.

1. Introduction

One of the most earliest application to a biological system is the discrete-time Nicholson and Bailey model which involved two insects, a host and a parasitoid. The development of the model goes to Nicholson and Bailey [1]. The parasitoid are the organism that kills the host organism and live freely as an adult stage but lays eggs in the larvae or pupae of the host. Host that are not parasitized give rise to their own offspring. The host that are completely parasitized die but the eggs of the parasitoid may be survive to the next generation. There are many unrealistic assumptions in such host-parasitoid systems such as a constant reproductivity of the host, a constant searching efficiency, and a homogeneous environment are unrealistic assumptions. Positive equilibrium state can be stabilized with addition of some realistic assumptions. In original Nicholson-Bailey model for a long time survival of parasitoid and its host many modification have been seen in literature.

The general host-parasitoid model is governed by the following two-dimensional discrete-time dynamical system:

$$\begin{aligned} N_{t+1} &= rN_t f\left(N_t, P_t\right), \\ P_{t+1} &= cN_t(1 - f\left(N_t, P_t\right)), \end{aligned} \tag{1}$$

where N_t is the density of host species in generation t, P_t is the density of parasitoid species in generation t, $f(N_t, P_t)$ is the fraction of hosts that are not parasitized, r is the number of eggs laid by a host that survive through the larvae, pupae, and adult stages, and c is the number of eggs laid by a parasitoid on a single host that survive through larvae, pupae, and adult stages. The function f can be interpreted as the probability that each individual host escapes the parasitoid, so that the complementary term $1-f$ in the second equation is the probability of being parasitized. Moreover, as $N_t = 0$ implies that $P_{t+1} = 0$, *i.e.*, the parasitoids cannot survive without the hosts. This is one reason why parasitoids are good biological control agents. In Nicholson-Bailey model it is assumed that $f(N_t, P_t) = \exp(-AP_t)$, where A is the searching efficiency of the parasitoid which is the probability that a given parasitoid will encounter a given host during its searching lifetime. Thus (1) reduces to the following Nicholson and Bailey model:

$$\begin{cases} N_{t+1} = rN_t \exp(-AP_t), \\ P_{t+1} = cN_t \left(1 - \exp(-AP_t)\right). \end{cases} \tag{2}$$

At the rate of low parasitoid densities the oscillatory behavior occurs in Nicholson-Bailey model, then the host population behaves approximately as follow:

$$N_{t+1} = rN_t,$$

that is, its growth remains unchecked. Moreover, this unrealistic behavior suddenly results increase in number of parasitoids until the host are killed by attack. To take into account the effect of parasitoid interference on the hostparasitoid model, the equation proposed by Hassell and Varley [6] can be used $A = aP_t^{-m}$, where P_t is the number of parasitoids in generation t, a stands for the quest constant, which represents the searching efficiency when $P_t = 1$, m is the mutual interference constant. Therefore, classical Nicholson-Bailey model can be modified as follows:

$$\begin{aligned} N_{t+1} &= rN_t \exp(-aP_t^{1-m}), \\ P_{t+1} &= cN_t \left(1 - \exp(-aP_t^{1-m})\right). \end{aligned} \tag{3}$$

The greater the values of mutual interference constant m in the model (3) yields stronger the stability chance, and as $m \to 0$ it behave like Nicholson-Bailey model, *i.e.*, unstable intersection will arise. For simplicity, we take $c = 1$ and $m = \frac{1}{2}$, then (3) reduces to the following special form of Hassell-Varley model [7]:

$$\begin{aligned} N_{t+1} &= rN_t \exp\left(-a\sqrt{P_t}\right), \\ P_{t+1} &= N_t\left(1 - \exp\left(-a\sqrt{P_t}\right)\right). \end{aligned} \tag{4}$$

In remaining discussion, we study the bifurcation analysis for confirmation of Neimark-Sacker bifurcation for positive steady-state of system (4), and feedback control strategy is used to stabilize the chaotic orbits at an unstable steady-state. For more detail of some interesting population models both in differential equations as well as in difference equations, we refer the interested reader to [2, 3, 4, 5]. Moreover, for some interesting results related to the qualitative behavior of difference equations, we refer the reader to [8, 9, 10, 11, 12].

2. Existence and stability of positive equilibrium

Let (N_*, P_*) be equilibrium point of system (4), then we have the following algebraic equations:

$$\begin{aligned} N_* &= rN_* \exp\left(-a\sqrt{P_*}\right), \\ P_* &= N_*\left(1 - \exp\left(-a\sqrt{P_*}\right)\right). \end{aligned}$$

Neglecting the trivial equilibrium $(N_*, P_*) = (0, 0)$, we have the following system of algebraic equations:

$$\begin{aligned} 1 &= r \exp\left(-a\sqrt{P_*}\right), \\ P_* &= N_*\left(1 - \exp\left(-a\sqrt{P_*}\right)\right). \end{aligned} \tag{5}$$

Solving system (5), we obtain $P_* = \left(\frac{\ln r}{a}\right)^2$, and $N_* = \frac{P_*}{1-\exp\left(-a\sqrt{P_*}\right)} = \frac{rP_*}{r-1}$. In order to confirm the positivity of $(N_*, P_*) = \left(\frac{r(\ln r)^2}{a^2(r-1)}, \left(\frac{\ln r}{a}\right)^2\right)$, it is assumed that $r > 1$. The Jacobian matrix $J(N_*, P_*)$ evaluated at the unique positive equilibrium point $(N_*, P_*) = \left(\left(\frac{r(\ln r)^2}{a^2(r-1)}, \left(\frac{\ln r}{a}\right)^2\right)\right)$ of system (4) is given by

$$J(N_*, P_*) = \begin{bmatrix} 1 & -\frac{r\ln r}{2(r-1)} \\ \frac{r-1}{r} & \frac{\ln r}{2(r-1)} \end{bmatrix}.$$

