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CERTAIN PROPERTIES OF AN INTEGRAL OPERATOR

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Abstract. In this paper we consider the integral operator Miller-Mocanu-Reade for analytic functions in the open unit disk and we obtain properties of this integral operator.

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1. Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{S}$ the subclass of A consisting of functions $f \in A$, which are univalent in $\mathcal{U}$.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in $\mathcal{U}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$ we note

$$H[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + \cdots\} \tag{2}$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots \right\}, \tag{3}$$

with $\mathcal{A}_1 = A$.

Let us denote $S_\alpha(\rho)$ the class spiral functions of type α and order ρ, where α, $\rho \in \mathbb{R}$,

$$S_\alpha(\rho) = \left\{ f \in A : Re \frac{e^{i\alpha} z f'(z)}{f(z)} > \rho \cos\alpha,\ |\alpha| < \frac{\pi}{2},\ \rho < 1,\ z \in \mathcal{U} \right\}. \tag{4}$$

We have $S_\alpha(0) = S_\alpha$, where S_α is the class spiral functions of type α.

In this paper we consider the integral operator Miller-Mocanu-Reade, $I_{\alpha,\beta,\gamma,\delta}$: $E \to \mathcal{H}(U)$, $E \subseteq \mathcal{H}(U)$ defined by:

$$I_{\alpha,\beta,\gamma,\delta}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma \phi(z)} \int_0^z f^\alpha(t) t^{\delta-1} \varphi(t) dt\right]^{\frac{1}{\beta}}, \tag{5}$$

where $\phi, \varphi \in H[1,n]$ with $\phi(z)\varphi(z) \neq 0$, $z \in \mathcal{U}$, α, β, γ, $\delta \in \mathbb{C}$, $\beta \neq 0$, $\alpha+\delta = \beta+\gamma$ and $Re\ (\alpha+\delta) > 0$, $f \in \mathcal{A}_n$, $f(z) = z + a_{n+1}z^{n+1} + \cdots$, $n \in \mathbb{N} - \{0\}$.

The integral operator $I_{\alpha,\beta,\gamma,\delta}$ was defined by S.S. Miller, P.T. Mocanu and M.O. Reade in 1978 [1] and studied in [2], [3], [4], [5].

For $\alpha = \beta = e^{i\sigma}$, $\sigma \in \mathbb{R}$, $\delta = \gamma$, $f \in S_\sigma(\rho)$, $\phi(z) = \varphi(z) = 1$, $z \in \mathcal{U}$, from (5) we obtain the integral operator

$$T_{\gamma,\sigma}(z) = \left[\frac{e^{i\sigma}+\gamma}{z^\gamma} \int_0^z [f(t)]^{e^{i\sigma}}\, t^{\gamma-1} dt\right]^{e^{-i\sigma}}, \tag{6}$$

for all $z \in \mathcal{U}$, that was studied by S.K. Bajpai in 1979 [7], which proved that if $f \in S_\sigma(\rho)$, $0 \leq \rho < 1$, $Re\ \gamma > -\rho\cos\sigma$, $|\sigma| < \frac{\pi}{2}$, then $T_{\gamma,\sigma} \in S_\sigma(\rho)$.

From (5), for $\alpha = \beta$, $\gamma = 0$, $\delta = 0$, $f \in A$, $\phi(z) = \varphi(z) = 1$, $z \in \mathcal{U}$, we obtain the integral operator Miller-Mocanu [8],

$$J_\beta(z) = \left[\beta \int_0^z t^{-1} f^\beta(t) dt\right]^{\frac{1}{\beta}}, \quad z \in \mathcal{U}. \tag{7}$$

In this paper we obtain properties of integral operator $I_{\alpha,\beta,\gamma,\delta}(f)$.

2. Preliminaries

We need the following lemmas.

Lemma 1. *Pascu [9]. Let α be a complex number, $Re\ \alpha > 0$ and $f \in A$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{8}$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z t^{\alpha-1} f'(t) dt\right]^{\frac{1}{\alpha}} \tag{9}$$

is regular and univalent in $\mathcal{U}$.

