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SOME NEW RESULTS ASSOCIATED WITH THE BESSEL-STRUVE KERNEL FUNCTION

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Abstract. The main object of this note is to present two unified integrals associated with the Bessel-Struve kernel function, which are expressed in terms of Wright hypergeometric function. Some integrals involving exponential functions, modified Bessel functions and Struve functions are also indicated as special cases of our main results. Finally, with the help of our main results and their special cases, we derive two reduction formulas for the Wright hypergeometric function.

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1. Introduction

In recent years, numerous (potentially useful) integral formulas associated with some well known Special functions (for example, hypergeometric functions, Bessel functions, Whittaker functions, Mittag-leffler functions, etc.) have been considered by several authors. (see[3],[5],[6],[7],[8],[9],[10]) In a Sequel of such type of works, in this paper, we further establish two new unified integral formulas involving Bessel-Struve kernel function, which are expressed in terms of wright hypergeometric functions.

For our present investigation, we recall here the following definitions of some well known Special functions :

The Wright hypergeometric is denoted by ${}_p\Psi_q$ and is defined by (see[2],[4])

$$ {}_p\Psi_q\left[\begin{array}{cccc} (\alpha_1\ ,\ A_1), & \ldots.., & (\alpha_p\ ,\ A_p); & \\ & & & z \\ (\beta_1\ ,\ B_1), & \ldots.., & (\beta_q\ ,\ B_q); & \end{array}\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_jk)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jk)}\,\frac{z^k}{k!}, \tag{1} $$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0.$$

A special case of (1) is

$$ {}_p\Psi_q \left[\begin{array}{cccc} (\alpha_1\ ,\ 1), & ....., & (\alpha_p\ ,\ 1); & \\ & & & z \\ (\beta_1\ ,\ 1), & ....., & (\beta_q\ ,\ 1); & \end{array} \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \ {}_pF_q \left[\begin{array}{cccc} \alpha_1, & ....., & \alpha_p\ ; & \\ & & & z \\ \beta_1, & ....., & \beta_q\ ; & \end{array} \right], \tag{2}$$

where ${}_pF_q$ is the generalized hypergeometric function defined by (see[2])

$$ {}_pF_q \left[\begin{array}{cccc} \alpha_1, & ....., & \alpha_p\ ; & \\ & & & z \\ \beta_1, & ....., & \beta_q\ ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

$$= {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots \beta_q;\ z), \tag{3}$$

where $(\lambda)_n$, is the well known Pochhammer's symbol (see[4]).
The Bessel-Struve kernel function $S_\alpha(\lambda z), \lambda \in C$ which is unique solution of the initial value problem $\ell_\alpha u(z) = \lambda^2 u(z)$ with the initial condition $u(0) = 1$ and $u'(0) = \frac{\lambda\Gamma(\alpha+1)}{\sqrt{\Pi}\Gamma(\alpha+3/2)}$ is given by (see[1])

$$S_{\alpha(\lambda z)} = \jmath_{\alpha(\imath\lambda z)} - \imath h_\alpha(\imath\lambda z), \forall z \in C$$

where $\jmath_\alpha$ and h_α are the normalized Bessel and Struve functions. The series representation, of the Bessel-Struve kernel function is given as follows:

$$S_\alpha(\lambda z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^n \Gamma(\alpha+1)\Gamma(n+1)/2}{\sqrt{\pi} n! \Gamma(n/2+\alpha+1)}. \tag{4}$$

Also we have the following relations of Bessel-Struve kernel function with exponential functions, modified Bessel functions and Struve functions :

$$S_{-1/2}(z) = e^z \tag{5}$$

$$S_{1/2}(z) = \frac{e^z - 1}{z} \tag{6}$$

$$S_0(z) = I_0(z) + L_0(z) \tag{7}$$

$$S_1(z) = \frac{2I_1(z) + L_1(z)}{z} \tag{8}$$

whereI_0, L_0 and I_1, L_1 are the modified Bessel and Struve functions of order zero and one respectively (see[4])
Furthermore, we recall here the following known result of Lavoie and Trottier. (see[7])

$$\int_0^1 x^{\alpha-1}\ (1-x)^{2\beta-1} \left(1-\frac{x}{3}\right)^{2\alpha-1} \left(1-\frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2\alpha} \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{9}$$

where $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

2. Main Results

This section deals with some integral formulas involving Bessel-Struve kernel function.

