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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS AND DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. The object of the present paper is to derive several interesting properties of the class $C_n(\lambda, \alpha)$ consisting of analytic univalent functions with negative coefficients by using Sălăgean operator. Coefficient inequalities, distortion theorems and closure theorems of functions in the class $C_n(\lambda, \alpha)$ are determined. Also radii of close to convexity, starlikeness and convexity for are determined. Furthermore, integral operators and modified Hadamard products of several functions belonging to the class $C_n(\lambda, \alpha)$ are studied her.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. For a function $f(z)$ in $\mathcal{A}$, let

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \end{aligned}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \ (n \in \mathbb{N} = \{1, 2, ...\}),$$

$$= z + \sum_{k=2}^{\infty} k^n a_k z^k \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.2}$$

The differential operator D^n was introduced by Salagean [5]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to $\mathcal{A}$ is in the class $K_n(\lambda, \alpha)$ if and only if

$$Re\left\{\frac{(D^n f(z))' + z(D^n f(z))''}{(D^n f(z))' + \lambda z(D^n f(z))''}\right\} > \alpha \ (n \in \mathbb{N}_0), \tag{1.3}$$

for some α $(0 \le \alpha < 1)$, λ $(0 \le \lambda < 1)$ and for all $z \in \mathbb{U}$.
Let T denote the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ (a_k \ge 0). \tag{1.4}$$

Further, we define the class $C_n(\lambda, \alpha)$ by

$$C_n(\lambda, \alpha) = K_n(\lambda, \alpha) \cap T. \tag{1.5}$$

We note that by specializing the parameters n, λ, and α, we obtain the following subclasses studied by various authors:
(i) $C_0(\lambda, \alpha) = C(\lambda, \alpha)$ (Altintas and Owa [1]);
(ii) $C_0(0, \alpha) = C(\alpha)$ (Silverman [8]);
(iii) $C_n(0, \alpha) = C_n(\alpha) =$

$$Re\left\{1 + \frac{z(D^n f(z))''}{(D^n f(z))'}\right\} > \alpha \ (n \in \mathbb{N}_0, \ 0 \le \alpha < 1 \text{ and for all } z \in \mathbb{U}).$$

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this paper that

$$0 \le \alpha < 1, \ 0 \le \lambda < 1, \ n \in \mathbb{N}_0 \text{ and } z \in \mathbb{U}.$$

Theorem 1. Let the function $f(z)$ be given by (1.6). Then $f(z) \in C_n(\lambda, \alpha)$ if and only if

$$\sum_{k=2}^{\infty} k^{n+1}\{k - \alpha[1 + \lambda(k-1)]\} a_k \le 1 - \alpha. \tag{2.1}$$

Proof. Suppose that (2.1) holds. Then we have

$$\left|\frac{(D^n f(z))' + z(D^n f(z))''}{(D^n f(z))' + \lambda z(D^n f(z))''} - 1\right| = \left|\frac{\sum\limits_{k=2}^{\infty} k^{n+1}(k-1)(1-\lambda)a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}}\right|$$

$$\leq \frac{\sum\limits_{k=2}^{\infty} k^{n+1}(k-1)(1-\lambda)a_k}{1 - \sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k} \leq 1-\alpha, \tag{2.2}$$

this shows that the values of $\frac{(D^n f(z))' + z(D^n f(z))''}{(D^n f(z))' + \lambda z(D^n f(z))''}$ lies in a circle centered at $\omega = 1$ whose radius is $1-\alpha$. Hence $f(z)$ satisfies the condition (1.3).

Conversely, assume that the function $f(z)$ defined by (1.4) be in the class $C_n(\lambda, \alpha)$. Then

$$Re\left\{\frac{(D^n f(z))' + z(D^n f(z))''}{(D^n f(z))' + \lambda z(D^n f(z))''}\right\} = Re\left\{\frac{1 - \sum\limits_{k=2}^{\infty} k^{n+2} a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}}\right\} > \alpha, \tag{2.3}$$

for $z \in \mathbb{U}$. Choose values of z on the real axis so that $\frac{(D^n f(z))' + z(D^n f(z))''}{(D^n f(z))' + \lambda z(D^n f(z))''}$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we obtain

$$1 - \sum_{k=2}^{\infty} k^{n+2} a_k \geq \alpha\left\{1 - \sum_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k\right\}, \tag{2.4}$$

which gives (2.1).

