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CARLSON TYPE INEQUALITY FOR CHOQUET-LIKE EXPECTATION

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Abstract. Carlson's inequality is one of the most important inequalities in probability, measure theory and analysis. The problem of finding a sharp inequality of Carlson type inequality for Choquet-like integral based on the multiplication operator has led to a challenging and an interesting subject for researchers. In this paper, we give a Carlson type inequality based on pseudo-analysis for two classes of Choquet-like integrals as generalizations of Choquet integral and Sugeno integral. In the first class, pseudo-operations are defined by a continuous strictly increasing function g. Another class concerns the Choquet-like integrals based on the operator sup and the pseudo-multiplication $\odot$.

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1. Introduction

Integral inequalities are an important aspect of the classical mathematical analysis. Some well-known inequalities such as the Jensen's inequality [15] and the Corlson's inequality [2, 7] play important roles not only in the theoretical area but also in application. The Sugeno integral was introduced by Sugeno [22] and then exploited by many authors [5, 6, 8, 14, 16, 17]. Recently, Romn-Flores et al. and others proved some inequalities for the Sugeno integral, see [9, 13, 18, 19]. Notice that the Sugeno integral is not an extension of the Lebesgue integral. On the other hand, some other integrals such as the Choquet integral [4, 10] and the Sugeno-Murofushi integral [23] are extensions of the Lebesgue integral but not of the Sugeno integral. Thus it would be an interesting topic to generalize an inequality from the frame of Lebesgue integral (or Sugeno integral) to that of some integrals which contain the Lebesgue integral and Sugeno integral as special cases [12].

The aim of this paper is to generalize Carlson's inequality to the frame of the Choquet-like integral [12].

The Carlson's inequality for the Lebesgue integral is of the form

$$\int_0^\infty f(x)dx \leqslant \sqrt{\pi}\left(\int_0^\infty f^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^\infty x^2 f^2(x)dx\right)^{\frac{1}{4}} \tag{1}$$

Michal Boczek and Marek Kaluszka [3] proved Carlson type inequality for the Chaquet integral.

Theorem 1. *Let $p, q \geqslant 1$ and $r, s > 0$. Suppose $f, g : \mathcal{X} \to [0, \infty)$ and $f, h : \mathcal{X} \to [0, \infty)$ are pairs of comonotone functions. If f is integralable on A, then*

$$\int f d\mu \leqslant K(\mu(A))^d \left(\int f^p g^p d\mu\right)^{\frac{r}{p(r+s)}} \left(\int f^q h^q d\mu\right)^{\frac{s}{q(r+s)}}, \tag{2}$$

where $K = \left(\int g d\mu\right)^{-\frac{r}{(r+s)}} \left(\int h d\mu\right)^{-\frac{s}{(r+s)}}$ and $d = 2 - \frac{1}{r+s}\left(\frac{r}{p} + \frac{s}{q}\right)$.

The paper is organized as follows: Section 2 recalls the concepts of the pseudo-additive measure, integral and recalls some basic aspects of the Choquet-like integral. In Section 3, we prove the Carlson type inequality for the Choquet-like integral. Finally, some conclusions are given.

2. Preliminaries

In this section, we recall some basic definitions and previous results which will be used in the sequel. For details, we refer to [12]. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [1]).

Definition 1. *Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is for $x, y, z, \mathbf{0}$ (zero element) $\in [a, b]$ if it satisfies the following requirements:*
(i) $x \oplus y = y \oplus x$;
(ii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(iii) $x \preceq y \Rightarrow x \oplus z \preceq y \oplus z$;
(iv) $\mathbf{0} \oplus x = x$;
Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2. *A binary operation function $\odot : [a,b]\times[a,b]\to[a,b]$ is called a pseudo-multipication, for $x,y,z,\mathbf{1}$ (unit element) $\in [a,b]$ if it satisfies the following requirements:*
(i) $x\odot y = y\odot x$;
(ii) $(x\odot y)\odot z = x\odot(y\odot z)$;
(iii) $x\preceq y \Rightarrow x\odot z\preceq y\odot z$ *for all* $z\in[a,b]_+$;
(iv) $(x\oplus y)\odot z = (x\odot z)\oplus(x\odot y)$;
(v) if the $\mathbf{1}\oplus x = x$;
(vi) if the $\lim_{n\to\infty} x_n$ *and* $\lim_{n\to\infty} y_n$ *exist and finite, then* $\lim_{n\to\infty}(x_n\odot y_n) = \lim_{n\to\infty} x_n \odot \lim_{n\to\infty} y_n$;
(vii) $a\odot x = 0 \Longleftrightarrow a=0$ *or* $x=0$.