Theorem 1. *The following integral formula holds true:*
For $\rho, \sigma, \lambda \in C$ and $x > 0$ with $\Re(\sigma) > 0, \Re(\rho) > 0$,

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda y(1-x/4)(1-x)^2\right) dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \frac{\Gamma(\alpha+1)\Gamma(\rho)}{\sqrt{\pi}}\ {}_2\Psi_2\left[\begin{array}{ccc} (1/2, 1/2), & (\sigma, 1); & \\ & & \lambda y \\ (\alpha+1, 1/2), & (\rho+\sigma, 1); & \end{array}\right]. \tag{10}$$

Proof. By making use of (4) in the integrand of (10) and then interchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda y(1-x/4)(1-x)^2\right) dx$$

$$= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(\lambda y)^n \Gamma(n+1)/2}{n!\Gamma(n/2+\alpha+1)} \int_0^1 x^{\rho-1}\ (1-x)^{2(\sigma+n)-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma+n-1} dx$$

Now using (9) in the above equation we get

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda y(1-x/4)(1-x)^2\right)dx$$
$$=\left(\frac{2}{3}\right)^{2\rho}\,\frac{\Gamma(\alpha+1)\Gamma(\rho)}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)/2\Gamma(\sigma+n)(\lambda y)^n}{\Gamma(n/2+\alpha+1)\Gamma(\rho+\sigma+n)n!} \tag{11}$$

which upon using (1) yields (10). This completes the proof of Theorem 1

Theorem 2. *The following integral formula holds true :*
For $\rho,\sigma,\lambda \in C$ and $x>0$ with $\Re(\sigma)>0, \Re(\rho)>0$,

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda xy(1-x/3)^2\right)dx$$
$$=\left(\frac{2}{3}\right)^{2\rho}\,\frac{\Gamma(\alpha+1)\Gamma(\sigma)}{\sqrt{\pi}}\,{}_2\Psi_2\left[\begin{array}{ccc}(1/2,1/2), & (\rho,1); & \frac{4\lambda y}{9}\\ (\alpha+1,1/2), & (\rho+\sigma,1); & \end{array}\right]. \tag{12}$$

where ${}_2\Psi_2$ is the Wright hypergeometric function defined by (1).

Proof. It is easy to see that a similar argument as in the proof of Theorem 1 will establish the integral formula (12).

Next we consider other variations of Theorem 1 and Theorem 2 in the form of corollaries:

Corollary 3. *In (11), on separating the hypergeometric series into its even and odd terms, we get the following integral formulas:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda y(1-x/4)(1-x)^2\right)dx$$
$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma(\alpha+1)\Gamma(\rho)\left\{{}_1\Psi_2\left[\begin{array}{cc}(\sigma,2); & \frac{\lambda^2y^2}{4}\\ (\alpha+1,1),(\rho+\sigma,2); & \end{array}\right]\right.$$
$$\left.+\left(\frac{\lambda y}{2}\right){}_2\Psi_3\left[\begin{array}{cc}(1,1),(\sigma+1,2); & \frac{\lambda^2y^2}{4}\\ (\alpha+3/2,1),(\rho+\sigma+1,2),(3/2,1); & \end{array}\right]\right\}. \tag{13}$$

where $\Re(\sigma)>0, \Re(\rho)>0$.

Corollary 4. *On expanding the R.H.S of (12) in series form and then separating the resulting series into its even and odd terms, we obtain*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda xy(1-x/3)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma(\alpha+1)\Gamma(\sigma)\left\{ {}_1\Psi_2\left[\begin{matrix}(\rho,2); \\ (\alpha+1,1),(\rho+\sigma,2);\end{matrix}\ \frac{4\lambda^2y^2}{81}\right]\right.$$

$$\left. + \left(\frac{2\lambda y}{9}\right){}_2\Psi_3\left[\begin{matrix}(1,1),(\rho+1,2); \\ (\alpha+3/2,1),(\rho+\sigma+1,2),(3/2,1);\end{matrix}\ \frac{4\lambda^2y^2}{81}\right]\right\}. \quad (14)$$

where $\Re(\sigma)>0, \Re(\rho)>0$.