Corollary 1. Let the function $f(z)$ defined by (1.4) be in the class $C_n(\lambda, \alpha)$. Then we have

$$a_k \leq \frac{1-\alpha}{k^{n+1}\{k - \alpha[1+\lambda(k-1)]\}} \ (k \geq 2). \tag{2.5}$$

The equality in (2.5) is attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{k^{n+1}\{k - \alpha[1+\lambda(k-1)]\}} z^k \ (k \geq 2). \tag{2.6}$$

3. Some properties of the class $C_n(\lambda, \alpha)$

Theorem 2. Let $0 \le \alpha < 1$, $0 \le \lambda_2 \le \lambda_1$ and $n \in \mathbb{N}_0$. Then

$$C_n(\lambda_1, \alpha) \subseteq C_n(\lambda_2, \alpha).$$

Proof. It follows from Theorem 1 that

$$\begin{aligned} & \sum_{k=2}^{\infty} k^{n+1}\{k - \alpha[1 + \lambda_1(k-1)]\}a_k \\ \le & \sum_{k=2}^{\infty} k^{n+1}\{k - \alpha[1 + \lambda_2(k-1)]\}a_k \le 1 - \alpha, \end{aligned}$$

for $f(z) \in C_n(\lambda_1, \alpha)$. Hence $f(z)$ in $C_n(\lambda_2, \alpha)$.

Theorem 3. Let $0 \le \alpha < 1$, $0 \le \lambda \le 1$ and $n \in \mathbb{N}_0$. Then

$$C_{n+1}(\lambda, \alpha) \subseteq C_n(\lambda, \alpha).$$

Proof. The proof follows immediately from Theorem 1.

4. Distortion theorems

Theorem 4. Let the function $f(z)$ given by (1.4) be in the class $C_n(\lambda, \alpha)$. Then we have

$$\left|D^i f(z)\right| \ge |z| - \frac{1-\alpha}{2^{n-i+1}[2 - \alpha(1+\lambda)]} |z|^2 \tag{4.1}$$

and

$$\left|D^i f(z)\right| \le |z| + \frac{1-\alpha}{2^{n-i+1}[2 - \alpha(1+\lambda)]} |z|^2, \tag{4.2}$$

for $z \in \mathbb{U}$, where $0 \le i \le n$. Then equalities in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{2^{n+1}[2 - \alpha(1+\lambda)]} z^2. \tag{4.3}$$

Proof. Note that $f(z) \in C_n(\lambda, \alpha)$ if and only if $D^i f(z) \in C_{n-i}(\lambda, \alpha)$, where

$$D^i f(z) = z - \sum_{k=2}^{\infty} k^i a_k z^k. \tag{4.4}$$

Using Theorem 1, we know that

$$2^{n-i+1}[2-\alpha(1+\lambda)] \sum_{k=2}^{\infty} k^i a_k \leq \sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\} a_k \leq 1-\alpha, \tag{4.5}$$

that is,that

$$\sum_{k=2}^{\infty} k^i a_k \leq \frac{1-\alpha}{2^{n-i+1}[2-\alpha(1+\lambda)]}. \tag{4.6}$$

It follows from (4.4) and (4.6) that

$$\left|D^i f(z)\right| \geq |z| - |z|^2 \sum_{k=2}^{\infty} k^i a_k \geq |z| - \frac{1-\alpha}{2^{n-i+1}[2-\alpha(1+\lambda)]} |z|^2 \tag{4.7}$$

and

$$\left|D^i f(z)\right| \leq |z| + |z|^2 \sum_{k=2}^{\infty} k^i a_k \leq |z| + \frac{1-\alpha}{2^{n-i+1}[2-\alpha(1+\lambda)]} |z|^2 . \tag{4.8}$$

This completes the proof of Theorem 4.
Taking $i = 0$ in Theorem 4, we have the following corollary

Corollary 2. Let the function $f(z)$ given by (1.4) be in the class $C_n(\lambda, \alpha)$. Then we have

$$|f(z)| \geq |z| - \frac{1-\alpha}{2^{n+1}[2-\alpha(1+\lambda)]} |z|^2 \tag{4.9}$$

and

$$|f(z)| \leq |z| + \frac{1-\alpha}{2^{n+1}[2-\alpha(1+\lambda)]} |z|^2 , \tag{4.10}$$

for $z \in \mathbb{U}$, where $0 \leq i \leq n$. Then equalities in (4.1) and (4.2) are attained for the function $f(z)$ given by (4.3).