The characteristic polynomial of the Jacobian matrix $J(N_*, P_*)$ is given by:

$$\mathbb{P}(\lambda) = \lambda^2 - \left(1 + \frac{\ln r}{2(r-1)}\right)\lambda + \frac{r \ln r}{2(r-1)}. \tag{6}$$

In order to study the stability analysis of unique positive equilibrium point of system (4), we have the following result.

Lemma 2.1. Assume that $\mathbb{F}(\lambda) = \lambda^2 - A\lambda + B$, and $\mathbb{F}(1) > 0$ with λ_1, λ_2 are root of $\mathbb{F}(\lambda) = 0$. Then the following results hold true:

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $\mathbb{F}(-1) > 0$ and $B < 1$.

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$, or $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if and only if $\mathbb{F}(-1) < 0$.

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $\mathbb{F}(-1) > 0$ and $B > 1$.

(iv) $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $\mathbb{F}(-1) = 0$ and $B \neq 0, 2$.

(v) λ_1 and λ_2 are complex and $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if $A^2 - 4B < 0$ and $B = 1$.

Suppose that λ_1 and λ_2 be eigenvalue of the Jacobian matrix $J(N_*, P_*)$ evaluated at unique positive equilibrium point of system (4), *i.e.*, roots of the characteristic polynomial (6). Then unique positive equilibrium point (N_*, P_*) of (4) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ and thus the sink is locally asymptotic stable equilibrium point. (N_*, P_*) is known as source or repeller if $|\lambda_1| > 1$ and $|\lambda_2| > 1$ and thus a source is always unstable. (N_*, P_*) is called a saddle point if $|\lambda_1| < 1$ and $|\lambda_2| > 1$, or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) and thus a saddle point is also unstable. (N_*, P_*) is known as non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

According to [7], the unique positive equilibrium point of (4) is locally asymptotically stable if $1 < r < r_0$, where r_0 be unique positive root of the equation $r \ln r - 2r + 2 = 0$, then we have $r_0 \approx 4.9215536345675055$. From (6), $\mathbb{P}(1) = \frac{1}{2}\ln r > 0$, so applying Lemma 2.1 the following result gives an explicit condition on r such that the roots of (6) are complex with unit modulus.

Lemma 2.2. The roots of characteristic polynomial (6) are complex with modulus one if $r = r_1$, where r_1 is unique positive root of the equation $r \ln r - 2r + 2 = 0$ in $]1, \infty[$ such that $r_1 \approx 4.9215536345675055$.

Proof. According to part (v) of Lemma 2.1, the roots of (6) are complex and lie on unit circle under the conditions $\left|\left(1 + \frac{\ln r}{2(r-1)}\right)\right| < 2$ and $\frac{r \ln r}{2(r-1)} = 1$, or equivalently we have

$$\frac{\ln r}{2(r-1)} < 1,\ r\ln r - 2r + 2 = 0. \tag{7}$$

Assume that $\Theta(r) = r\ln r - 2r + 2$ and $\Psi(r) = \frac{\ln r}{2(r-1)} - 1$. Then it follows that $\Theta(r) < 0$ for all $r \in]1, r_1[$, $\Theta(r) > 0$ for all $r \in]r_1, \infty[$, $\Theta'(r) > 0$ for $r > 2.718281828459045$, where $r_1 \approx 4.9215536345675055$ and $\Theta(4.9215536345675055) \approx 0$. Thus $r_1 \approx 4.9215536345675055$ be unique positive root of $\Theta(r) = 0$ in $]1, \infty[$. On the other hand, $\lim_{r\to 1} \Psi(r) = -\frac{1}{2}$, $\lim_{r\to\infty} \Psi(r) = -1$ and $\Psi'(r) = \frac{r-1-r\ln r}{2r(r-1)^2} < 0$ for all $r \in]1, \infty[$ because $r - 1 < r\ln r$ for all $r > 1$.

3. Neimark-Sacker bifurcation

We investigate the parametric conditions for Neimark-Sacker bifurcation for the unique positive equilibrium point (N_*, P_*) of system (4) by taking r as bifurcation parameter. Similar results found in [13, 14, 15, 16, 17, 18, 19, 20]. Under the influence of Neimark-Sacker bifurcation dynamically invariant closed curves are produced. The characteristic equation of the linearized system of (4) has two complex conjugate roots with modulus 1 under the conditions (7) of Lemma 2.2. Next, we assume that

$$\Omega_{NS} = \left\{(a, r) : \frac{\ln r}{2(r-1)} < 1,\ r\ln r - 2r + 2 = 0,\ r > 1,\ a > 0\right\}.$$

Then it follows from Lemma 2.2 that

$$\Omega_{NS} = \{(a, r) : r \approx 4.9215536345675055,\ a > 0\}.$$

Choosing the parameters (a, r_1) in an arbitrary fashion from the set Ω_{NS} with $r_1 \approx 4.9215536345675055$. System (4) can be described by the following two-dimensional map:

$$\begin{pmatrix} N \\ P \end{pmatrix} \to \begin{pmatrix} rNe^{-a\sqrt{P}} \\ N(1 - e^{-a\sqrt{P}}) \end{pmatrix}. \tag{8}$$