Corollary 5. *On applying the result* $(\lambda_n)=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ *in (13) and then by using (3), we get the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda y(1-x/4)(1-x)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma(\alpha+1)B(\rho,\sigma)\left\{\frac{1}{\Gamma(\alpha+1)}{}_2F_3\left[\begin{matrix}(\sigma/2),(\frac{\sigma+1}{2}); \\ (\alpha+1),\left(\frac{\rho+\sigma}{2}\right),\left(\frac{\rho+\sigma+1}{2}\right);\end{matrix}\ \frac{\lambda^2y^2}{4}\right]\right.$$

$$\left. + \frac{\lambda y\sigma}{\sqrt{\pi}\Gamma(\alpha+3/2)(\rho+\sigma)}{}_3F_4\left[\begin{matrix}(1),\left(\frac{\sigma+1}{2}\right),\left(\frac{\sigma+2}{2}\right); \\ (\alpha+3/2),\left(\frac{\rho+\sigma+1}{2}\right),\left(\frac{\rho+\sigma+2}{2}\right),(3/2);\end{matrix}\ \frac{\lambda^2y^2}{4}\right]\right\}. \quad (15)$$

where $\Re(\sigma)>0, \Re(\rho)>0,$ *and* ${}_2F_3, {}_3F_4$ *are the generalized hypergeometric function defined by (3).*

Corollary 6. *Further,on applying The result* $(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ *in (14) and then by using (3), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1} S_\alpha\left(\lambda xy(1-x/3)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2\rho}\Gamma(\alpha+1)B(\rho,\sigma)\left\{\frac{1}{\Gamma(\alpha+1)}{}_2F_3\left[\begin{matrix}(\rho/2),(\frac{\rho+1}{2}); \\ (\alpha+1),\left(\frac{\rho+\sigma}{2}\right),\left(\frac{\rho+\sigma+1}{2}\right);\end{matrix}\ \frac{4\lambda^2y^2}{81}\right]\right.$$

$$+\frac{4\lambda y\rho}{9\sqrt{\pi}\Gamma(\alpha+3/2)(\rho+\sigma)}{}_3F_4\left[\begin{array}{c}(1),\left(\frac{\rho+1}{2}\right),\left(\frac{\rho+2}{2}\right);\\(\alpha+3/2),\left(\frac{\rho+\sigma+1}{2}\right),\left(\frac{\rho+\sigma+2}{2}\right),(3/2);\end{array}\frac{4\lambda^2y^2}{81}\right]\Bigg\}. \tag{16}$$

where $\Re(\sigma)>0, \Re(\rho)>0$.

3. Special Cases

In this section, we derive some interesting integral formulas involving exponential functions, Bessel functions and Struve function as follows:

Corollary 7. *On setting* $\alpha=-1/2$ *and* $\lambda=1$ *in (10), and then by using (5), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\,e^{\left(y(1-x/4)(1-x)^2\right)}dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma(\rho)\,{}_1\Psi_1\left[\begin{array}{c}(\sigma,1);\\(\rho+\sigma,1);\end{array}y\right]. \tag{17}$$

where $\Re(\sigma)>0, \Re(\rho)>0$.

Corollary 8. *Further, on setting* $\alpha=-1/2$ *and* $\lambda=1$ *in (12), and then by using (5), we arrive at*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\,e^{\left(xy(1-x/3)^2\right)}dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma(\sigma)\,{}_1\Psi_1\left[\begin{array}{c}(\rho,1);\\(\rho+\sigma,1);\end{array}\frac{4\lambda}{9}\right]. \tag{18}$$

where $\Re(\sigma)>0, \Re(\rho)>0$.