Remark 1. *Mesiar [12] showed that, if $\odot$ is a pseudo-multiplication corresponding to a given pseudo-addition $\oplus$ fulling axioms (i)-(vii) and if its identity element e is not an idempotent of $\oplus$, then there is a unique continuous strictly increasing function $g:[0,\infty]\to[0,\infty]$ with $g(0)=0$ and $g(\infty)=\infty$, such that $g(e)=1$ and*

$$a\oplus b = g^{-1}(g(a)+g(b)) \quad \oplus \text{ is called a g-addition,}$$

$$a\odot b = g^{-1}(g(a).g(b)) \quad \oplus \text{ is called a g-multipication.}$$

Mesiar [12] also proved that if the identity element e of the pseudo-multiplication is also an idempotent of $\oplus$ (i.e., $e\oplus e = e$), then $\oplus = \vee$(sup, i.e. the logical addition). In this case, the logical multiplication $\wedge$ and the g-multiplication are the candidates of $\odot$, among others. For $x\in[0,\infty]$ and $p\in(0,\infty)$ we introduce the pseudo-power $x_\odot^{(p)}$ as follows: If $p=n$ is a natural number, then $x_\odot^{(p)} = \underbrace{x\odot x\odot ...\odot x}_{n-times}$. If p is not a natural number, then the corresponding power is defined by

$$x_\odot^{(p)} = \sup\left\{ y_\odot^{(m)} | y_\odot^{(n)} \leqslant x, \textit{where}\, m,n \textit{ are natural number such that } \frac{m}{n}\leqslant p\right\}.$$

Evidently, if $x\odot y = g^{-1}(g(x).g(y))$, then $x_\odot^{(p)} = g^{-1}(g^p(x))$.

Remark 2. *Restricting to the interval $[0,1]$ a pseudo-multiplication and a pseudo-addition with additional properties of associativity and commutativity can be considered as the t-norm T and the t-conorms S (see[11]), respectively.*

Definition 3. *A monotone measure μ on a measurable space (Ω,F) is a set function $\mu : F\to[0,\infty]$ satisfying*
(i)$\mu(\emptyset)=0$;
(ii)$\mu(\Omega)>0$;
(iii)$\mu(A)\leqslant\mu(B)$ *whenever* $A\subseteq B$.

Moreover, μ is called real if $\|\mu\| = \mu(\mathcal{X}) < \infty$ and μ is said to be an additive measure if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$. The triple $(\mathcal{X}, \mathcal{F}, \mu)$ is also called a monotone measure space if μ is a monotone measure on $\mathcal{F}$. We call μ a monotone probability, if $\|\mu\| = 1$. When μ is a monotone probability, the triple $(\Omega, \mathcal{F}, \mu)$ is called a monotone probability space.

Definition 4. *For a fixed measurable space $(\mathcal{X}, \mathcal{F})$, i.e., a non-empty set $\mathcal{X}$ equipped with a $\sigma-$algebra $\mathcal{F}$, a function $f : \mathcal{X} \to [0, \infty]$ is called $\mathcal{F}-$ measurable if for each $b \in \mathcal{B}([0, \infty])$. The $\sigma-$algebra of Borel subsets of $[0, \infty]$, the preimage $f^{-1}(b)$ is an element of $\mathcal{F}$.*

Definition 5. *Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a monotone measure space and $f : \mathcal{X} \to [0, \infty]$ be a $\mathcal{F}-$measurable function. The Choquet expectation (integral) of f with respect to (w.r.t.) real monotone measure μ is defined by*

$$\mathbb{E}_c^{\mu} = \int_0^{\infty} \mu(\mathcal{X} \cap f \geqslant y)dy,$$

where the integral on the right-hand side is the (improper) Riemann integral.

Mesiar [12] developed a type of integral, the so-called Choquet-like integral, which generalizes the concepts of some well- known integrals, including the Sugeno integral and the Choquet integral.