Lemma 2. *General Schwarz Lemma [10]. Let f the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If the function f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \ z \in \mathcal{U}_R, \tag{10}$$

the equality (in the inequality (10) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3. Main results

Theorem 3. *Let α, β, γ, δ be complex numbers, $\beta \neq 0$, $\beta + \gamma = \alpha + \delta \neq 0$, $a = Re(\alpha + \delta) > 0$ and the functions $\phi, \varphi \in H[1, n]$ with $\phi(z)\varphi(z) \neq 0$, $z \in \mathcal{U}$, the function $f \in \mathcal{A}_n$, $f(z) = z + a_{n+1}z^{n+1} + \cdots$, L, M positive real numbers.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \quad z \in \mathcal{U}, \tag{11}$$

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| < L, \quad z \in \mathcal{U} \tag{12}$$

and

$$|\alpha|M + L \leq \frac{(2a+n)^{\frac{2a+n}{2a}}}{2n^{\frac{n}{2a}}}, \quad n \in \mathbb{N} - \{0\}, \tag{13}$$

then

$$I_{\alpha,\beta,\gamma,\delta}(f)(z) = \frac{1}{\phi^{\frac{1}{\beta}}(z)} z\left(1 + b_2 z + b_3 z^2 + \cdots\right)^{\frac{\beta+\gamma}{\beta}}, \quad z \in \mathcal{U} \tag{14}$$

and

$$z^{\frac{\gamma}{\beta+\gamma}}\phi^{\frac{1}{\beta+\gamma}}(z) I^{\frac{\beta}{\beta+\gamma}}_{\alpha,\beta,\gamma,\delta}(f)(z) \tag{15}$$

belongs to class $\mathcal{S}$.

Proof. From (5) we have

$$I_{\alpha,\beta,\gamma,\delta}(f)(z) = \left[\frac{\beta+\gamma}{(\alpha+\delta)z^{\gamma}\phi(z)}\right]^{\frac{1}{\beta}} \left\{\left[(\alpha+\delta)\int_0^z t^{\alpha+\delta-1}\left(\frac{f(t)}{t}\right)^{\alpha}\varphi(t)dt\right]^{\frac{1}{\alpha+\delta}}\right\}^{\frac{\alpha+\delta}{\beta}} \tag{16}$$

for all $z \in \mathcal{U}$.

We consider the function

$$G_{\alpha,\delta}(z) = \left[(\alpha+\delta)\int_0^z t^{\alpha+\delta-1}\left(\frac{f(t)}{t}\right)^{\alpha}\varphi(t)dt\right]^{\frac{1}{\alpha+\delta}}, \quad z \in \mathcal{U}, \tag{17}$$

where $\alpha+\delta = \beta+\gamma$.

Let the function

$$p(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha}\varphi(t)dt, \quad z \in \mathcal{U}, \tag{18}$$

which is regular in $\mathcal{U}$ and $p(0) = p'(0) - 1 = 0$.

We have

$$\frac{zp''(z)}{p'(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \frac{z\varphi'(z)}{\varphi(z)}, \quad z \in \mathcal{U} \tag{19}$$

and hence, we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \le \frac{1-|z|^{2a}}{a}\left[|\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + \left|\frac{z\varphi'(z)}{\varphi(z)}\right|\right], \tag{20}$$

for all $z \in \mathcal{U}$.

Applying Lemma 2, from (11) and (12) we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M|z|^n, \quad z \in \mathcal{U}, \tag{21}$$

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| \le L|z|^n, \quad z \in \mathcal{U}. \tag{22}$$

From (20) and (21), (22) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \le \frac{1-|z|^{2a}}{a}|z|^n(|\alpha|M + L), \quad z \in \mathcal{U}. \tag{23}$$

We consider the function $Q : [0,1] \to \mathbb{R}$, $Q(x) = \frac{(1-x^{2a})x^n}{a}$, where $x = |z|$, $x \in [0,1]$.

We have

$$\max_{x\in[0,1]} Q(x) = \frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{2a+n}{2a}}},\ n \in \mathbb{N} - \{0\}. \tag{24}$$

By (13), (24) and (23) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \leq 1, \tag{25}$$

for all $z \in \mathcal{U}$.

