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STARLIKENESS AND CONVEXITY OF ORDER α AND TYPE β FOR P-VALENT HYPERGEOMETRIC FUNCTIONS

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Abstract. Given the hypergeometric function $F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n$, we place conditions on a,b and c to guarante that $z^pF(a,b;c;z)$ will be in various subclasses of p-valent starlike and p-valent convex functions of order α and type β $(0\leq\alpha<p, 0<\beta\leq1)$. Operators related to the hypergeometric function are also examined.

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1. Introduction

Let $S(p)$ be the class of functions of the form:

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \quad (p\in\mathbb{N}=\{1,2,....\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U=\{z\in\mathbb{C}: |z|<1\}$. A function $f(z)\in S(p)$ is called p-valent starlike of order α if $f(z)$ satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{1.2}$$

for $0\leq\alpha<p$, $p\in\mathbb{N}$ and $z\in U$. We denote by $S_p^*(\alpha)$ the class of all p-valent starlike functions of order α and $S_p^*(0)=S_p^*$. Denote by $S_p^*(\alpha,\beta)$ the subclass consisting of functions $f(z)\in S(p)$ which satisfy

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{\frac{zf'(z)}{f(z)}+p-2\alpha}\right|<\beta \tag{1.3}$$

for $0\leq\alpha<p, 0<\beta\leq1$, $p\in\mathbb{N}$ and $z\in U$. Also a function $f(z)\in S(p)$ is called p-valent convex of order α if $f(z)$ satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.4}$$

for $0 \leq \alpha < p$, $p \in \mathbb{N}$ and $z \in U$. We denote by $K_p(\alpha)$ the class of all p-valent convex functions of order α and $K_p(0) = K_p$. Also denote by $K_p(\alpha, \beta)$ the subclass consisting of functions $f(z) \in S(p)$ which satisfy

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\alpha} \right| < \beta \tag{1.5}$$

for $0 \leq \alpha < p, 0 < \beta \leq 1$, $p \in \mathbb{N}$ and $z \in U$.

It follows from (1.3) and (1.5) that

$$f(z) \in K_p(\alpha, \beta) \Leftrightarrow \frac{zf'(z)}{p} \in S_p(\alpha, \beta). \tag{1.6}$$

Denoting by $T(p)$ the subclass of $S(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \quad (a_{p+n} \geq 0; p \in \mathbb{N}). \tag{1.7}$$

We denote by $T_p^*(\alpha), T_p^*(\alpha, \beta), C_p(\alpha)$ and $C_p(\alpha, \beta)$ the classes obtained by taking intersections, respectively, of the classes $S_p^*(\alpha), S_p^*(\alpha, \beta), K_p(\alpha)$ and $K_p(\alpha, \beta)$ with the class $T(p)$

$$T_p^* = S_p^* \cap T(p)$$
$$T_p^*(\alpha) = S_p^*(\alpha) \cap T(p)$$
$$T_p^*(\alpha, \beta) = S_p^*(\alpha, \beta) \cap T(p)$$

$$C_p = K_p \cap T(p)$$

$$C_p(\alpha) = K_p(\alpha) \cap T(p)$$

and

$$C_p(\alpha, \beta) = K_p(\alpha, \beta) \cap T(p).$$

The class $S_p^*(\alpha)$ was studied by Patil and Thakare [8]. The classes $T_p^*(\alpha)$ and $C_p(\alpha)$ were studied by Owa [7], and the classes $T_p^*(\alpha, \beta)$ and $C_p(\alpha, \beta)$ were studied by Hossen [4] (see also [1]).