Taking $i = 1$ in Theorem 4, we have the following corollary

Corollary 3. Let the function $f(z)$ given by (1.4) be in the class $C_n(\lambda, \alpha)$. Then we have

$$\left|f'(z)\right| \geq 1 - \frac{1-\alpha}{2^{n}[2-\alpha(1+\lambda)]} |z| , \tag{4.11}$$

and

$$\left|f^{'}(z)\right| \leq 1+\frac{1-\alpha}{2^{n}[2-\alpha(1+\lambda)]}\left|z\right|, \tag{4.12}$$

for $z \in \mathbb{U}$. the equalities in (4.12) and (4.13) are attained for the function $f(z)$ given by (4.3).

Corollary 4. Let the function $f(z)$ be given by (1.4) be in the class $C_n(\lambda,\alpha)$. Then the unit disc $\mathbb{U}$ is mapped onto a domain that contains the disc

$$|\omega| < \frac{2^{n+1}[2-\alpha(1+\lambda)]-(1-\alpha)}{2^{n+1}[2-\alpha(1+\lambda)]}. \tag{4.13}$$

The result is sharp with the extremal function $f(z)$ given by (4.3).

5. Closure theorems

Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ be given by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \ (a_{k,j} \geq 0;\ z \in \mathbb{U}). \tag{5.1}$$

We shall prove the following results for the class $C_n(\lambda,\alpha)$.

Theorem 5. Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ given by (5.1) be in the class $C_n(\lambda,\alpha)$ for every $j = 1, 2, ..., m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{m} c_j f_j(z) \ (c_j \geq 0) \tag{5.2}$$

is also in the same class $C_n(\lambda,\alpha)$, where

$$\sum_{j=1}^{m} c_j = 1. \tag{5.3}$$

Proof. According to the definition of $h(z)$, we can write

$$h(z) = z - \sum_{k=2}^{\infty}(\sum_{j=1}^{m} c_j a_{k,j}) z^k. \tag{5.4}$$

Futher, since $f_j(z)$ are in $C_n(\lambda,\alpha)$ for every $j=1,2,...,m$ we get

$$\sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}a_{k,j} \leq 1-\alpha, \tag{5.5}$$

for every $j=1,2,...,m$. Hence we can see that

$$\sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}(\sum_{j=1}^{m} c_j a_{k,j}) \tag{5.6}$$

$$= \sum_{j=1}^{m} c_j \left(\sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}a_{k,j} \right)$$

$$\leq \left(\sum_{j=1}^{m} c_j \right)(1-\alpha) = 1-\alpha,$$

which implies that $h(z)$ in $C_n(\lambda,\alpha)$. Thus we have the theorem.
Taking $m=2$, $c_1=\mu$, $c_2=1-\mu$ in Theorem 5, we have the following corollary

Corollary 5. The class $C_n(\lambda,\alpha)$ is closed under convex linear combination.

Proof. Let the functions $f_j(z)$ $(j=1,2)$ be given by (5.1) be in the class $C_n(\lambda,\alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \tag{5.7}$$

is in the class $C_n(\lambda,\alpha)$.
As a consequence of Corollary 5, there exists the extreme points of the class $C_n(\lambda,\alpha)$.

Theorem 6. Let $f_1(z)=z$ and

$$f_k(z) = z - \frac{1-\alpha}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}} z^k \ (k \geq 2), \tag{5.8}$$

for $0 \leq \alpha < 1$, $0 \leq \lambda < 1$ and $n \in \mathbb{N}_0$. Then $f(z)$ is in the class $C_n(\lambda,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{5.9}$$

where $\mu_k \geq 0$ $(k \geq 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)=z-\sum_{k=2}^{\infty}\frac{1-\alpha}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}\mu_k z^k. \tag{5.10}$$

Then it follows that

$$\sum_{k=2}^{\infty}\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\frac{(1-\alpha)\mu_k}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}} \tag{5.11}$$

$$=\sum_{k=2}^{\infty}\mu_k=1-\mu_1\leq 1.$$

So by theorem 1, $f(z)$ in $C_n(\lambda,\alpha)$.
Conversely, assume that the function $f(z)$ defined by (1.4) belongs to the class $C_n(\lambda,\alpha)$. Then

$$a_k\leq\frac{1-\alpha}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}\ (k\geq 2). \tag{5.12}$$

Setting

$$\mu_k=\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}a_k\ (k\geq 2), \tag{5.13}$$

and

$$\mu_1=1-\sum_{k=2}^{\infty}\mu_k, \tag{5.14}$$

we can see that $f(z)$ can be expressed in the form (5.9). This completes the proof of Theorem 6.