Now, from (25) and Lemma 1, it results that

$$G_{\alpha,\delta} \in \mathcal{S}, \quad G_{\alpha,\delta}(z) = z + b_2 z + b_3 z^2 + \cdots, \tag{26}$$

hence, for $\alpha + \delta = \beta + \gamma$, from (16) and (26) we have

$$I_{\alpha,\beta,\gamma,\delta}(f)(z) = \frac{1}{\phi^{\frac{1}{\beta}}(z)} z\left(1 + b_2 z + b_3 z^2 + \cdots\right)^{\frac{\beta+\gamma}{\beta}}, \quad z \in \mathcal{U} \tag{27}$$

and

$$z^{\frac{\gamma}{\beta+\gamma}} \phi^{\frac{1}{\beta+\gamma}}(z) I_{\alpha,\beta,\gamma,\delta}^{\frac{\beta}{\beta+\gamma}}(f)(z) \in \mathcal{S}. \tag{28}$$

Remark 1. *From Theorem 3, for $\phi(z) = 1$ and $\gamma = 0$ we have $I_{\alpha,\beta,0,\delta}(f)(z)$ belongs to class $\mathcal{S}$.*

Corollary 4. *Let α, β, γ, δ be complex numbers, $\alpha = \beta = e^{i\sigma}$, $\sigma \in \mathbb{R}$, $\delta = \gamma$, $a = Re(e^{i\sigma} + \gamma) > 0$ and the functions $\phi(z) = \varphi(z) = 1$, $z \in \mathcal{U}$ and the function $f \in A$, $f(z) = z + a_2 z^2 + \cdots$, M positive real number.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \quad z \in \mathcal{U}, \tag{29}$$

$$M \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}, \tag{30}$$

then

$$T_{\gamma,\sigma}(z) = z(1 + b_2 z + \cdots)^{\frac{e^{i\sigma}+\gamma}{e^{i\sigma}}}, \quad z \in \mathcal{U} \tag{31}$$

and $T_{0,\sigma}(z)$ belongs to class $\mathcal{S}$.

Proof. Using (20) and Theorem 3 for $n = 1$, we obtain Corollary 4.

Corollary 5. *Let β be a complex number, $Re\ \beta > 0$, the functions $\phi(z) = \varphi(z) = 1$, $z \in \mathcal{U}$ and the function $f \in A$, $f(z) = z + a_2 z^2 + \cdots$, M positive real number.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \quad z \in \mathcal{U}, \tag{32}$$

$$|\beta| \leq \frac{(2Re\ \beta + 1)^{\frac{2Re\ \beta + 1}{2Re\ \beta}}}{2M}, \tag{33}$$

then the integral operator Miller-Mocanu, J_β, belongs to class $\mathcal{S}$,

$$J_\beta(z) = z + b_2 z^2 + \cdots, \quad z \in \mathcal{U}. \tag{34}$$

Proof. We have $\gamma = 0$, $\delta = 0$, $\alpha = \beta$, $\phi(z) = \varphi(z) = 1$ and $a = Re\ \beta > 0$. Applying Theorem 3 and using (19), we obtain Corollary 5.

References

[1] S.S. Miller, P.T. Mocanu, M.O. Reade, *Starlike integral operators*, Pacific J. Math., 79(1978), 157-168.

[2] S.S. Miller, P.T. Mocanu, *Integral operators on certain classes of analytic functions*, Univalent Functions, Fractional Calculus and their Applications, Halstead Press, J. Wiley and Sons, New York (1989), 153-166.

[3] S.S. Miller, P.T. Mocanu, *Classes of univalent integral operators*, J. Math. Anal. Appl., 157, 1(1991), 147-165.

[4] S.S. Miller, P.T. Mocanu, *Briot-Bouquet differential equations and differential subordinations*, Complex Variables, 33(1997), 217-237.

[5] P.T. Mocanu, *Some integral operators and starlike functions*, Rev. Roum. Math. Pures and Appl., 31, 3(1986), 231-235.

[6] P.T. Mocanu, T. Bulboacă, G.St., Sălăgean, *Geometric Theory of Univalent Functions*, Cluj, 1999.

[7] S.K. Bajpai, *An analogue of R.J. Libera's result*, Rend. Math., 12, 7(1979), 285-289.

[8] S.S. Miller, P.T. Mocanu, *Differential Subordinations, Theory and Applications, Monographs and Text Books in Pure and Applied Mathematics*, 255, Marcel Dekker, New York, 2000.

[9] N.N. Pascu, *On a univalence criterion II*, Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1985), 153-154.

[10] O. Mayer, *The Functions Theory of One Variable Complex*, Bucureşti, 1981.

[11] Z. Nehari, *Conformal Mapping*, Mc Graw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975).

[12] C. Pommerenke, *Univalent functions*, Göttingen, 1975.

[13] A.W. Goodman, *Univalent functions*, Mariner Publishing Company Inc., 1984.

[14] V.Pescar, V. D. Breaz, *The univalence of integral operators*, Sofia, 2008, Academic Publishing House, Monograph.

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