For $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, ...$, the Gaussian hypergeometric function is defined by:

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in U), \tag{1.8}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1).........(\lambda + n - 1) & (n \in \mathbb{N}). \end{cases} \tag{1.9}$$

The series in (1.8) represents an analytic function in U and has an analytic continuation throughout the finite complex plane except at most for the cut $[1, \infty)$. We note that $F(a, b; c; 1)$ converges for $Re(c - a - b) > 0$ and is related to the Gamma function by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{1.10}$$

Corresponding to the function $F(a, b; c; z)$ we define

$$h_p(a, b; c; z) = z^p F(a, b; c; z). \tag{1.11}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$h_p(a, b; c; z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p} (b)_{n-p}}{(c)_{n-p} (1)_{n-p}} z^n. \tag{1.12}$$

In [3] El-Ashwah et al. gave necessary and sufficient conditions for $z^p F(a, b; c; z)$ to be in the classes $T_p^*(\alpha)$ and $C_p(\alpha)$ $(0 \leq \alpha < p)$ and has also examined a linear operator acting on hypergeometric functions. Also in [10] Silverman gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in the classes $T_1^*(\alpha) = T^*(\alpha)$ and $C_1(\alpha) = C(\alpha)$ $(0 \leq \alpha < 1)$ and has also examined a linear operator acting on hypergeometric functions. Also in [6] Mostafa obtained analogous results for the classes $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$. For the other interesting developments for $zF(a, b; c; z)$ in connection with various subclasses of univalent functions, the reader can refer to the works of Carlson and Shaffer [2], Merkes and Scott [5] and Ruscheweyh and Singh [9].

In the present paper, we determine necessary and sufficient conditions for $h_p(a, b; c; z)$ to be in the classes $T_p^*(\alpha, \beta)$ and $C_p(\alpha, \beta)$.

Furthermore, we consider an integral operator related to the hypergeometric function.

2. Main Results

To establish our main results, we shall need the following lemmas.

Lemma 1 [4]. Let the function $f(z)$ defined by (1.1).

(i) A sufficient condition for $f(z) \in S(p)$ to be in the class $S_p^*(\alpha, \beta)$ is that

$$\sum_{n=p+1}^{\infty} \{n(1+\beta) - [p(1-\beta) + 2\alpha\beta]\} |a_n| \leq 2\beta(p-\alpha).$$

(ii) A sufficient condition for $f(z) \in S(p)$ to be in the class $K_p(\alpha, \beta)$ is that

$$\sum_{n=p+1}^{\infty} \frac{n}{p} \{n(1+\beta) - [p(1-\beta) + 2\alpha\beta]\} |a_n| \leq 2\beta(p-\alpha).$$

Lemma 2 [4]. Let the function $f(z)$ defined by (1.7). Then

(i) $f(z) \in T(p)$ is in the class $T_p^*(\alpha, \beta)$ if and only if

$$\sum_{n=p+1}^{\infty} \{n(1+\beta) - [p(1-\beta) + 2\alpha\beta]\} a_n \leq 2\beta(p-\alpha).$$

(ii) $f(z) \in T(p)$ is in the class $C_p(\alpha, \beta)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{n}{p} \{n(1+\beta) - [p(1-\beta) + 2\alpha\beta]\} a_n \leq 2\beta(p-\alpha).$$

Theorem 1. If $a, b > 0$ and $c > a+b+1$, then a sufficient condition for $h_p(a, b; c; z)$ to be in the class $S_p^*(\alpha, \beta)$ $(0 \leq \alpha < p, 0 < \beta \leq 1)$ is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[1 + \frac{ab(1+\beta)}{2\beta(p-\alpha)(c-a-b-1)}\right] \leq 2. \tag{2.1}$$

Condition (2.1) is necessary and sufficient for F_p defined by $F_p(a, b; c; z) = z^p(2 - F(a, b; c; z))$ to be in the class $T_p^*(\alpha, \beta)$.

Proof. Since $h_p(a, b; c; z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n$, according to Lemma 1 (i), we need only show that

$$\sum_{n=p+1}^{\infty} \{n(1+\beta) - [p(1-\beta) + 2\alpha\beta]\} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \leq 2\beta(p-\alpha).$$

Now

$$\sum_{n=p+1}^{\infty} \{n(1+\beta) - [p(1-\beta) + 2\alpha\beta]\} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}}$$

$$= \sum_{n=1}^{\infty} \{(n+p)(1+\beta) - [p(1-\beta) + 2\alpha\beta]\} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= (1+\beta) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} + 2\beta(p-\alpha) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}. \qquad (2.2)$$

Noting that $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ and then applying (1.10), we may express (2.2) as

$$\frac{ab}{c}(1+\beta) \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + 2\beta(p-\alpha) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{ab}{c}(1+\beta) \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + 2\beta(p-\alpha) \left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{ab(1+\beta)}{c-a-b-1} + 2\beta(p-\alpha)\right] - 2\beta(p-\alpha).$$

But this last expression is bounded above by $2\beta(p-\alpha)$ if and only if (2.1) holds.