Corollary 6. The extreme points of the class $C_n(\lambda,\alpha)$ are the functions $f_k(z)$ $(k\geq 1)$ given by Theorem 6.

6. Radii of close-to-convexity, starlikeness and convexity

Theorem 7. Let the function $f(z)$ defined by (1.4) be in the class $C_n(\lambda,\alpha)$.
Then $f(z)$ is close-to-convex of order η $(0\leq\eta<1)$ *in* $|z|\leq r_1(n,\ \lambda,\ \alpha,\ \eta)$, where

$$r_1(n,\lambda,\ \alpha,\ \eta)=\inf_k\left\{\frac{(1-\eta)k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right\}^{\frac{1}{k-1}}\ (k\geq 2). \tag{6.1}$$

The result is sharp, the extremal function given by (2.6).

Proof. We must show that

$$\left|f^{'}(z)-1\right| \leq 1-\eta \ for \ \ |z| \leq r_1, \tag{6.2}$$

where r_1 is given by (6.1). Indeed we find from (1.4) that

$$\left|f^{'}(z)-1\right| \leq \sum_{k=2}^{\infty} k a_k \left|z\right|^{k-1}.$$

Thus

$$\left|f^{'}(z)-1\right| \leq 1-\eta,$$

if

$$\sum_{k=1}^{\infty} \left(\frac{k}{1-\eta}\right) a_k \left|z\right|^{k-1} \leq 1. \tag{6.3}$$

But by using Theorem 1, (6.3) will be true if

$$\left(\frac{k}{1-\eta}\right) \left|z\right|^{k-1} \leq \left(\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right).$$

Then

$$|z| \leq \left\{\frac{(1-\eta)k^{n}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right\}^{\frac{1}{k-1}} \ (k \geq 2). \tag{6.4}$$

The result follows easily from (6.4).

Theorem 8. Let the function $f(z)$ defined by (1.4) be in the class $C_n(\lambda,\alpha)$. Then $f(z)$ is starlike of order η $(0 \leq \eta < 1)$ in $|z| \leq r_2(n,\ \lambda,\ \alpha,\ \eta)$, where

$$r_2(n,\lambda,\ \alpha,\ \eta) = \inf_{k} \left\{\frac{(1-\eta)k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{(k-\eta)(1-\alpha)}\right\}^{\frac{1}{k-1}} \ (k \geq 2). \tag{6.5}$$

The result is sharp, the extremal function given by (2.6).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\eta \ for \ \ |z| \leq r_2(n,\lambda,\ \alpha,\eta), \tag{6.6}$$

where r_2 is given by (6.5). Indeed we find from the definition of (1.4) that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{k=1}^{\infty}(k-1)a_k\left|z\right|^{k-1}}{1-\sum\limits_{k=1}^{\infty}a_k\left|z\right|^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1-\eta,$$

if

$$\sum_{k=1}^{\infty}\left(\frac{k-\eta}{1-\eta}\right)a_k\,|z|^{k-1}\leq 1. \tag{6.7}$$

But by using Theorem 1, (6.7) will be true if

$$\left(\frac{k-\eta}{1-\eta}\right)|z|^{k-1}\leq\left(\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right).$$

Then

$$|z|\leq\left\{\frac{(1-\eta)k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{(k-\eta)(1-\alpha)}\right\}^{\frac{1}{k-1}}\ (k\geq 2). \tag{6.8}$$

Corollary 7. Let the function $f(z)$ defined by (1.4) be in the class $C_n(\lambda,\alpha)$. Then $f(z)$ is convex of order η $(0\leq\eta<1)$ in $|z|\leq r_3(n,\lambda,\alpha,\eta)$, where

$$r_3(n,\lambda,\ \alpha,\ \eta)=\inf_k\left\{\frac{(1-\eta)k^{n}\{k-\alpha[1+\lambda(k-1)]\}}{(k-\eta)(1-\alpha)}\right\}^{\frac{1}{k-1}}\ (k\geq 2). \tag{6.9}$$

The result is sharp, the extremal function given by (2.6).