It is easy to see that map (8) has a unique positive fixed point $\left(\frac{r(\ln r)^2}{a^2(r-1)}, \left(\frac{\ln r}{a}\right)^2\right)$ which is also the unique positive equilibrium point of system (4). Since $(a, r_1) \in \Omega_{NS}$ and $r_1 \approx 4.9215536345675055$. Taking $\tilde{r}$ as bifurcation parameter and considering the perturbation of (8) as follows:

$$\begin{pmatrix} N \\ P \end{pmatrix} \to \begin{pmatrix} (r_1 + \tilde{r})Ne^{-a\sqrt{P}} \\ N(1 - e^{-a\sqrt{P}}) \end{pmatrix}, \tag{9}$$

where $|\tilde{r}| \ll 1$ is taken as small perturbation parameter. Next we consider the transformations $u = N - \frac{r(\ln r)^2}{a^2(r-1)}$, $v = P - \left(\frac{\ln r}{a}\right)^2$ so that map (8) is transferred into the following form:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}, \tag{10}$$

where

$$f_1(u,v) = a_{13}uv + a_{14}v^2 + b_1 uv^2 + b_2 v^3 + +O\left((|u|+|v|)^4\right),$$
$$f_2(u,v) = a_{23}uv + a_{24}v^2 + d_1 uv^2 + d_2 v^3 + O\left((|u|+|v|)^4\right),$$
$$a_{11} = 1,\ a_{12} = -\frac{(r_1+\tilde{r})\ln(r_1+\tilde{r})}{2(r_1+\tilde{r}-1)},\ a_{21} = \frac{r_1+\tilde{r}-1}{r_1+\tilde{r}},\ a_{22} = \frac{\ln(r_1+\tilde{r})}{2(r_1+\tilde{r}-1)},$$
$$a_{13} = -\frac{a^2}{2\ln(r_1+\tilde{r})},\ a_{14} = \frac{a^2(r_1+\tilde{r})\left(1+\ln(r_1+\tilde{r})\right)}{8\ln(r_1+\tilde{r})(r_1+\tilde{r}-1)},\ b_1 = \frac{a^4\left(1+\ln(r_1+\tilde{r})\right)}{8\left(\ln(r_1+\tilde{r})\right)^3},$$
$$b_2 = -\frac{a^4(r_1+\tilde{r})\left[(\ln(r_1+\tilde{r}))^2 + 3\ln(r_1+\tilde{r}) + 3\right]}{48(r_1+\tilde{r}-1)\left(\ln(r_1+\tilde{r})\right)^3},\ a_{23} = \frac{a^2}{2(r_1+\tilde{r})\ln(r_1+\tilde{r})},$$
$$a_{24} = -\frac{\ln(r_1+\tilde{r})+1}{8\ln(r_1+\tilde{r})(r_1+\tilde{r}-1)},\ d_1 = -\frac{a^4(\ln(r_1+\tilde{r})+1)}{8(r_1+\tilde{r})(\ln(r_1+\tilde{r}))^3},$$
$$d_2 = -\frac{a^4\left[(\ln(r_1+\tilde{r}))^2 + 3\ln(r_1+\tilde{r}) + 3\right]}{48(r_1+\tilde{r}-1)\left(\ln(r_1+\tilde{r})\right)^3}.$$

The characteristic equation of Jacobian matrix of linearized system of (10) evaluated at the equilibrium $(0,0)$ can be written as follows:

$$\lambda^2 - p(\tilde{r})\lambda + q(\tilde{r}) = 0, \tag{11}$$

where

$$p(\tilde{r}) = 1 + \frac{\ln(r_1+\tilde{r})}{2(r_1+\tilde{r}-1)},\ q(\tilde{r}) = \frac{(r_1+\tilde{r})\ln(r_1+\tilde{r})}{2(r_1+\tilde{r}-1)}.$$

Since $(a, r_1) \in \Omega_{NS}$, the roots of (11) are conjugate complex numbers λ_1, λ_2 with $|\lambda_1| = |\lambda_2| = 1$. Then it follows that:

$$\lambda_1, \lambda_2 = \frac{p(\tilde{r})}{2} \pm \frac{\iota}{2}\sqrt{4q(\tilde{r}) - p^2(\tilde{r})}.$$

Then we obtain

$$\left(\frac{d|\lambda_1|}{d\tilde{r}}\right)_{\tilde{r}=0} = \left(\frac{d|\lambda_2|}{d\tilde{r}}\right)_{\tilde{r}=0} = \frac{r_1 - 1 - \ln r_1}{2\sqrt{2}(r_1-1)^{\frac{3}{2}}\sqrt{r_1\ln r_1}} \approx 0.037843691260996154 > 0.$$

Furthermore, we have $p(0) = 1 + \frac{\ln r_1}{2(r_1-1)} \neq 0, 1$. Moreover, $(a, r_1) \in \Omega_{NS}$ implies that $-2 < p(0) < 2$. Thus $p(0) \neq \pm 2, 0, 1$ gives $\lambda_1^m, \lambda_2^m \neq 1$ for all $m = 1, 2, 3, 4$ at $\tilde{r} = 0$. Hence, roots of (11) do not lie in the intersection of the unit circle with the coordinate axes when $\tilde{r} = 0$ and $r_1 \approx 4.9215536345675055$.