There are two classes of Choquet-like integral: the Choquet-like integral (denoted by $\mathbb{E}^{\mu}_{Cl,g}$) based on a g-addition and a g-multiplication and the Choquet-like integral based on $\vee$ and a corresponding pseudo-multiplication $\odot$. Observe that for a $\oplus-$measure, Choquet-like integrals coincide with the corresponding pseudo-additive integrals.

Theorem 2. *Let $\odot$ and $\oplus$ be generated by a generator g. Then the Choquet-like expectation of a measurable function $f : \mathcal{X} \to [0, \infty]$ w.r.t. a real monotone measure μ can be represented as*

$$\mathbb{E}^{\mu}_{Cl,g}[f] = g^{-1}\left(\mathbb{E}_C^{g(\mu)}[g(f)]\right) = g^{-1}\left(\int_0^{\infty} g(\mu(\mathcal{X} \cap \{g(f) \geqslant y\}))dy\right).$$

Notice that sometimes we call this kind of Choquet-like integral a $g-$Choquet integral ($g-C-$integral for short). It is plain that the $g-C-$integral is the original Choquet integral (expectation) whenever $g = i$ (the identity mapping).

Theorem 3. *[22] Let $\odot$ be a pseudo-multiplication corresponding to $\vee$ and fullling (i)-(vii). Then the Choquet-like integral (so-called $\odot - \mathbb{S}_{\mu}-$integral) of a measurable function $f : \mathcal{X} \to [0, \infty]$ w.r.t. a real monotone measure l can be represented as*

$$\odot - \mathbb{S}_{\mu}[f] = \sup_{a \in [0,\infty]} a \odot \mu(\mathcal{X} \cap \{f \geqslant a\}).$$

It is plain that the $\odot - \mathbb{S}_\mu$ integral is the original Sugeno integral whenever $\odot = \wedge$.

Restricting now to the unit interval $[0,1]$ we shall consider the measurable function $f : \mathcal{X} \to [0,1]$ with $\|\mu\| = 1$. Observe that, in this case, we have the restriction of the pseudo-multiplication $\odot$ to $[0,1]^2$ called a semicopula or a conjunctor, i.e., a binary operation $\circledast : [0,1]^2 \to [0,1]$ which is non-decreasing in both components, has 1 as neutral element and satisfies $a \circledast b \leqslant \min(a,b)$ for all $(a,b) \in [0,1]^2$, (see [21, 24]). In a special case, for a fixed strict $t-$norm T, the corresponding $\odot - \mathbb{S}_\mu-$integral is the so-called Sugeno-Weber integral [24]. If $\odot$ is the standard product, then the Shilkret integral [21] can be recognized.

The $\odot - \mathbb{S}_\mu-$ integral on the $[0,1]$ scale related to the semicopula $\circledast$ is given by

$$\circledast - \mathbb{S}_\mu[f] = \sup_{a \in [0,\infty]} a \circledast \mu(\mathcal{X} \cap \{f \geqslant a\}).$$

This type of integral was called seminormed integral.

Recently, Girotto and Holzer [10] proved the following Chebyshev type inequality for Choquet integral (expectation).

Theorem 4. *Let $(\mathcal{X}, \mathcal{F})$ be a measurable space and Y and Z be $\mathcal{F}-$Borel measurable functions. If Y are Z are comonotone and two real-valued functions, then the following version of Chebyshev inequality:*

$$\|\mu\| \mathbb{E}^\mu_C[YZ] \geqslant \mathbb{E}^\mu_C[Y] \mathbb{E}^\mu_C[Z],$$

holds for any real monotone set function μ on $\mathcal{F}$, when $Y, Z \geqslant 0$, and for any real (finitely) additive measure μ on $\mathcal{F}$, when Y and Z are Choquet integrable.

Before stating our main result, we need a definition and a theorem from [1].

Definition 6. *Let $A, B : [0,\infty)^2 \to [0,\infty)$ be two binary operations. Then A dominates B (or B is dominated by A), denoted by $A \gg B$, if*

$$A(B(a,b), B(c,d)) \geqslant B(A(a,c), A(b,d))$$

holds for any $a, b, c, d \in [0,\infty)$.

Theorem 5. *Let $u, v : \mathcal{X} \to [0,\infty)$ be two comonotone functions. Then the inequality*

$$\|\mu\| \odot (\mathbb{E}^\mu_{Cl,g}[u \odot v]) \geqslant (\mathbb{E}^\mu_{Cl,g}[u]) \odot (\mathbb{E}^\mu_{Cl,g}[v])$$

holds for the g-Choquet integral if the generator g is a real-valued function.