7. Integral operators

Theorem 9. Let the function $f(z)$ defined by (1.4) be in the class $C_n(\lambda,\alpha)$ and let c be a real number such that $c>-1$. Then the function $F(z)$ defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt, \tag{7.1}$$

also belongs to the class $C_n(\lambda,\alpha)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z)=z-\sum_{k=2}^{\infty}b_k z^k, \tag{7.2}$$

where

$$b_k=\left(\frac{c+1}{c+k}\right)a_k\ , \tag{7.3}$$

therefore, we have

$$\sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}b_k = \sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}\left(\frac{c+1}{c+k}\right)a_k$$

$$\leq \sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}a_k \leq 1-\alpha, \tag{7.4}$$

since $f(z) \in C_n(\lambda, \alpha)$. Hence, by Theorem 1, we have $F(z) \in C_n(\lambda, \alpha)$.

Theorem 10. Let c be a real number such that $c > -1$. If $F(z) \in C_n(\lambda, \alpha)$, then the function $f(z)$ defined by (7.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \left(\frac{(c+1)k^n\{k-\alpha[1+\lambda(k-1)]\}}{(c+k)(1-\alpha)}\right)^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.5}$$

The result is sharp.

Proof. Let $F(z) = z - \sum_{k=2}^{\infty} a_k z^k (a_k \geq 0)$. If follows from (7.1) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{(c+1)} = z - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right)a_k z^k (c > -1). \tag{7.6}$$

In order to obtain the requird result, it stuffices to show that $\left|f'(z)-1\right| < 1$ in $|z| < R^*$, where R^* is given by (7.5). Now

$$\left|f'(z)-1\right| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}. \tag{7.7}$$

Thus

$$\left|f'(z)-1\right| \leq 1,$$

if

$$\sum_{k=1}^{\infty}\left(\frac{k(c+k)}{c+1}\right) a_k |z|^{k-1} \leq 1. \tag{7.8}$$

But by using Theorem 1, (7.8) will be true if

$$\left(\frac{k(c+k)}{c+1}\right)|z|^{k-1} \leq \left(\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)}\right) \quad (k \geq 2), \tag{7.9}$$

or if

$$|z| \leq \left(\frac{(c+1)k^n\{k-\alpha[1+\lambda(k-1)]\}}{(c+k)(1-\alpha)}\right)^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.10}$$

Therefore $f(z)$ is univalent in $|z| < R^*$. Sharpness follows if we take

$$f(z) = z - \frac{(c+k)(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}(c+1)} z^k \ (k \geq 2). \tag{7.11}$$

8. Modified Hadamard products

Let the functions $f_j(z)$ $(j = 1, 2)$ be defined by (5.1).The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined here by

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1}a_{k,2}z^k = (f_2 * f_1)(z). \tag{8.1}$$

Theorem 11. Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $C_n(\lambda, \alpha)$. Then $(f_1 * f_2)(z) \in C_n(\lambda,\ \beta(n,\ \lambda,\ \alpha))$, where

$$\beta(n, \lambda, \alpha) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n+1}\{2-\alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}. \tag{8.2}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z - \frac{1-\alpha}{2^{n+1}\{2-\alpha(1+\lambda)\}} z^2 (z \in \mathbb{U}). \tag{8.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest β such that

$$\sum_{k=2}^{\infty} \frac{k^{n+1}\{k-\beta[1+\lambda(k-1)]\}}{1-\beta} a_{k,1}a_{k,2} \leq 1. \tag{8.4}$$

Since $f_j(z) \in C_n(\lambda, \alpha)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=2}^{\infty} \frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha} a_{k,1} \leq 1, \tag{8.5}$$

and

$$\sum_{k=2}^{\infty} \frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha} a_{k,2} \leq 1. \tag{8.6}$$

By the Cauchy Schwarz inequality, we have

$$\sum_{k=2}^{\infty} \frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{8.7}$$

Thus it is sufficient to show that

$$\frac{k^{n+1}\{k-\beta[1+\lambda(k-1)]\}}{1-\beta}a_{k,1}a_{k,2} \leq$$

$$\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\sqrt{a_{k,1}a_{k,2}} \ (k \geq 2), \tag{8.8}$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\beta)\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)\{k-\beta[1+\lambda(k-1)]\}} \ (k \geq 2). \tag{8.9}$$

Hence, in light of the inequality (8.9), it is sufficient to prove that

$$\frac{1-\alpha}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}} \leq$$

$$\frac{(1-\beta)\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)\{k-\beta[1+\lambda(k-1)]\}} \ (k \geq 2). \tag{8.10}$$