Corollary 9. *On taking* $\alpha=-1/2$ *and* $\lambda=1$ *in (13), and then by using (5), we get*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\,e^{\left(y(1-x/4)(1-x)^2\right)}dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \Gamma(\rho)\sqrt{\pi} \left\{ {}_1\Psi_2 \left[\begin{array}{c} (\sigma, 2); \\ (1/2, 1), (\rho+\sigma, 2); \end{array} \frac{y^2}{4} \right] + \left(\frac{y}{2}\right) {}_1\Psi_2 \left[\begin{array}{c} (\sigma+1, 2); \\ (3/2, 1), (\rho+\sigma+1, 2); \end{array} \frac{y^2}{4} \right] \right\}. \tag{19}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 10. *Further on taking* $\alpha = -1/2$ *and* $\lambda = 1$ *in (14), and then by using (5), we arrive at*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1}\ e^{\left(xy(1-x/3)^2\right)} dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \Gamma(\sigma)\sqrt{\pi} \left\{ {}_1\Psi_2 \left[\begin{array}{c} (\rho, 2); \\ (1/2, 1), (\rho+\sigma, 2); \end{array} \frac{4y^2}{81} \right] + \left(\frac{2y}{9}\right) {}_1\Psi_2 \left[\begin{array}{c} (\rho+1, 2); \\ (3/2, 1), (\rho+\sigma+1, 2); \end{array} \frac{4y^2}{81} \right] \right\}. \tag{20}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 11. *On setting* $\alpha = 1/2$ *and* $\lambda = 1$ *in (10), and then using (6), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-3} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-2} \left(\ e^{y(1-x/4)(1-x)^2} - 1\right) dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \frac{y\Gamma(\rho)}{2} {}_2\Psi_2 \left[\begin{array}{c} (\sigma, 1), (1/2, 1/2); \\ (\rho+\sigma, 1), (3/2, 1/2); \end{array} y \right]. \tag{21}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 12. *Further, on setting* $\alpha = 1/2$ *and* $\lambda = 1$ *in (12), and then using (6), we arrive at*

$$\int_0^1 x^{\rho-2}\ (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-3} \left(1-\frac{x}{4}\right)^{\sigma-1} \left(\ e^{xy(1-x/3)^2} - 1\right) dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} \frac{y\Gamma(\sigma)}{2} {}_2\Psi_2 \left[\begin{array}{c} (\rho, 1), (1/2, 1/2); \\ (\rho+\sigma, 1), (3/2, 1/2); \end{array} \frac{4y}{9} \right]. \tag{22}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 13. *On setting $\alpha = 1/2$ and $\lambda = 1$ in (13), and then using (6), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-3}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-2}\left(\ e^{y(1-x/4)(1-x)^2}-1\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{y\sqrt{\pi}\Gamma(\rho)}{2}\left\{{}_1\Psi_2\left[\begin{matrix}(\sigma,2); \\ (3/2,1),(\rho+\sigma,2);\end{matrix}\ \frac{y^2}{4}\right]\right.$$

$$\left.+\left(\frac{y}{2}\right){}_2\Psi_3\left[\begin{matrix}(1,1),(\sigma+1,2); \\ (2,1),(\rho+\sigma+1,2),(3/2,1);\end{matrix}\ \frac{y^2}{4}\right]\right\}. \tag{23}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 14. *Further, on setting $\alpha = 1/2$ and $\lambda = 1$ in (14), and then using (6), we arrive at*

$$\int_0^1 x^{\rho-2}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-3}\left(1-\frac{x}{4}\right)^{\sigma-1}\left(\ e^{xy(1-x/3)^2}-1\right)dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{y\sqrt{\pi}\Gamma(\sigma)}{2}\left\{{}_1\Psi_2\left[\begin{matrix}(\rho,2); \\ (3/2,1),(\rho+\sigma,2);\end{matrix}\ \frac{4y^2}{81}\right]\right.$$

$$\left.+\left(\frac{2y}{9}\right){}_2\Psi_3\left[\begin{matrix}(1,1),(\rho+1,2); \\ (2,1),(\rho+\sigma+1,2),(3/2,1);\end{matrix}\ \frac{4y^2}{81}\right]\right\}. \tag{24}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 15. *On setting $\alpha = 0$ and $\lambda = 1$ in (10), and then using (7), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\ (1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[I_0\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)+L_0\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho)}{\sqrt{\pi}}{}_2\Psi_2\left[\begin{matrix}(\frac{1}{2},\frac{1}{2}), & (\sigma,1); \\ (1,\frac{1}{2}), & (\rho+\sigma,1);\end{matrix}\ y\right]. \tag{25}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$, *and* I_0, L_0 *are the modified Bessel and Struve functions of order zero.*

Corollary 16. *Further on setting $\alpha = 0$ and $\lambda = 1$ in (12), and then using (7), we arrive at*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[I_0\left(xy\left(1-\frac{x}{3}\right)^2\right)+L_0\left(xy\left(1-\frac{x}{3}\right)^2\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\ \frac{\Gamma(\sigma)}{\sqrt{\pi}}\,{}_2\Psi_2\left[\begin{matrix}(\frac{1}{2},\frac{1}{2}), & (\rho,1); \\ (1,\frac{1}{2}), & (\rho+\sigma,1);\end{matrix}\ \frac{4y}{9}\right]. \tag{26}$$

where $\Re(\sigma)>0, \Re(\rho)>0,$ *and* I_0, L_0 *are the modified Bessel and Struve functions of order zero.*