Theorem 6. *Let* $u, v : \mathcal{X} \to [0, \infty]$ *be two comonotone functions and* $\star : [0, \infty)^2 \to [0, \infty)$ *be continuous and non-decreasing in both arguments. If* $\odot$ *is a pseudo-multipicatin (with neutral element e) corresponding to* $\vee$ *satisfying*

$$(a \star b) \odot c) \geqslant [(a \odot c) \star b] \vee [a \star (b \odot c)],$$

then the inequality

$$\odot - \mathbb{S}_\mu[\frac{u}{\|\mu\|} \star v] \geq (\odot - \mathbb{S}_\mu[\frac{u}{\|u\|}) \star (\odot - \mathbb{S}_\mu[v]) \tag{3}$$

holds for the $\odot - \mathbb{S}_\mu$ *integral and any real monotone set function* μ *such that* $a \odot \|\mu\| \leqq a$ *for all* a *and* $\odot - \mathbb{S}_\mu[\frac{u}{\|\mu\|}], \odot - \mathbb{S}_\mu[v]$ *are finite.*

3. Main results

In this section, we state some results of Choquet-like integrals

Theorem 7. *Let* $u_1, u_2, ..., u_n$ *be functions. If each both of them are comonotone, then the inequality*

$$\|\mu\| \odot (\mathbb{E}^\mu_{Cl,g}[u_1 \odot u_2 ... \odot u_n]) \geqslant (\mathbb{E}^\mu_{Cl,g}[u_1]) \odot (\mathbb{E}^\mu_{Cl,g}[u_2]) ... \odot (\mathbb{E}^\mu_{Cl,g}[u_n])$$

holds for the g-Choquet integral if the generator g is a real-valued function.

Proof. Since u_1, u_2 are comonotone, then by Theorem 2.11 we have

$$\|\mu\| \odot (\mathbb{E}^\mu_{Cl,g}[u \odot v]) \geqslant (\mathbb{E}^\mu_{Cl,g}[u]) \odot (\mathbb{E}^\mu_{Cl,g}[v]).$$

Moreover, the comonotonicity of $u_1 \odot u_2$ and u_3 imply that

$$\begin{aligned} \|\mu\| \odot (\mathbb{E}^\mu_{Cl,g}[u_1 \odot u_2 \odot u_3]) &\geqslant (\mathbb{E}^\mu_{Cl,g}[u_1 \odot u_2]) \odot (\mathbb{E}^\mu_{Cl,g}[u_3]) \\ &\geqslant (\mathbb{E}^\mu_{Cl,g}[u_1]) \odot (\mathbb{E}^\mu_{Cl,g}[u_2]) \odot (\mathbb{E}^\mu_{Cl,g}[u_3]) \end{aligned}$$

The proof follows by induction.

In fact, with this corollary we prove a version of Jensen's inequality for the Choquet-like integral. Assuming $\|\mu\| = 1$ and $u_1, u_2, ...u_n$ are the same functions, we obtain this sequel.

Corollary 8. *Let* $u : \mathcal{X} \to [0, \infty]$ *be a increasing function, then the inequality*

$$(\mathbb{E}^\mu_{Cl,g}[u^n_\odot]) \geqslant (\mathbb{E}^\mu_{Cl,g}[u])^n_\odot$$

holds for the g-Choquet integral (If the generator g is a real-valued function).

Using Corollary 3.2, if $\mu = 1$, then by the same method of proof Theorem 3.1, an extension of Jensen type inequality for Choquet-like integral is recognizable.

Corollary 9. *Let $u : \mathcal{X} \to [0,\infty]$ be an integrable function and $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and non-decreasing in both arguments. If $\odot$ is a pseudo-multiplication (with neutral element e) corresponding to $\vee$ satisfying*

$$(a \star b) \odot c) \geqslant [(a \odot c) \star b] \vee [a \star (b \odot c)],$$

then the inequality

$$\odot - \mathbb{S}_\mu[u^n_\star] \geq (\odot - \mathbb{S}_\mu[u])^n_\star \tag{4}$$

holds for the $\odot - \mathbb{S}_\mu$ integral and any real monotone set function μ such that $a \odot \|\mu\| \leqq a$ for all a and $\odot - \mathbb{S}_\mu[v]$ is finite.