It follows from (8.10) that

$$\beta \leq 1 - \frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}^2 - [1+\lambda(k-1)](1-\alpha)^2}(k \geq 2). \tag{8.11}$$

Now defining the function $G(k)$ by

$$G(k) = 1 - \frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}^2 - [1+\lambda(k-1)](1-\alpha)^2}, \tag{8.12}$$

we see that $G(k)$ is an increasing function of k $(k \geq 2)$, letting $k = 2$ in (8.12), we obtain

$$\beta \leq G(2) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n+1}\{2-\alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}, \tag{8.13}$$

which evidently completes the proof of Theorem 11.

Corollary 1. For $f_1(z)$ and $f_2(z)$ as in Theorem 11, the function

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}a_{k,2}}z^k, \tag{8.14}$$

belongs to the class $C_n(\lambda, \alpha)$.
This result follows from the Cauchy-Schwarz inequality (8.7). It is sharp for the same functions as in Theorem 11.

Theorem 12. Let the functions $f_j(z)\,(j=1,2)$ defined by (5.1), $f_1(z)\in C_n(\lambda,\alpha)$ and $f_2(z)\in C_n(\lambda,\gamma)$. Then $(f_1*f_2)(z)\in C_n(\lambda,\eta(n,\,\lambda,\,\alpha,\,\gamma))$, where

$$\eta(n,\lambda,\alpha,\gamma)=1-\frac{(1-\lambda)(1-\alpha)(1-\gamma)}{2^{n+1}\{2-\alpha(1+\lambda)\}\{2-\gamma(1+\lambda)\}-(1+\lambda)(1-\alpha)(1-\gamma)}. \tag{8.15}$$

The result is best possible for the functions

$$f_1(z)=z-\frac{1-\alpha}{2^{n+1}\{2-\alpha(1+\lambda)\}}z^2, \tag{8.16}$$

and

$$f_2(z)=z-\frac{1-\gamma}{2^{n+1}\{2-\gamma(1+\lambda)\}}z^2. \tag{8.17}$$

Proof. Proceeding as in the proof of Theorem 11, we get

$$\eta(n,\lambda,\alpha,\gamma)\le B(k)=1-$$

$$\frac{(k-1)(1-\lambda)(1-\alpha)(1-\gamma)}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}\{k-\gamma[1+\lambda(k-1)]\}-[1+\lambda(k-1)](1-\alpha)(1-\gamma)}\ (k\ge 2) \tag{8.18}$$

Since the function $B(k)$ is an increasing function of k ($k\ge 2$), setting $k=2$ in (8.18) we get

$$\eta(n,\lambda,\alpha,\gamma)\ge B(2)=1-\frac{(1-\lambda)(1-\alpha)(1-\gamma)}{2^{n+1}\{2-\alpha(1+\lambda)\}\{2-\gamma(1+\lambda)\}-(1+\lambda)(1-\alpha)(1-\gamma)}. \tag{8.19}$$

This completes the proof of Theorem 12.

Corollary 9. Let the functions $f_j(z)$ ($j=1,\,2,\,3$) defined by (5.1) be in the class $C_n(\lambda,\alpha)$. Then $(f_1*f_2*f_3)(z)\in C_n(\lambda,\,\zeta(n,\,\lambda,\,\alpha))$, where

$$\zeta(n,\lambda,\alpha)=1-\frac{(1-\lambda)(1-\alpha)^3}{4^{n+1}\{2-\alpha(1+\lambda)\}^3-(1+\lambda)(1-\alpha)^3}. \tag{8.20}$$

The result is sharp for the functions $f_j(z)$ ($j=1,2,3$) given by (8.3).

Proof. Form Theorem 11, we have $(f_1*f_2)(z)\in C_n(\lambda,\,\beta(n,\,\lambda,\,\alpha))$, where β is given by (8.2). By using Theorem 12, we get $(f_1*f_2*f_3)(z)\in C_n(\lambda,\,\zeta(n,\,\lambda,\,\alpha))$, where

$$\zeta(n,\lambda,\alpha)=1-\frac{(1-\lambda)(1-\alpha)(1-\beta)}{2^{n+1}\{2-\alpha(1+\lambda)\}\{2-\beta(1+\lambda)\}-(1+\lambda)(1-\alpha)(1-\beta)} \tag{8.21}$$

$$=1-\frac{(1-\lambda)(1-\alpha)^3}{4^{n+1}\{2-\alpha(1+\lambda)\}^3-(1+\lambda)(1-\alpha)^3}. \tag{8.22}$$

This completes the proof of corollary 9.