Corollary 17. *On setting $\alpha = 0$ and $\lambda = 1$ in (13), and then using (7), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[I_0\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)+L_0\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma(\rho)\left\{{}_1\Psi_2\left[\begin{matrix}(\sigma,2); \\ (1,1),(\rho+\sigma,2);\end{matrix}\ \frac{y^2}{4}\right]\right.$$

$$\left.+\left(\frac{y}{2}\right){}_2\Psi_3\left[\begin{matrix}(1,1),(\sigma+1,2); \\ (3/2,1),(\rho+\sigma+1,2),(3/2,1);\end{matrix}\ \frac{y^2}{4}\right]\right\}. \tag{27}$$

where $\Re(\sigma)>0, \Re(\rho)>0.$

Corollary 18. *Further on setting $\alpha = 0$ and $\lambda = 1$ in (14), and then using (7), we arrive at*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[I_0\left(xy\left(1-\frac{x}{3}\right)^2\right)+L_0\left(xy\left(1-\frac{x}{3}\right)^2\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\Gamma(\sigma)\left\{{}_1\Psi_2\left[\begin{matrix}(\rho,2); \\ (1,1),(\rho+\sigma,2);\end{matrix}\ \frac{4y^2}{81}\right]\right.$$

$$\left.+\left(\frac{2y}{9}\right){}_2\Psi_3\left[\begin{matrix}(1,1),(\rho+1,2); \\ (3/2,1),(\rho+\sigma+1,2),(3/2,1);\end{matrix}\ \frac{4y^2}{81}\right]\right\}. \tag{28}$$

where $\Re(\sigma)>0, \Re(\rho)>0.$

Corollary 19. *On setting $\alpha = 1$ and $\lambda = 1$ in (10), and then using (8), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-3}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-2}\left[2I_1\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)+L_1\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{y\Gamma(\rho)}{\sqrt{\pi}}\,{}_2\Psi_2\left[\begin{array}{ll} (\frac{1}{2},\frac{1}{2}), & (\sigma,1); \\ & & y \\ (2,\frac{1}{2}), & (\rho+\sigma,1); \end{array}\right]. \tag{29}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$, and I_1, L_1 are the modified Bessel and Struve functions of order zero.

Corollary 20. *Further, on setting $\alpha = 1$ and $\lambda = 1$ in (12), and then using (8), we arrive at*

$$\int_0^1 x^{\rho-2}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-3}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[2I_1\left(xy\left(1-\frac{x}{3}\right)^2\right)+L_1\left(xy\left(1-\frac{x}{3}\right)^2\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho}\frac{y\Gamma(\sigma)}{\sqrt{\pi}}\,{}_2\Psi_2\left[\begin{array}{ll} (\frac{1}{2},\frac{1}{2}), & (\rho,1); \\ & & \frac{4y}{9} \\ (2,\frac{1}{2}), & (\rho+\sigma,1); \end{array}\right]. \tag{30}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 21. *On setting $\alpha = 1$ and $\lambda = 1$ in (13), and then using (8), we obtain the following integral formula:*

$$\int_0^1 x^{\rho-1}\,(1-x)^{2\sigma-3}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\sigma-2}\left[2I_1\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)+L_1\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)\right]dx$$

$$=\left(\frac{2}{3}\right)^{2\rho} y\Gamma(\rho)\left\{{}_1\Psi_2\left[\begin{array}{l} (\sigma,2); \\ & \frac{y^2}{4} \\ (2,1),(\rho+\sigma,2); \end{array}\right]\right.$$
$$\left.+\left(\frac{y}{2}\right){}_2\Psi_3\left[\begin{array}{l} (1,1),(\sigma+1,2); \\ & \frac{y^2}{4} \\ (3/2,1),(\rho+\sigma+1,2),(3/2,1); \end{array}\right]\right\}. \tag{31}$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Corollary 22. *Further, on setting $\alpha = 1$ and $\lambda = 1$ in (14), and then using (8), we arrive at*