The following result is a general version of the Carlson type inequality for the Choquet-like integral.

Theorem 10. *Let X, Y and Z be three non-negative measurable functions and let the pseudo-operations be generated by generator g. Suppose $X, Y : \mathcal{X} \to [0,\infty)$ and $X, Z : \mathcal{X} \to [0,\infty)$ are pairs of comonotone functions and $p, q \geq 1$ and $r, s \geq 0$. If f is integrable on A, then*

$$\mathbb{E}^\mu_{Cl-g}[X]_\odot \leq K \odot g^{-1}\left(\mu(A)^d\right) \odot \left[\mathbb{E}^\mu_{Cl-g}[X^p \odot Y^p]_\odot\right]^{\frac{r}{p(r+s)}} \odot \left[\mathbb{E}^\mu_{Cl-g}[X^q \odot Z^q]_\odot\right]^{\frac{s}{q(r+s)}}, \tag{5}$$

where

$$K = \left[\mathbb{E}^\mu_{Cl-g}[Y]_\odot\right]^{\frac{-r}{r+s}} \odot \left[\mathbb{E}^\mu_{Cl-g}[Z]_\odot\right]^{\frac{-s}{r+s}}$$

and $d = 2 - \frac{1}{r+s}(\frac{r}{p} + \frac{s}{q})$.

Proof. We have

$$\mathbb{E}^\mu_{Cl-g}[X]_\odot = g^{-1}\left(\mathbb{E}^{g(\mu)}_C[g(X)]\right) = g^{-1}\left(\mathbb{E}^{g(\mu)}_C[g \circ X]\right).$$

By Theorem 1.1

$$\begin{aligned} g^{-1}\left(\mathbb{E}_C^{g(\mu)}[g(X)]\right) &= g^{-1}\left(\mathbb{E}_C^{g(\mu)}[g\circ X]\right) \\ &\leq g^{-1}\Bigg(K.\mu(A)^d\left(E_C^{g(\mu)}[g\circ X]^p[g\circ Y]^p\right)^{\frac{r}{p(r+s)}} \\ &\quad \left(E_C^{g(\mu)}[g\circ X]^q[g\circ Z]^q\right)^{\frac{s}{q(r+s)}}\Bigg) \\ &= g^{-1}\Bigg(g(g^{-1}(K)).g(g^{-1}(\mu(A)^d)) \\ &\quad g\left(g^{-1}\left(\left(E_C^{g(\mu)}[g\circ X]^p[g\circ Y]^p\right)^{\frac{r}{p(r+s)}}\right)\right) \\ &\quad g\left(g^{-1}\left(\left(E_C^{g(\mu)}[g\circ X]^q[g\circ Z]^q\right)^{\frac{s}{q(r+s)}}\right)\right)\Bigg). \end{aligned} \tag{6}$$

Using inequality (3.3) and Remark 2.3, we have

$$\begin{aligned} g^{-1}\left(\mathbb{E}_C^{g(\mu)}[g(X)]\right) &\leq g^{-1}\Bigg(g(g^{-1}(\left[\mathbb{E}_C^{\mu}[g\circ Y]\right]^{\frac{-r}{r+s}}\left[\mathbb{E}_C^{\mu}[g\circ Z]\right]^{\frac{-s}{r+s}})).g(g^{-1}(\mu(A)^d)) \\ &\quad g\left(g^{-1}\left(\left(E_C^{g(\mu)}[g\circ X]^p[g\circ Y]^p\right)^{\frac{r}{p(r+s)}}\right)\right) \\ &\quad g\left(g^{-1}\left(\left(E_C^{g(\mu)}[g\circ X]^q[g\circ Z]^q\right)^{\frac{s}{q(r+s)}}\right)\right)\Bigg) \\ &= g^{-1}(\left[\mathbb{E}_C^{g(\mu)}[g\circ Y]\right]^{\frac{-r}{r+s}}\odot\left[\mathbb{E}_C^{\mu}[g\circ Z]\right]^{\frac{-s}{r+s}}))\odot g^{-1}(\mu(A)^d)) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)}[g\circ X]^p[g\circ Y]^p\right)^{\frac{r}{p(r+s)}}\right) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)}[g\circ X]^q[g\circ Z]^q\right)^{\frac{s}{q(r+s)}}\right). \end{aligned}$$

