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# SOME OPERATORS IN IDEAL TOPOLOGICAL SPACES VIA COZERO SETS

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Abstract. An ideal on a set $X$ is a nonempty collection of subsets of $X$ with heredity property which is also closed finite unions. The concept of ideal topological spaces via cozero sets was introduced by Al-Omari [1]. In this paper, we introduce and study an operator $\Phi : \mathcal{P}(X) \to \tau$ defined as follows for every $A \in X$, $\Phi(A) = \{x \in X$ : there exists a cozero set $U$ containing $x$ such that $U - A \in \mathcal{I}\}$ and observes that $\Phi(A) = X - (X - A)_z$. We construct a topology $\tau_z^*$ for $X$ by using the cozero sets and an ideal $\mathcal{I}$ on $X$. Moreover, we obtain some characterizations of $\Phi(A)$.

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## 1. Introduction and Preliminaries

The notion of ideal topological spaces was first studied by Kuratowski [7] and Vaidyanathaswamy [11]. Compatibility of the topology $\tau$ with an ideal $\mathcal{I}$ was first defined by Njåstad [9]. In 1990, Jankovic and Hamlett [4, 5] investigated further properties of ideal topological spaces. An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies the following properties:

1. $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.
2. $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set $U$ containing $x\}$ is called the local function of $A$ with respect to $\mathcal{I}$ and $\tau$ (see [4, 7]). We simply write $A^*$ instead of $A^*(\mathcal{I}, \tau)$ in case there is no chance for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than $\tau$, generating by the base $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau$ and $J \in \mathcal{I}\}$. It is known in [4] that $\beta(\mathcal{I}, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by $\tau^*$. Recall that $A$ is said to be $*$-dense in itself (resp. $\tau^*$-closed, $*$-perfect) if $A \subseteq A^*$ (resp. $A^* \subseteq A$, $A = A^*$). For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of $A$ in $(X, \tau^*)$, respectively.

A subset $H$ of a topological space $(X, \tau)$ is called a cozero set if there is a continuous real-valued function $g$ on $X$ such that $H = \{x \in X : g(x) \neq 0\}$. The complement of a cozero set is called a zero set. Recently papers [2, 3, 6, 10] have introduced some new classes of functions via cozero sets. Since the intersection of two cozero sets is a cozero set, the collection of all cozero subsets of $(X, \tau)$ is a base for a topology $\tau_z$ on $X$, called the complete regularization of $\tau$. It is clear that $\tau_z \subseteq \tau$ in general. Furthermore, the space $(X, \tau)$ is completely regular if and only if $\tau_z = \tau$. In general for any topological space $\tau_z \subseteq \tau$, we note that $(X, \tau_z)$ is completely regular.

We set $Int_z(A) = \cup\{U : U \subseteq A, U$ is a cozero set $\}$ and $Cl_z(A) = \cap\{F : A \subseteq F, F$ is a zero set $\}$.

**Proposition 1.1.** [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A$ be a cozero set then $Int_z(A) = A$ and if $A$ is zero set then $Cl_z(A) = A$.

Conversely $Int_z(A) = A$ dose not imply $A$ is cozero set and $Cl_z(A) = A$ dose not imply that $A$ is zero set.

**Lemma 1.2.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $x \in Cl_z(A)$ if and only if every cozero set $U_x$ containing $x$, $U_x \cap A \neq \emptyset$.

*Proof.* Let $x \in Cl_z(A)$. Supposed that $U_x \cap A = \emptyset$, where $U_x$ is a cozero set containing $x$. Then $A \subseteq X - U_x$ and $X - U_x$ is a zero set containing $A$. Therefore, $x \in X - U_x$ and this a contradiction. Conversely supposed that $U_x \cap A \neq \emptyset$ for every cozero set $U_x$ containing $x$. Suppose that $x \notin Cl_z(A)$. Then there exists a zero set $F$ such that $A \subseteq F$ and $x \notin F$. Therefore, $x \in X - F$. So $A \cap (X - F) = \emptyset$ for the cozero set $X - F$ containing $x$. It is a contradiction.

**Definition 1.3.** [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset $A$ of $X$, we define the following set: $A_z(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \mathcal{CZ}(x)\}$, where $\mathcal{CZ}(x)$ is the set of all cozero set in $X$ containing $x$. In case there is no confusion $A_z(\mathcal{I}, \tau)$ is briefly denoted by $A_z$ and is called the $z$-local function of $A$ with respect to $\mathcal{I}$ and $\tau$.