In order to obtain the canonical form of (10) at $\tilde{r} = 0$, we take $\alpha = \frac{p(0)}{2}$, $\beta = \frac{1}{2}\sqrt{4q(0) - p^2(0)}$ and consider the following mapping:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ \alpha - a_{11} & -\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{12}$$

Under mapping (12), the canonical form of (10) can be expressed as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \widetilde{f}(x,y) \\ \widetilde{g}(x,y) \end{pmatrix}, \tag{13}$$

where

$$\widetilde{f}(x,y) = \frac{a_{13}}{a_{12}}uv + \frac{a_{14}}{a_{12}}v^2 + \frac{b_1}{a_{12}}uv^2 + \frac{b_2}{a_{12}}v^3,$$

$$\widetilde{g}(x,y) = \left(\frac{a_{13}(\alpha - a_{11})}{\beta a_{12}} - \frac{a_{23}}{\beta}\right) uv + \left(\frac{a_{14}(\alpha - a_{11})}{\beta a_{12}} - \frac{a_{24}}{\beta}\right) v^2$$
$$+ \left(\frac{b_1(\alpha - a_{11})}{\beta a_{12}} - \frac{d_1}{\beta}\right) uv^2 + \left(\frac{b_2(\alpha - a_{11})}{\beta a_{12}} - \frac{d_2}{\beta}\right) v^3 + O\left((|x| + |y|)^4\right),$$

$u = a_{12}x$ and $v = (\alpha - a_{11})x - \beta y$. Next, we define the following nonzero real number:

$$L = \left(\left[-Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\xi_{20}\xi_{11}\right) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\lambda_2\xi_{21})\right]\right)_{\tilde{r}=0},$$

where

$$\xi_{20} = \frac{1}{8}\left[\widetilde{f}_{xx} - \widetilde{f}_{yy} + 2\widetilde{g}_{xy} + \iota\left(\widetilde{g}_{xx} - \widetilde{g}_{yy} - 2\widetilde{f}_{xy}\right)\right],$$

$$\xi_{11} = \frac{1}{4}\left[\widetilde{f}_{xx} + \widetilde{f}_{yy} + \iota\left(\widetilde{g}_{xx} + \widetilde{g}_{yy}\right)\right],$$

$$\xi_{02} = \frac{1}{8}\left[\widetilde{f}_{xx} - \widetilde{f}_{yy} - 2\widetilde{g}_{xy} + \iota\left(\widetilde{g}_{xx} - \widetilde{g}_{yy} + 2\widetilde{f}_{xy}\right)\right],$$

$$\xi_{21} = \frac{1}{16}\left[\widetilde{f}_{xxx} + \widetilde{f}_{xyy} + \widetilde{g}_{xxy} + \widetilde{g}_{yyy} + \iota\left(\widetilde{g}_{xxx} + \widetilde{g}_{xyy} - \widetilde{f}_{xxy} - \widetilde{f}_{yyy}\right)\right].$$

Due to above analysis, we have the following conclusion.

Theorem 3.1. Assume that $r_1 \approx 4.9215536345675055$ holds and $L \neq 0$, then system (4) undergoes Neimark-Sacker bifurcation at the unique positive equilibrium point (N_*, P_*) when the parameter r varies in a small neighborhood of $r_1 \approx 4.9215536345675055$. Furthermore, if $L < 0$, then an attracting invariant closed curve bifurcates from the equilibrium point for $r > r_1$, and if $L > 0$, then a repelling invariant closed curve bifurcates from the equilibrium point for $r < r_1$.

4. Chaos control

In order to stabilize the unstable equilibrium of system (4) we introduce feedback strategy [21, 22]. Consider the following controlled system corresponding to (4):

$$\begin{aligned} N_{t+1} &= rN_t \exp\left(-a\sqrt{P_t}\right) - U_t, \\ P_{t+1} &= N_t\left(1 - \exp\left(-a\sqrt{P_t}\right)\right), \end{aligned} \tag{14}$$

where $U_t = K\left(N_t - \frac{r(\ln r)^2}{a^2(r-1)}\right) + W\left(P_t - \left(\frac{\ln r}{a}\right)^2\right)$ is taken as controlling force and K and W are the feedback gains. The Jacobian matrix of controlled system (14) evaluated at unique positive equilibrium point $(N_*, P_*) = \left(\frac{r(\ln r)^2}{a^2(r-1)}, \left(\frac{\ln r}{a}\right)^2\right)$ is given as:

$$C_J(N_*, P_*) = \begin{bmatrix} 1-K & -\frac{r\ln r}{2(r-1)} - W \\ \frac{r-1}{r} & \frac{\ln r}{2(r-1)} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix $C_J(N_*, P_*)$ is given by

$$\lambda^2 - \left(1 + \frac{\ln r}{2(r-1)} - K\right)\lambda + \frac{\ln r}{2(r-1)}(r-K) + \frac{W(r-1)}{r} = 0. \tag{15}$$