$$\int_0^1 x^{\rho-2}\,(1-x)^{2\sigma-1}\left(1-\frac{x}{3}\right)^{2\rho-3}\left(1-\frac{x}{4}\right)^{\sigma-1}\left[2I_1\left(xy\left(1-\frac{x}{3}\right)^2\right)+L_1\left(xy\left(1-\frac{x}{3}\right)^2\right)\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} y\Gamma(\sigma) \left\{ {}_1\Psi_2 \left[\begin{array}{cc} (\rho,2); & \\ & \frac{4y^2}{81} \\ (2,1),(\rho+\sigma,2); & \end{array} \right] \right.$$

$$\left. + \left(\frac{2y}{9}\right) {}_2\Psi_3 \left[\begin{array}{cc} (1,1),(\rho+1,2); & \\ & \frac{4y^2}{81} \\ (3/2,1),(\rho+\sigma+1,2),(3/2,1); & \end{array} \right] \right\}. \quad (32)$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

Remark 1. *With the help of the result* $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ *in (17) and (18), respectively, and then by using the definition of Whittaker function :*

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-z/2} {}_1F_1\left(\frac{1}{2}+\mu-k\ ; 2\mu+1\ ; z\right),$$

we obtain the following very interesting integral formulas:

$$\int_0^1 x^{\rho-1} (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} e^{y(1-x/4)(1-x)^2} dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} B(\rho,\sigma)\ e^{y/2} y^{-\frac{(\rho+\sigma)}{2}} M_{\frac{\rho-\sigma}{2},\frac{\rho+\sigma-1}{2}}(y). \quad (33)$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

$$\int_0^1 x^{\rho-1} (1-x)^{2\sigma-1} \left(1-\frac{x}{3}\right)^{2\rho-1} \left(1-\frac{x}{4}\right)^{\sigma-1} e^{xy(1-x/3)^2} dx$$

$$= \left(\frac{2}{3}\right)^{2\rho} B(\rho,\sigma)\ e^{2y/9} \left(\frac{4y}{9}\right)^{-\frac{(\rho+\sigma)}{2}} M_{\frac{\rho-\sigma}{2},\frac{\rho+\sigma-1}{2}}(4y/9). \quad (34)$$

where $\Re(\sigma) > 0, \Re(\rho) > 0$.

4. Reducibility of the Wright hypergeometric function

Here we present two interesting reduction formulas for the Wright hypergeometric function as follows:

$${}_2\Psi_2 \left[\begin{array}{ccc} (\frac{1}{2},\frac{1}{2}), & (\sigma,1); & \\ & & \lambda y \\ (\alpha+1,\frac{1}{2}), & (\rho+\sigma,1); & \end{array} \right] = \sqrt{\pi} \left\{ {}_1\Psi_2 \left[\begin{array}{cc} (\sigma,2); & \\ & \frac{\lambda^2 y^2}{4} \\ (\alpha+1,1),(\rho+\sigma,2); & \end{array} \right] \right.$$

$$+\left(\frac{\lambda y}{2}\right) {}_2\Psi_3\left[\begin{array}{cc} (1,1),(\sigma+1,2); & \frac{\lambda^2 y^2}{4} \\ (\alpha+3/2,1),(\rho+\sigma+1,2),(3/2,1); & \end{array}\right]\Bigg\}. \quad (35)$$

$$ {}_2\Psi_2\left[\begin{array}{ccc} (\frac{1}{2},\frac{1}{2}), & (\rho,1); & \frac{4\lambda y}{9}. \\ (\alpha+1,\frac{1}{2}), & (\rho+\sigma,1); & \end{array}\right] = \sqrt{\pi}\Bigg\{ {}_1\Psi_2\left[\begin{array}{cc} (\rho,2); & \frac{4\lambda^2 y^2}{81} \\ (\alpha+1,1),(\rho+\sigma,2); & \end{array}\right]$$

$$+\left(\frac{2\lambda y}{9}\right) {}_2\Psi_3\left[\begin{array}{cc} (1,1),(\rho+1,2); & \frac{4\lambda^2 y^2}{81} \\ (\alpha+3/2,1),(\rho+\sigma+1,2),(3/2,1); & \end{array}\right]\Bigg\}. \quad (36)$$

The result (35) can be established by comparing (10) and (13), and the result (36) can be established by comparing (12) and (14).

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