Since $F_p(a,b;c;z) = z^p - \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n$, the necessity of (2.1) for F_p to be in the class $T_p^*(\alpha,\beta)$ follows from Lemma 2 (i).

In the next theorem, we find constraints on a, b and c that lead to necessary and sufficient conditions for $h_p(a,b;c;z)$ to be in the class $T_p^*(\alpha,\beta)$.

Theorem 2. If $a, b > -1, c > 0$, and $ab < 0$, then a necessary and sufficient condition for $h_p(a,b;c;z)$ to be in the class $T_p^*(\alpha,\beta)$ is that $c \geq a+b+1 - \frac{ab(1+\beta)}{2\beta(p-\alpha)}$. The condition $c \geq a+b+1-\frac{ab}{p}$ is necessary and sufficient for $h_p(a,b;c;z)$ to be in the class T_p^*.

Proof. Since

$$h_p(a,b;c;z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n$$

$$= z^p + \frac{ab}{c} \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} z^n$$

$$= z^p - \left|\frac{ab}{c}\right| \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} z^n, \tag{2.3}$$

according to Lemma 2 (i) we must show that

$$\sum_{n=p+1}^{\infty} \{n(1+\beta) - [p(1-\beta) - 2\alpha\beta]\} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \leq \left|\frac{c}{ab}\right| 2\beta(p-\alpha). \tag{2.4}$$

Note that the left side of (2.4) diverges if $c \leq a+b+1$. Now

$$\sum_{n=0}^{\infty} \{(n+p+1)(1+\beta) - [p(1-\beta) - 2\alpha\beta]\} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= (1+\beta)\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + 2\beta(p-\alpha)\frac{c}{ab}\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}(1+\beta) + 2\beta(p-\alpha)\frac{c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right].$$

Hence, (2.4) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[(1+\beta) + 2\beta(p-\alpha)\frac{(c-a-b-1)}{ab}\right]$$

$$\leq 2\beta(p-\alpha)\left[\frac{c}{|ab|} + \frac{c}{ab}\right] = 0. \tag{2.5}$$

Thus, (2.5) is valid if and only if

$$(1+\beta) + 2\beta(p-\alpha)\frac{(c-a-b-1)}{ab} \leq 0,$$

or, equivalently,

$$c \geq a+b+1 - \frac{ab(1+\beta)}{2\beta(p-\alpha)}.$$

Another application of Lemma 2 (i) when $\alpha = 0$ and $\beta = 1$ completes the proof of Theorem 2.

Our next theorems will parallel Theorems 1 and 2 for the p-valent convex case.

Theorem 3. If $a, b > 0$ and $c > a+b+2$, then a sufficient condition for $h_p(a,b;c;z)$ to be in the class $K_p(\alpha.\beta)$, $0 \leq \alpha < p, 0 < \beta \leq 1$, is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{(2p+1)(1+\beta) - [p(1-\beta) + 2\alpha\beta]}{2\beta p(p-\alpha)}\frac{ab}{(c-a-b-1)} + \right.$$

$$\left.\frac{(1+\beta)(a)_2(b)_2}{2\beta p(p-\alpha)(c-a-b-2)_2}\right] \leq 2. \tag{2.6}$$

Condition (2.6) is necessary and sufficient for $F_p(a,b;c;z) = z^p(2 - F(a,b;c;z))$ to be in the class $C_p(\alpha,\beta)$.