Thus

$$\begin{aligned} g^{-1}\left(\mathbb{E}_C^{g(\mu)}[g(X)]\right) &\leq g^{-1}(\left[\mathbb{E}_C^{g(\mu)} g(g^{-1}([g\circ Y]\right]^{\frac{-r}{r+s}})) \odot \left[\mathbb{E}_C^{g(\mu)} g(g^{-1}([g\circ Z]\right]^{\frac{-s}{r+s}})) \\ &\quad \odot g^{-1}(\mu(A)^d)) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)} g(g^{-1}([g\circ X]^p)) g(g^{-1}([g\circ Y]^p))\right)^{\frac{r}{p(r+s)}}\right) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)} g(g^{-1}([g\circ X]^q)) g(g^{-1}([g\circ Z]^q))\right)^{\frac{s}{q(r+s)}}\right) \\ &= g^{-1}(\left[\mathbb{E}_C^{g(\mu)} g[Y]_\odot\right]^{\frac{-r}{r+s}}) \odot \left[\mathbb{E}_C^{g(\mu)} g[Z]_\odot\right]^{\frac{-r}{r+s}})) \odot g^{-1}(\mu(A)^d)) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)} g[X]_\odot^p) g[Y]_\odot^p))\right)^{\frac{r}{p(r+s)}}\right) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)} g[X]_\odot^q)) g[Z]_\odot^q))\right)^{\frac{s}{q(r+s)}}\right) \\ &= K \odot g^{-1}(\mu(A)^d)) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)} g\left(g^{-1}\left(g[X]_\odot^p) g[Y]_\odot^p\right)\right)\right)^{\frac{r}{p(r+s)}}\right) \\ &\quad \odot g^{-1}\left(\left(E_C^{g(\mu)} g\left(g^{-1}\left(g[X]_\odot^q)) g[Z]_\odot^q\right)\right)\right)^{\frac{s}{q(r+s)}}\right). \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}_{Cl-g}^{\mu}[X]_\odot \leq & K \odot g^{-1}(\mu(A)^d)) \\ & \odot g^{-1}\left(\left(E_C^{g(\mu)} g([X]_\odot^p \odot [Y]_\odot^p\right)^{\frac{r}{p(r+s)}}\right) \\ & \odot g^{-1}\left(\left(E_C^{g(\mu)} g([X]_\odot^q \odot [Z]_\odot^q\right)^{\frac{s}{q(r+s)}}\right). \end{aligned}$$

Therefore, we have

$$\mathbb{E}_{Cl-g}^{\mu}[X]_\odot \leq K \odot g^{-1}\left(\mu(A)^d\right) \odot \left[\mathbb{E}_{Cl-g}^{\mu}[X^p \odot Y^p]_\odot\right]^{\frac{r}{p(r+s)}} \odot \left[\mathbb{E}_{Cl-g}^{\mu}[X^q \odot Z^q]_\odot\right]^{\frac{s}{q(r+s)}}$$

which completes the proof.

Example 1. *Let $g(x) = x^\alpha, \alpha > 0$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^\alpha + y^\alpha}$ and $x \odot y = xy$. Then the inequality (3.1) reduces on the following inequality*

$$\sqrt[\alpha]{\mathbb{E}_C^{\mu^\alpha}[X^\alpha]} \leq \sqrt[\alpha]{(\mu(A)^d)}\ \sqrt[\frac{\alpha}{r+s}]{\left[\mathbb{E}_C^{\mu^\alpha}[Y^\alpha]\right]^{-r}\left[\mathbb{E}_C^{\mu^\alpha}[Z^\alpha]\right]^{-s}\left[\mathbb{E}_C^{\mu\alpha}[(XY)^{\alpha p}]\right]^{\frac{r}{p}}\left[\mathbb{E}_C^{\mu\alpha}[(XZ)^{\alpha q}]\right]^{\frac{s}{q}}}$$

where $d = 2 - \frac{1}{r+s}(\frac{r}{p} + \frac{s}{q})$.