Let λ_1 and λ_2 be the roots of (15), then we have

$$\lambda_1 + \lambda_2 = 1 + \frac{\ln r}{2(r-1)} - K, \tag{16}$$

and

$$\lambda_1\lambda_2 = \frac{\ln r}{2(r-1)}(r-K) + \frac{W(r-1)}{r}. \tag{17}$$

In order to obtain the lines of marginal stability we must assume that $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. Under these restrictions the absolute values of λ_1 and λ_2 must be less than 1. First we suppose that $\lambda_1\lambda_2 = 1$, then it follows from (17) that:

$$L_1 : \frac{\ln r}{2(r-1)}K - \frac{r-1}{r}W = \frac{r\ln r}{2(r-1)} - 1. \tag{18}$$

Next, suppose that $\lambda_1 = 1$ and from (16) and (17), we obtain the following line for marginal stability:

$$L_2 : \left(\frac{\ln r - 2r + 2}{2(r-1)}\right) K - \frac{r-1}{r} W = \frac{\ln r}{2(r-1)}. \tag{19}$$

Finally, if $\lambda_1 = -1$, then from (16) and (17) we get

$$L_3 : \left(\frac{\ln r + 2r - 2}{2(r-1)}\right) K - \frac{r-1}{r} W = 2 + \frac{(1+r)\ln r}{2(r-1)}. \tag{20}$$

It is easy to see that stable eigenvalues lie within the triangular region bounded by the straight lines L_1, L_2 and L_3 (see Fig. 1).

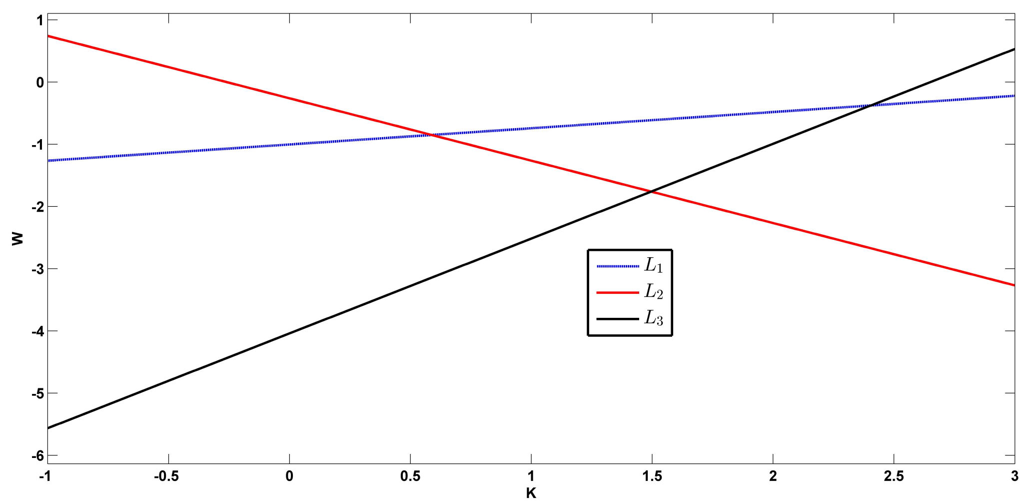
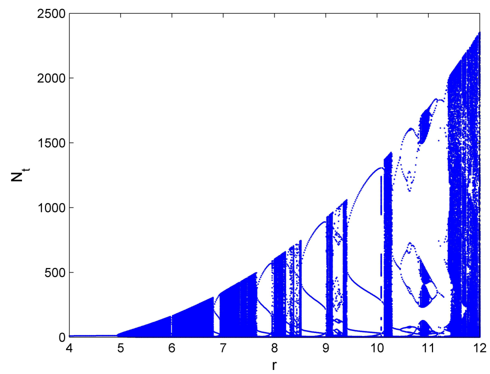


Figure 1: Stability region of system (14) for $a = 0.95$ and $r = 4.8$

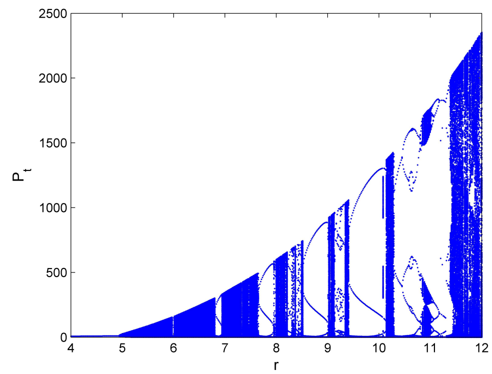
5. Numerical simulation and discussion

Example 5.1. First, we consider system (4) such that $a = 0.5$, $r \in [4, 12]$ and with initial conditions $N_0 = 12.7$, $P_0 = 10.1$. In this case, system (4) undergoes Neimar-Sacker bifurcation and bifurcation diagrams are shown in Fig. 2. From Fig. 2a and Fig. 2b, it easy to see that both populations undergo Neimark-Sacker bifurcation when r varies in the interval $[4, 12]$. The corresponding maximum Lyapunov exponents (MLE) are plotted in Fig. 2c. In Fig. 3 the local amplification of Fig.