**Lemma 1.4.** [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A$, $B$ any subsets of $X$. Then the following properties hold:

1. $(\emptyset)_z = \emptyset$.
2. $(A_z)_z \subseteq A_z$.
3. $A_z \cup B_z = (A \cup B)_z$.

**Theorem 1.5.** [1] Let $(X, \tau)$ be a topological space, $\mathcal{I}$ and $\mathcal{J}$ be ideals on $X$, and let $A$ and $B$ be subsets of $X$. Then the following properties hold:

1. If $A \subseteq B$, then $A_z \subseteq B_z$.
2. If $\mathcal{I} \subseteq \mathcal{J}$, then $A_z(\mathcal{I}) \supseteq A_z(\mathcal{J})$.
3. $A_z = Cl_z(A_z) \subseteq Cl_z(A)$.
4. If $A \subseteq A_z$, then $A_z = Cl_z(A_z) = Cl_z(A)$.
5. If $A \in \mathcal{I}$, then $A_z = \emptyset$.

**Corollary 1.6.** [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A, I$ subsets of $X$ with $I \in \mathcal{I}$. Then $(A \cup I)_z = A_z = (A - I)_z$.

**Remark 1.7.** In [1] Al-Omari obtained that $Cl_z(A) = A \cup A_z$ is a Kuratowski closure operator. We will denote by $\tau_z^*$ the topology generated by $Cl_z$, that is, $\tau_z^* = \{U \subseteq X : Cl_z(X - U) = X - U\}$.

**Theorem 1.8.** [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\beta(\mathcal{CZ}, \mathcal{I}) = \{V - I : V$ is a cozero set of $(X, \tau), I \in \mathcal{I}\}$ is a basis for $\tau_z^*$.

**Theorem 1.9.** [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the following properties are equivalent:

1. $\mathcal{CZ} \cap \mathcal{I} = \{\emptyset\}$;

2. If $I \in \mathcal{I}$, then $Int_z(I) = \{\emptyset\}$;

3. For every cozero set $G$, $G \subseteq G_z$;

4. $X = X_z$.

**Lemma 1.10.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $A$ is cozero set then $\mathcal{CZ} \cap \mathcal{I} = \{\emptyset\}$ if and only if $A_z = Cl_z(A)$.

*Proof.* Let $\mathcal{CZ} \cap \mathcal{I} = \{\emptyset\}$. Let $A$ be a nonempty cozero sets then by Theorem 1.5 we have $A_z \subseteq Cl_z(A)$. Let $x \in Cl_z(A)$, then for all cozero set $U_x$ containing $x$ we have $U_x \cap A \neq \emptyset$. Again $U_x \cap A$ is a nonempty cozero set, so $U_x \cap A \notin \mathcal{I}$, since $\mathcal{CZ} \cap \mathcal{I} = \{\emptyset\}$. Hence $x \in A_z$. Therefore, $A_z = Cl_z(A)$. Conversely for $A$ cozero set we have $A_z = Cl_z(A)$. Then $X = X_z$ and this implies that $\mathcal{CZ} \cap \mathcal{I} = \{\emptyset\}$ by Theorem 1.9.

## 2. Φ-operator in ideal topological spaces

**Definition 2.1.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. An operator $\Phi : \mathcal{P}(X) \to \tau$ is defined as follows for every $A \in X$, $\Phi(A) = \{x \in X :$ there exists a cozer set $U$ containing $x$ such that $U - A \in \mathcal{I}\}$ and observes that $\Phi(A) = X - (X - A)_z$.

Several basic facts concerning the behavior of the operator $\Phi$ are included in the following theorem.