Example 2. *Let* $g(x) = e^x$. *The corresponding pseudo-operations are* $x \oplus y = \ln(e^x + e^y)$ *and* $x \odot y = x + y$. *Then the inequality (3.1) reduces on the following inequalities*

$$\ln\left(\mathbb{E}_C^{e^\mu}[e^x]\right) \leq d \ln \mu(A) + \frac{-r}{r+s}\left(\ln \mathbb{E}_C^{e^\mu}[e^y]\right) + \frac{-s}{r+s}\left(\ln \mathbb{E}_C^{e^\mu}[e^z]\right) + \frac{r}{p(r+s)}\left(\ln \mathbb{E}_C^{e^\mu}[e^{x^p+y^p}]\right) + \frac{s}{q(r+s)}\left(\ln \mathbb{E}_C^{e^\mu}[e^{x^q+z^q}]\right).$$

I.e., we have

$$\mathbb{E}_C^{e^\mu}[e^x] \leq \mu(A)^d . \left(\mathbb{E}_C^{e^\mu}[e^y]\right)^{\frac{-r}{r+s}} . \left(\mathbb{E}_C^{e^\mu}[e^z]\right)^{\frac{-s}{r+s}} + \left(\mathbb{E}_C^{e^\mu}[e^{x^p+y^p}]\right)^{\frac{r}{p(r+s)}} + \left(\mathbb{E}_C^{e^\mu}[e^{x^q+z^q}]\right)^{\frac{s}{q(r+s)}}.$$

Theorem 11. *Fix a real monotone measure* μ. *Let* X, Y *and* Z *be three non-negative measurable functions and* $\star : [0,\infty)^2 \to [0,\infty)$ *be continuous and non-decreasing in both arguments. If* $\odot$ *is a pseudo-multipicatin (with neutral element* e*) corresponding to* $\vee$ *satisfying*

$$(a \star b) \odot c) \geqslant [(a \odot c) \star b] \vee [a \star (b \odot c)],$$

then the inequality

$$\odot - \mathbb{S}_\mu[X] \leq (\odot - \mathbb{S}_\mu(Y))^{\frac{-r}{r+s}} (\odot - \mathbb{S}_\mu(Z))^{\frac{-s}{r+s}} (\odot - \mathbb{S}_\mu(X^p \star Y^p))^{\frac{r}{p(r+s)}} (\odot - \mathbb{S}_\mu(X^q \star Z^q))^{\frac{r}{q(r+s)}}.$$

holds for the $\odot - \mathbb{S}_\mu$ *integral and any real monotone set function* μ *such that* $a \odot \|\mu\| \leqq a$ *for all* a *and* $\odot - \mathbb{S}_\mu[\frac{u}{\|\mu\|}], \odot - \mathbb{S}_\mu[v]$ *are finite.*

Proof. For a given $c \geq 0$, the following Jensen type inequality

$$(\odot - \mathbb{S}_\mu[X])^c \leq \odot - \mathbb{S}_\mu[X^c]$$

holds. Then From the inequality (3.1) we have

$$(\odot - \mathbb{S}_\mu[X \star Y])^r (\odot - \mathbb{S}_\mu[X \star Z])^s \leq (\odot - \mathbb{S}_\mu[X^p \star Y^p])^{\frac{r}{p}} (\odot - \mathbb{S}_\mu[X^q \star Y^q])^{\frac{s}{q}}.$$

Since X and Y are comonotone functions, the following Chebyshev type inequality for Choquet-like expectation

$$\odot - \mathbb{S}_\mu[X \star Y] \geq (\odot - \mathbb{S}_\mu[X]) \star (\odot - \mathbb{S}_\mu[Y])$$

holds. The functions $X$and Z are also comonotone functions, so from the inequality (3.2) we get

$$(\odot - \mathbb{S}_\mu[X])^{r+s} \star (\odot - \mathbb{S}_\mu[Y])^{r} \star (\odot - \mathbb{S}_\mu[Z])^{s} \leq (\odot - \mathbb{S}_\mu[X^p \star Y^p])^{\frac{r}{p}} (\odot - \mathbb{S}_\mu[X^q \star Y^q])^{\frac{s}{q}},$$

this completes the proof.

Let $\odot$ be the standard product (i.e., Shilkret integral[21]) in Theorem 3.2. Then the following result holds.