Theorem 13. Let the functions $f_j(z)\,(j=1,2)$ defined by (5.1) be in the class $C_n(\lambda,\alpha)$. Then the function

$$h(z) = z - \sum_{k=2}^{\infty}\left(a_{k,1}^2 + a_{k,2}^2\right) z^k \tag{8.23}$$

belong to the class $C_n(\lambda,\phi(n,\lambda,\alpha))$, where

$$\phi(n,\lambda,\alpha) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^n\{2-\alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}. \tag{8.24}$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ given by (8.3).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right]^2 a_{k,1}^2 \le$$

$$\left[\sum_{k=2}^{\infty}\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}a_{k,1}\right]^2 \le 1, \tag{8.25}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right]^2 a_{k,2}^2 \le$$

$$\left[\sum_{k=2}^{\infty}\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}a_{k,2}\right]^2 \le 1. \tag{8.26}$$

It follows from (8.25) and (8.26) that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right]^2\left(a_{k,1}^2 + a_{k,2}^2\right) \le 1. \tag{8.27}$$

Therefore, we need to find the largest $\phi = \phi(n,\lambda,\alpha)$ such that

$$\frac{k^{n+1}\{k-\phi[1+\lambda(k-1)]\}}{1-\phi} \le$$

$$\frac{1}{2}\left[\frac{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right]^2 \;(k\ge 2), \tag{8.28}$$

that is,

$$\phi \leq 1 - \frac{2(k-1)(1-\lambda)(1-\alpha)^2}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}^2 - 2[1+\lambda(k-1)](1-\alpha)^2} \ (k \geq 2). \tag{8.29}$$

Since

$$D(k) = 1 - \frac{2(k-1)(1-\lambda)(1-\alpha)^2}{k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}^2 - 2[1+\lambda(k-1)](1-\alpha)^2}, \tag{8.30}$$

is an increasing function of $k(k \geq 2)$,setting $k = 2$ in (8.30) we get

$$\phi \leq D(2) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^n\{2-\alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}, \tag{8.31}$$

and Theorem 13 follows at once.

Theorem 14. Let the function $f_1(z) = z - \sum_{k=2}^{\infty} a_{k,1} z^k$ $(a_{k,1} \geq 0)$ be in the class $C_n(\lambda, \alpha)$ and $f_2(z) = z - \sum_{k=2}^{\infty} |a_{k,2}| z^k$, with $|a_{k,2}| \leq 1$, $k = 2, 3, ...$ .Then $(f_{1*}f_2)(z) \in C_n(\lambda, \alpha)$.

Proof. Since

$$\sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\} |a_{k,1}a_{k,2}| = \sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\} a_{k,1} |a_{k,2}|$$

$$\begin{aligned} &\leq \sum_{k=2}^{\infty} k^{n+1}\{k-\alpha[1+\lambda(k-1)]\} a_{k,1} \\ &\leq 1-\alpha, \end{aligned}$$

by Theorem 1, it follows that $(f_{1*}f_2)(z) \in C_n(\lambda, \alpha)$.

9. Definitions and applications of fractional calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g. [2], [9] and [10]. We find it to be convenient to recall here the following definitions which were used recently by Owa [4] and by Srivastava and Owa [7]).

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt \ (\mu > 0), \tag{9.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z)=\frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\mu}}dt \ (0\leq \mu<1), \tag{9.2}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-t)^{-\mu}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Definition 3. Under the hypotheses of definition 2, the fractional derivative of order $n+\mu$ is defined by

$$D_z^{n+\mu} f(z)=\frac{d^n}{dz^n}D_z^{\mu} f(z) \ (0\leq \mu<1;\ n\in \mathbb{N}_0)\ . \tag{9.3}$$

Theorem 15. Let the function $f(z)$ defined by (1.6) be in the class $C_n(\lambda,\alpha)$. Then we have and

$$\left|D_z^{-\mu}(D^i f(z))\right|\leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1+\frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)](2+\mu)}|z|\right\} \tag{9.4}$$

and

$$\left|D_z^{-\mu}(D^i f(z))\right|\geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)](2+\mu)}|z|\right\}. \tag{9.5}$$

for $\mu>0$ and $z\in\mathbb{U}$. The result is sharp.