Proof. In view of Lemma 1 (ii), we need only show that

$$\sum_{n=p+1}^{\infty} n\left\{n(1+\beta) - [p(1-\beta)+2\alpha\beta]\right\} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \leq 2\beta p(p-\alpha).$$

Now

$$\sum_{n=0}^{\infty}(n+p+1)\left\{(n+p+1)(1+\beta) - [p(1-\beta)+2\alpha\beta]\right\} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$= (1+\beta)\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \{2p(1+\beta) - [p(1-\beta)+2\alpha\beta]\}\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$+2p\beta(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$= (1+\beta)\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + \{2p(1+\beta) - [p(1-\beta)+2\alpha\beta]\}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}$$

$$+2\beta p(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$= (1+\beta)\sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + \{(2p+1)(1+\beta) - [p(1-\beta)+2\alpha\beta]\}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}$$

$$+2\beta p(p-\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$= (1+\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}} + \{(2p+1)(1+\beta) - [p(1-\beta)+2\alpha\beta]\}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}$$

$$+ 2\beta p(p-\alpha)\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}. \tag{2.7}$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we may write (2.7) as

$$\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}(1+\beta) + \{(2p+1)(1+\beta) - [p(1-\beta)+2\alpha\beta]\}\frac{ab}{c}.$$

$$\cdot\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+2\beta p(p-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right].$$

Upon simplification, we see that this last expression is bounded above by $2\beta p(p-\alpha)$ if and only if (2.6) holds. That (2.6) is also necessary for F_p to be in the class $C_p(\alpha,\beta)$ follows from Lemma 2 (ii).

Theorem 4. If $a,b>-1, ab<0$ and $c>a+b+2$, then a necessary and sufficient condition for $h_p(a,b;c;z)$ to be in the class $C_p(\alpha,\beta)$ is that

$$(a)_2(b)_2(1+\beta)+\{(2p+1)(1+\beta)-[p(1-\beta)+2\alpha\beta]\}\,ab(c-a-b-2)$$
$$+2\beta p(p-\alpha)(c-a-b-2)_2\geq 0. \tag{2.8}$$

Proof. Since $h_p(a,b;c;z)$ has the form (2.3), we see from Lemma 2 (ii) that our conclusion is equivalent to

$$\sum_{n=p+1}^{\infty} n\left\{n(1+\beta)-[p(1-\beta)+2\alpha\beta]\right\}\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}\leq\left|\frac{c}{ab}\right|2\beta p(p-\alpha). \tag{2.9}$$

Note that $c>a+b+2$ if the left hand side of (2.9) converges. Now,

$$\sum_{n=p+1}^{\infty} n\left\{n(1+\beta)-[p(1-\beta)+2\alpha\beta]\right\}\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}$$
$$=\sum_{n=0}^{\infty}(n+p+1)\left\{(n+p+1)(1+\beta)-[p(1-\beta)+2\alpha\beta]\right\}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$=(1+\beta)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+[p(1+3\beta)-2\alpha\beta]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$
$$+2\beta p(p-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$=(1+\beta)\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+$$
$$[p(1+3\beta)+(1+\beta)-2\alpha\beta]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+2\beta p(p-\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$=\tfrac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\left\{(a+1)(b+1)(1+\beta)+[p(1+3\beta)+(1+\beta)-2\alpha\beta](c-a-b-2)\right.$$

$$+\frac{2\beta p(p-\alpha)}{ab}(c-a-b-2)_2\Bigg\} - \frac{2\beta p(p-\alpha)c}{ab}.$$

This last expression is bounded above by $\left|\frac{c}{ab}\right| 2\beta p(p-\alpha)$ if and only if

$$(a+1)(b+1)(1+\beta) + [p(1+3\beta)+(1+\beta)-2\alpha\beta](c-a-b-2)$$
$$+\frac{2\beta p(p-\alpha)}{ab}(c-a-b-2)_2 \leq 0,$$

which is equivalent to (2.8).

Putting $p=\beta=1$ in Theorem 4, we obtain the following corollary.

Corollary 1. If $a,b>-1,\ ab<0$, and $c>a+b+2$, then $zF(a,b;c;z)$ is in the class $C(\alpha)$ $(0\leq\alpha<1)$, if and only if

$$(a)_2(b)_2 + (3-\alpha)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geq 0.$$

Remark 1. Corollary 1 corrects the result given by Silverman [10, Theorem 4].