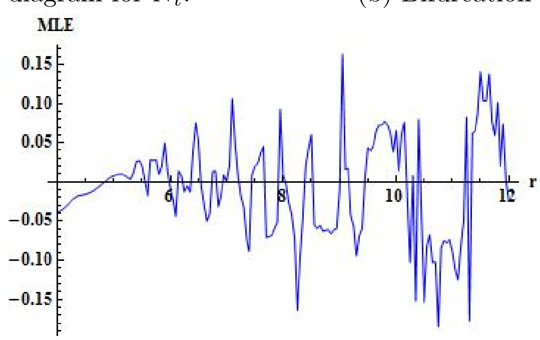
2 is shown when r is taken in a subinterval $[4.4, 6]$ of $[4, 12]$. Furthermore, in Fig. 4 another local amplification of Fig. 2 is shown when r is taken in a subinterval $[10, 12]$ of $[4, 12]$. In Fig. 5, phase portraits of system (4) with varying r as $r = 4.89$, $r = 4.9$, $r = 4.921553635$, $r = 4.93$, $r = 4.95$ and $r = 4.98$ while keeping $a = 0.5$ and initial conditions $N_0 = 12.7$, $P_0 = 10.1$ are shown in Figures 5a, 5b, 5c, 5d, 5e and 5f, respectively.



(a) Bifurcation diagram for N_t.

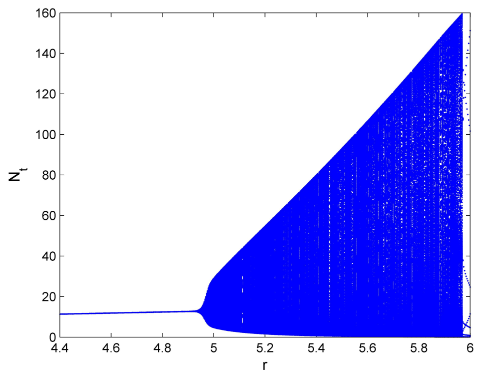


(b) Bifurcation diagram for P_t.

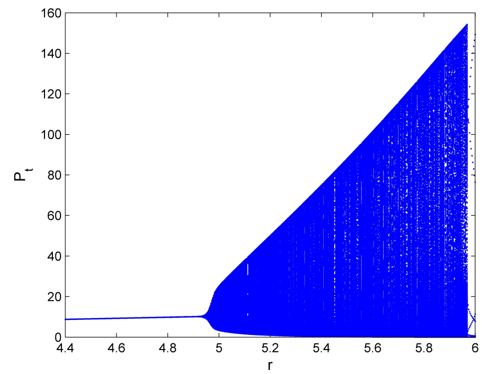


(c) Maximum Lypunov exponents

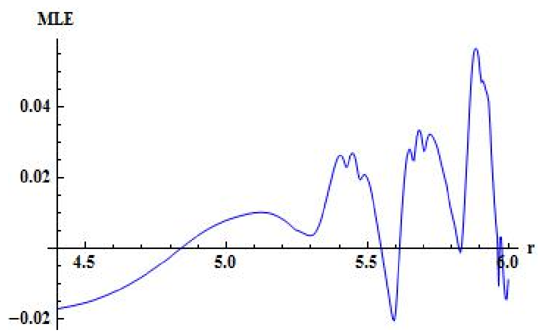
Figure 2: Plots for the system (4)



(a) Local amplification corresponding to Fig. 2a.

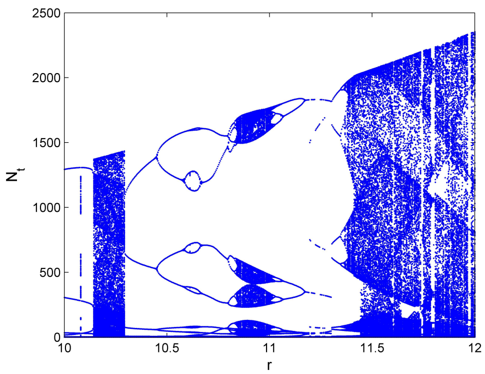


(b) Local amplification corresponding to Fig. 2b.

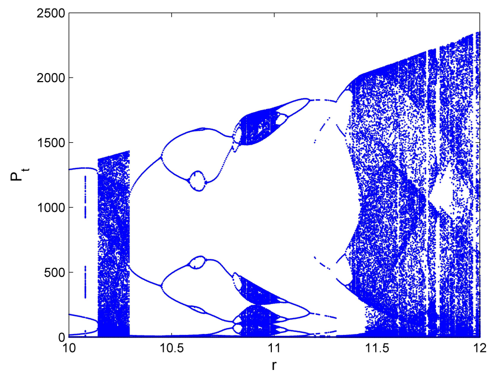


(c) Local amplification corresponding to Fig. 2c

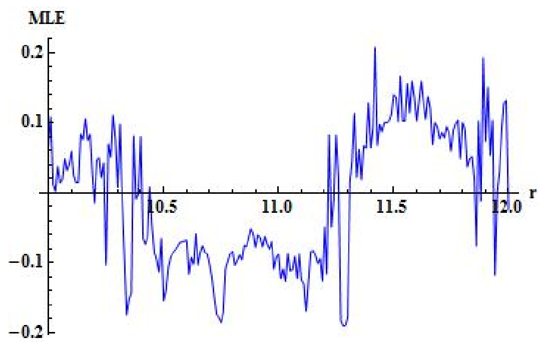
Figure 3: Local amplification corresponding to Fig. 2 for $r \in [4.4, 6]$.



(a) Local amplification corresponding to Fig. 2a.

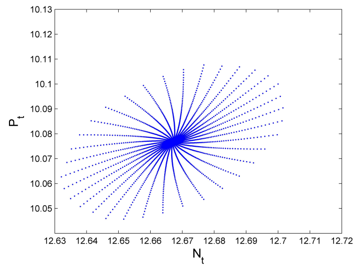


(b) Local amplification corresponding to Fig. 2b.