Corollary 12. *Let X, Y and Z be three non-negative measurable function and $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and non-decreasing in both arguments. Then the inequality*

$$Sh_\mu[X] \leq (Sh_\mu(Y))^{\frac{-r}{r+s}} (Sh_\mu(Z))^{\frac{-s}{r+s}} \\ (Sh_\mu(X^p \star Y^p))^{\frac{r}{p(r+s)}} (Sh_\mu(X^q \star Z^q))^{\frac{r}{q(r+s)}}. \qquad (-14)$$

holds for the Sh_μ integral and any real monotone set function μ such that $a.\|\mu\| \leqq a$ for all a and $Sh_\mu[\frac{u}{\|\mu\|}], Sh_\mu[v]$ are finite.

Notice that if $\odot$ is minimum (i.e., for Sugeno integral) in Theorem 3.2, then (3.1) holds readily. Then the following result holds.

Corollary 13. *Let X, Y and Z be three non-negative measurable functions and $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and non-decreasing in both arguments. then the inequality*

$$Su_\mu[X] \leq (Su_\mu(Y))^{\frac{-r}{r+s}} (Su_\mu(Z))^{\frac{-s}{r+s}} \\ (Su_\mu(X^p \star Y^p))^{\frac{r}{p(r+s)}} (Su_\mu(X^q \star Z^q))^{\frac{r}{q(r+s)}}$$

holds for the Su_μ integral and any real monotone set function μ such that $a.\|\mu\| \leqq a$ for all a and $Su_\mu[\frac{u}{\|\mu\|}], Su_\mu[v]$ are finite.

Notice that by working on $[0,1]$ in Theorem 3.2 and $\odot = \circledast$ is semicopula (t-seminorm) and the following result holds [24].

Corollary 14. *Fix a real monotone measure μ. Let X, Y and Z be three non-negative measurable functions and $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and non-decreasing in both arguments. If $\odot$ is a pseudo-multiplication (with neutral element e) corresponding to $\vee$ satisfying*

$$(a \circledast b) \circledast c) \geqslant [(a \circledast c) \circledast b] \vee [a \circledast (b \circledast c)],$$

then the inequality

$$\mathbb{S}_{\mu}^{\circledast}[X] \leq (\mathbb{S}_{\mu}^{\circledast}(Y))^{\frac{-r}{r+s}} (\mathbb{S}_{\mu}^{\circledast}(Z))^{\frac{-s}{r+s}} (\mathbb{S}_{\mu}^{\circledast}(X^p \circledast Y^p))^{\frac{r}{p(r+s)}} (\mathbb{S}_{\mu}^{\circledast}(X^q \circledast Z^q))^{\frac{r}{q(r+s)}}$$

holds for the $\mathbb{S}_{\mu}^{\circledast}$ *integral and any real monotone set function* μ *such that* $a \circledast \|\mu\| \leqq a$ *for all* a *and* $\mathbb{S}_{\mu}^{\circledast}[\frac{u}{\|\mu\|}], \mathbb{S}_{\mu}^{\circledast}[v]$ *are finite.*

Example 3. *Putting* $X = 1$ *and* $r = s$ *in Corollary 3.9. If* $1 \star x = x$*, then we have*

$$Su_{\mu}[X] \leq (Su_{\mu}(Y))^{\frac{-r}{2r}} (Su_{\mu}(Z))^{\frac{-r}{2r}} (Su_{\mu}(1 \star Y^p))^{\frac{r}{p(2r)}} (Su_{\mu}(1 \star Z^q))^{\frac{r}{q(2r)}}.$$

It follows that

$$(Su_{\mu}[X])^2 \leq \frac{1}{(Su_{\mu}(Y))((Su_{\mu}(Z))} (Su_{\mu}(Y^p))^{\frac{1}{p}} (Su_{\mu}(Z^q))^{\frac{1}{q}}.$$

This result is a version of Holder inequality for Sugeno integral.

Conclusion: We have shown a version of Carlson's inequality for two classes of Choquet-like integrals. At first, two classes of Choquet-like integrals were introduced. Then, we prepared extensions of these inequalities from the Choquet expectation and the Sugeno integral to the two classes of Choquet-like integrals. We have proposed new versions of Carlson's inequality for different kinds of Choquet-like integrals, that generalize some results already known from the literature for the Choquet and for the Sugeno integrals. Note that the area of integral inequalities is a living area important for applications, especially when approximations have to be considered, with several fresh generalizations either in the classical Lebesgue integral setting, see for example [20], or the setting of Sugeno integrals[6, 7].

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