3. An integral operator

In this section, we obtain results in connection with a particular integral operator $G_p(a,b;c;z)$ acting on $F(a,b;c;z)$ as follows:

$$G_p(a,b;c;z) = p\int_0^z t^{p-1}F(a,b;c;t)dt$$

$$= z^p + \sum_{n=1}^{\infty}\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n}z^{n+p}. \tag{3.1}$$

We note that $\frac{zG_p'}{p} = h_p$.

To prove Theorem 5, we shall need the following lemma.

Lemma 3 [3]. (i) If $a,b>0$ and $c>a+b$, then a sufficient condition for $G_p(a,b;c;z)$ defined by (3.1) to be in the class S_p^* is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq 2.$$

(ii) If $a,b>-1, c>0$, and $ab<0$, then $G_p(a,b;c;z)$ defined by (3.1) is in the class $T(p)$ or in the class $S(p)$ if and only if $c>max\{a,b\}$.

Now $G_p(a,b;c;z) \in K_p(\alpha,\beta)$ if and only if $\frac{z}{p}G'_p(a,b;c;z) = h_p(a,b;c;z) \in S^*_p(\alpha,\beta)$. This follows upon observing that $\frac{zG'_p}{p} = h_p$, $\frac{z}{p}G''_p = h'_p - \frac{1}{p}G'_p$, and so

$$1 + \frac{zG''_p}{G'_p} = \frac{zh'_p}{h_p}.$$

Thus any p-valent starlike about h_p leads to a p-valent convex *funcabout* G_p. Thus from Theorems 1, 2 and Lemma 3, we obtain the following theorem.

Theorem 5. (i) If $a, b > 0$ and $c > a + b + 1$, then a sufficient condition for $G_p(a,b;c;z)$ defined in (3.1) to be in the class $K_p(\alpha,\beta)(0 \le \alpha < p, 0 < \beta \le 1)$ is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{ab(1+\beta)}{2\beta(p-\alpha)(c-a-p-1)}\right] \le 2.$$

(ii) If $a, b > -1, ab < 0$, and $c > a + b + 2$, then a necessary and sufficient condition for $G_p(a,b;c;z)$ to be in the class $C_p(\alpha,\beta)$ is that $c \ge a + b + 1 - \frac{ab(1+\beta)}{2\beta(p-\alpha)}$.

Remark 2. (i) Putting $\beta = 1$ in all the above results we obtain the results, obtained by El-Ashwah et al.[3];

(ii)Putting $p = \beta = 1$ in all the above results we obtain the results, obtained by Silverman [10];

(iii) Putting $p = 1$ in all the above results we obtain the analogous results, obtained by Mostafa [6].

References

[1] M. K. Aouf, p-Valent regular functions with negative coefficients of order α, Bull. Institute Math. Acad. Sinica 17(1989), no. 3, 255-265.

[2] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, J. Math. Anal. Appl. 15(1984), 737-745.

[3] R. M. El-Ashwah, M. K. Aouf and A. O. Mostafa, Starlike and convexity properties for p-valent hypergeometric functions, Acta Math. Univ. Comenianae, 33(2010), no. 1, 55-64.

[4] H. M. Hossen, Quasi-Hadamard product of certain p-valent functions, Demonostratio Math., 33(2000), no. 2, 277-281.

[5] E. Merkes and B. T. Scott, Starlike hypergeometric functions, Proc. Amer. Math. Soc. 12(1961), 885-888.

[6] A. O. Mostafa, A study on starlikeness and convexity properties for hypergeomertic functions, Math. Comput. Modelling, 59(2009), no. 8, 2821-2826.

[7] S. Owa, On certain classes of p-valent functions with negative coefficients, Simon Stevin, 59(1985), 385-402.

[8] D. A. Patil and N.K. Thakare, On convex hulls and extreme points of p-valent starlike and convex classes with applications, Bull. Math. Soc. Sci. Math. R.S.Roumanie (N.S.), 27(1983), no.75, 145-160.

[9] St. Ruscheweyh and V. Singh, On the order of starlikeness of hypergeometric functions, J. Math. Anal. Appl. 113(1986), 1-11.

[10] H. Silverman, Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl. 172(1993), 574-581.

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