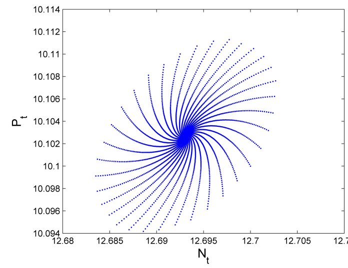


(c) Local amplification corresponding to Fig. 2c

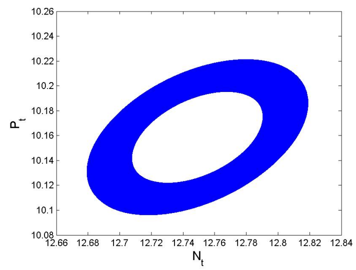
Figure 4: Local amplification corresponding to Fig. 2 for $r \in [10, 12]$.



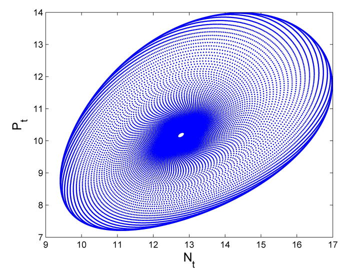
(a) Phase portrait for $r = 4.89$



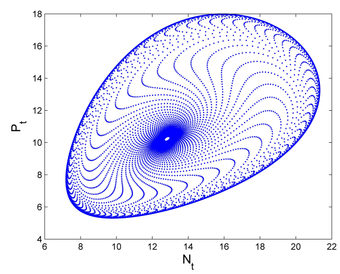
(b) Phase portrait for $r = 4.9$



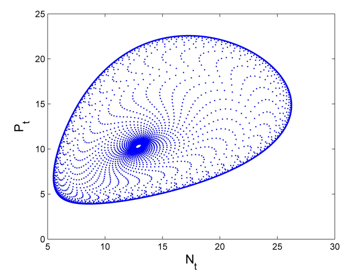
(c) Phase portrait for $r = 4.921553635$



(d) Phase portrait for $r = 4.93$



(e) Phase portrait for $r = 4.95$

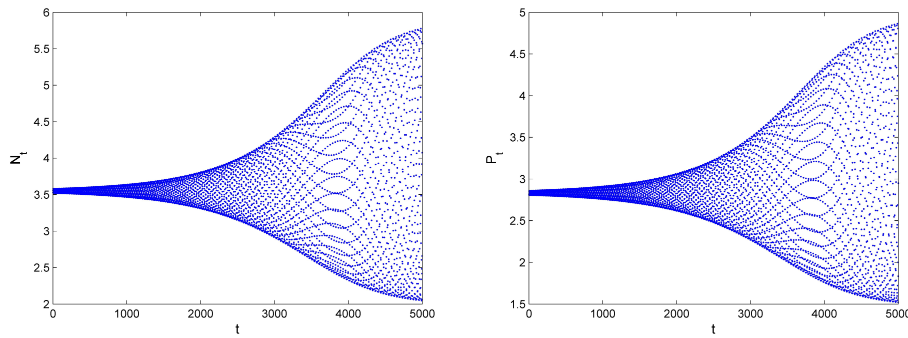


(f) Phase portrait for $r = 4.98$

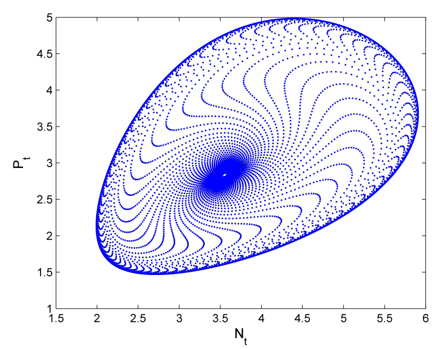
Figure 5: Phase portraits of system (4) for different values of r with $a = 0.5$ and initial conditions $N_0 = 12.7,\ P_0 = 10.1$

Example 5.2. Next, we take $a = 0.95$, $r = 4.95$ and initial conditions $(N_0, P_0) = (3.53, 2.81)$. In this case, the unique positive equilibrium point $(3.5519621500321303, 2.834394038914528)$ of system (4) is unstable. In Fig. 5, plot of N_t is shown in Fig. 6a, plot of P_t is shown in Fig. 6b and phase portrait is shown in Fig. 6c for system (4). In order to make the equilibrium point $(3.5519621500321303, 2.834394038914528)$ locally asymptotically stable, we use the state feedback control strategy. For this, we consider the corresponding controlled system (14) in which the feedback controlling force is taken as $U_t = K(N_t -$

$3.5519621500321303) + W(P_t - 2.834394038914528)$ with feedback gains $K = 0.01$ and $W = -0.02$. In Fig. 6, plot of N_t is shown in Fig. 7a, plot of P_t is shown in Fig. 7b and phase portrait is shown in Fig. 7c for system (14).