Proof. Note that $f(z)\in C_n(\lambda,\alpha)$ if and only if $D^i f(z)\in C_{n-i}(\lambda,\alpha)$, $D^i f(z)$ is given by (4.4). Using Theorem 1, we know that

$$2^{n-i+1}[2-\alpha(1+\lambda)]\sum_{k=2}^{\infty}k^i a_k\leq \sum_{k=2}^{\infty}k^{n+1}\{k-\alpha[1+\lambda(k-1)]\}a_k\leq 1-\alpha, \tag{9.6}$$

that is,that

$$\sum_{k=2}^{\infty}k^i a_k\leq \frac{1-\alpha}{2^{n-i+1}[2-\alpha(1+\lambda)]}. \tag{9.7}$$

Let

$$F(z)=\Gamma(2+\mu)z^{-\mu}D_z^{-\mu}(D^i f(z)=z-\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}k^i a_k z^k. \tag{9.8}$$

Then

$$F(z) = z - \sum_{k=2}^{\infty} \Psi(k) k^i a_k z^k, \tag{9.9}$$

where

$$\Psi(k) = \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)} \ (k \geq 2). \tag{9.10}$$

Since $\Psi(k)$ is an decreasing function of k , then

$$0 < \Psi(k) \leq \Psi(2) = \frac{2}{2+\mu} \ (k \geq 2). \tag{9.11}$$

From (9.9) and (9.11), we have

$$|F(z)| \geq |z| - \Psi(2) |z|^2 \sum_{k=1}^{\infty} k^i a_k \tag{9.12}$$

$$|F(z)| = \left|\Gamma(2+\mu) z^{-\mu} D_z^{-\mu}(D^i f(z))\right| \geq |z| - \Psi(2) |z|^2 \sum_{k=1}^{\infty} k^i a_k$$

$$\geq |z| - \frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)](2+\mu)} |z|^2 \tag{9.13}$$

and

$$|F(z)| = \left|\Gamma(2+\mu) z^{-\mu} D_z^{-\mu}(D^i f(z))\right| \leq |z| + \Psi(2) |z|^2 \sum_{k=1}^{\infty} k^i a_k$$

$$\leq |z| + \frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)](2+\mu)} |z|^2 . \tag{9.14}$$

which proves the inequalities of Theorem 15. Further equalities are attained for the function $f(z)$ given by

$$D_z^{-\mu}(D^i f(z)) \geq \frac{z^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 - \frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)](2+\mu)} z \right\}, \tag{9.15}$$

or

$$D^i f(z) = z - \frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)]} z^2. \tag{9.16}$$

Using arguments similiar to those in the proof of Theorem 15, we obtain the following theorem.

Theorem 16. Let the function $f(z)$ defined by (1.4) be in the class $C_n(\lambda, \alpha)$. Then we have

$$\left|D_z^{\mu}(D^i f(z))\right| \le \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1+\frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)](2-\mu)}\,|z|\right\} \tag{9.17}$$

and

$$\left|D_z^{\mu}(D^i f(z))\right| \ge \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{1-\alpha}{2^{n-i}[2-\alpha(1+\lambda)](2-\mu)}\,|z|\right\}. \tag{9.18}$$

for $0 \le \mu < 1$ and $z \in \mathbb{U}$. The result is sharp for the function $f(z)$ given by (9.16).

Remarks.
(1) Putting $i = n = 0$ in Theorem 15, we obtain the result obtained by Altintas and Owa [1, Theorem 9];
(2) Putting $\mu = 0$ in Theorem 16, we obtain the result of Theorem 4;
(3) Putting $i = n = 0$ in Theorem 16, we obtain the results obtained by Altintas and Owa [1, Theorem 11];
(4) Putting $i = n = \mu = 0$ in Theorem 16, we obtain the results obtained by Altintas and Owa [1, Theorem 4, inequality (2.22)];
(5) Putting $n = \mu = 0$ and $i = 1$ in Theorem 16, we obtain the results obtained by Altintas and Owa [1, Theorem 4, inequality (2.23)];
(6) Putting $\mu = 0$ and $i = 0$ in Theorem 16, we obtain the result of Corollary 2;
(7) Putting $\mu = 0$ and $i = 1$ in Theorem 16, we obtain the result of Corollary 3.

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