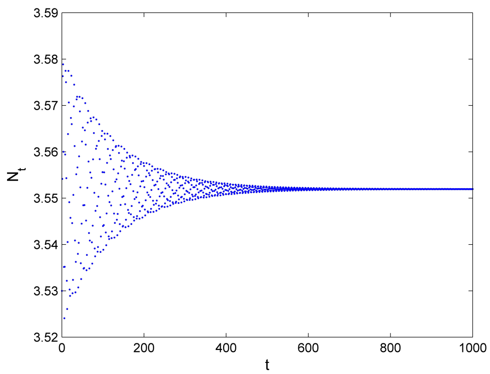


(a) Plot of N_t for system (4) (b) Plot of P_t for system (4)

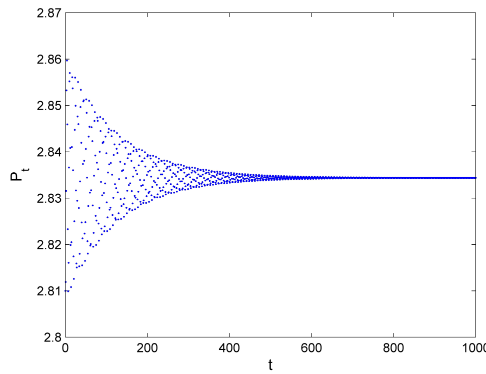


(c) Phase portrait for system (4)

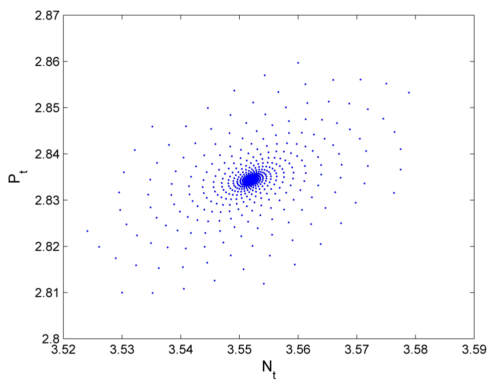
Figure 6: Plots for system (4) with $a = 0.95$, $r = 4.95$ and initial conditions $(N_0, P_0) = (3.53, 2.81)$



(a) Plot of N_t for system (14)



(b) Plot of P_t for system (14)



(c) Phase portrait for system (14)

Figure 7: Plots for system (14) with $a = 0.95$, $r = 4.95$, $K = 0.01$, $W = -0.02$ and initial conditions $(N_0, P_0) = (3.53, 2.81)$

References

[1] V. A. Bailey, A. J. Nicholson, *The balance of animal populations*, I. Proc. Zool. Soc. Lond., 3(1935), 551–598.

[2] Linda J.S. Allen, *An Introduction to Mathematical Biology*, Prentice Hall, (2007).

[3] F. Brauer, C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer, (2000).

[4] L. Edelstein-Keshet,*Mathematical Models in Biology*, McGraw-Hill, (1988).

[5] S. Elaydi, *Discrete chaos: With applications in science and engineering*, second edition, Chapman and Hall/CRC, (2008).

[6] M. P. Hassell, G. C. Varley, *New inductive population model for insect parasites and its bearing on biological control*, Nature, 223(1969), 1133–1137.

[7] M. N. Qureshi, A. Q. Khan, Q. Din, *Asymptotic behavior of a Nicholson-Bailey model*, Adv. Differ. Equ., 2014(2014), 62.

[8] Q. Din, *Dynamics of a host-pathogen model with constant mortality rate*, Nonlinear Anal. Model. Control, 22(2)(2017), 173–187.

[9] Q. Din, *Global stability of a population model*, Chaos Soliton Fract., 59(2014), 119–128.

[10] Q. Din, *Global behavior of a plant-herbivore model*, Adv. Differ. Equ., 2015(2015), 1–12.

[11] Q. Din, *Global behavior of a host-parasitoid model under the constant refuge effect*, Appl. Math. Model., 40(4)(2016), 2815–2826.

[12] Q. Din, M.A. Khan, U. Saeed, *Qualitative Behaviour of Generalised Beddington Model*, Z. Naturforsch. A, 71(2)(2016), 145–155.

[13] Q. Din, *Global stability of Beddington model*, Qual. Theor. Dyn. Syst., (2016), DOI: 10.1007/s12346-016-0197-9, 25 pages.

[14] Q. Din, *Global stability and Neimark-Sacker bifurcation of a host-parasitoid model*, Int. J. Syst. Sci., 48(2017), 1194–1202.

[15] Q. Din, *Neimark-Sacker bifurcation and chaos control in Hassell-Varley model*, J. Differ. Equations Appl., (2017), DOI:10.1080/10236198.2016.1277213, 22 pages.

[16] Q. Din, Ö.A. Gümüş, H. Khalil, *Neimark-Sacker Bifurcation and Chaotic Behaviour of a Modified Host-Parasitoid Model*, Z. Naturforsch. A, 72(1)(2017), 25–37.

[17] Q. Din, *Complexity and chaos control in a discrete-time prey-predator model*, Commun. Nonlinear Sci. Numer. Simul., 49(2017), 113–134.

[18] Z. He, X. Lai, *Bifurcation and chaotic behavior of a discrete-time predator–prey system*, Nonlinear Anal. RWA, 12(2011), 403–417.

[19] X. Liu, D. Xiao, *Complex dynamic behaviors of a discrete-time predator–prey system*, Chaos Soliton Fract., 32(2007), 80–94.

[20] Z. Jing, J. Yang, *Bifurcation and chaos in discrete-time predator–prey system*, Chaos Soliton Fract., 27(2006), 259–277.

[21] G. Chen, X. Dong, *From Chaos to Order: Perspectives, Methodologies, and Applications*, World Scientific, Singapore, (1998).

[22] S. N. Elaydi, *An Introduction to Difference Equations*, 3rd ed., Springer-Verlag, New York, (2005).

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