**Theorem 2.2.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties hold:

1. If $A \subseteq B$, then $\Phi(A) \subseteq \Phi(B)$.

2. If $A, B \in \mathcal{P}(X)$, then $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

3. If $U \in \tau_z^*$, then $U \subseteq \Phi(U)$.

4. If $A \subseteq X$, then $\Phi(A) \subseteq \Phi(\Phi(A))$.

5. If $A \subseteq X$, then $\Phi(A) = \Phi(\Phi(A))$ if and only if $(X - A)_z = ((X - A)_z)_z$.

6. If $A \in \mathcal{I}$, then $\Phi(A) = X - X_z$.

7. If $A \subseteq X$, then $A \cap \Phi(A) = Int_z(A)$.

8. If $A \subseteq X$, $I \in \mathcal{I}$, then $\Phi(A - I) = \Phi(A)$.

9. If $A \subseteq X$, $I \in \mathcal{I}$, then $\Phi(A \cup I) = \Phi(A)$.

10. If $(A - B) \cup (B - A) \in \mathcal{I}$, then $\Phi(A) = \Phi(B)$.

*Proof.* (1) This follows from Theorem 1.5 (1).
(2) It follows from (1) that $\Phi(A \cap B) \subseteq \Phi(A)$ and $\Phi(A \cap B) \subseteq \Phi(B)$. Hence $\Phi(A \cap B) \subseteq \Phi(A) \cap \Phi(B)$. Now let $x \in \Phi(A) \cap \Phi(B)$. There exist $U, V \in \mathcal{CZ}(x)$ such that $U - A \in \mathcal{I}$ and $V - B \in \mathcal{I}$. Let $G = U \cap V \in \mathcal{CZ}(x)$ and we have $G - A \in \mathcal{I}$ and $G - B \in \mathcal{I}$ by heredity. Thus $G - (A \cap B) = (G - A) \cup (G - B) \in \mathcal{I}$ by additivity, and hence $x \in \Phi(A \cap B)$. We have shown $\Phi(A) \cap \Phi(B) \subseteq \Phi(A \cap B)$ and the proof is complete.
(3) If $U \in \tau_z^*$, then $X - U$ is $\tau_z^*$-closed which implies $(X - U)_z \subseteq X - U$ and hence $U \subseteq X - (X - U)_z = \Phi(U)$.
(4) This follows from (3).
(5) This follows from the facts:

1. $\Phi(A) = X - (X - A)_z$.

2. $\Phi(\Phi(A)) = X - [X - (X - (X - A)_z)]_z = X - ((X - A)_z)_z$.

(6) By Corollary 1.6 we obtain that $(X - A)_z = X_z$ if $A \in \mathcal{I}$.
(7) If $x \in A \cap \Phi(A)$, then $x \in A$ and there exists a cozero set $U_x$ containing such that $U_x - A \in \mathcal{I}$. Then by Theorem 1.8, $U_x - (U_x - A)$ is an $\tau_z^*$-open neighborhood of $x$ and $x \in Int_z(A)$. On the other hand, if $x \in Int_z(A)$, there exists a basic $\tau_z^*$-open neighborhood $V_x - I$ of $x$, where $V_x$ is a cozero set and $I \in \mathcal{I}$, such that $x \in V_x - I \subseteq A$ which implies $V_x - A \subseteq I$ and hence $V_x - A \in \mathcal{I}$. Hence $x \in A \cap \Phi(A)$.
(8) This follows from Corollary 1.6 and $\Phi(A - I) = X - [X - (A - I)]_z = X - [(X - A) \cup I]_z = X - (X - A)_z = \Phi(A)$.
(9) This follows from Corollary 1.6 and $\Phi(A \cup I) = X - [X - (A \cup I)]_* = X - [(X - A) - I]_z = X - (X - A)_z = \Phi(A)$.
(10) Assume $(A - B) \cup (B - A) \in \mathcal{I}$. Let $A - B = I$ and $B - A = J$. Observe that $I, J \in \mathcal{I}$ by heredity. Also observe that $B = (A - I) \cup J$. Thus $\Phi(A) = \Phi(A - I) = \Phi[(A - I) \cup J] = \Phi(B)$ by (8) and (9).

**Corollary 2.3.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then

1. $U \subseteq \Phi(U)$ for every cozero set set $U$.
2. $U \subseteq Cl_z(\Phi(U))$ for every cozero set set $U$.
3. $U \cap A \subseteq Cl_z(\Phi(U \cap A))$ if $A \subseteq Cl_z(\Phi(A))$ and $U$ is a cozero set.

*Proof.* (1). We know that $\Phi(U) = X - (X - U)_z$. Now $(X - U)_z \subseteq Cl_z(X - U) = X - U$, since $X - U$ is zero set. Therefore, $U = X - (X - U) \subseteq X - (X - U)_z = \Phi(U)$.
(2). If follows from (1).
(3). Let $U$ be cozero set and $A \subseteq Cl_z(\Phi(A))$. By Theorem 2.2 and (1), we have

$$\begin{aligned} U \cap A &\subseteq U \cap Cl_z(\Phi(A)) \\ &\subseteq Cl_z(U \cap \Phi(A)) \\ &\subseteq Cl_z(\Phi(U) \cap \Phi(A)) \\ &= Cl_z(\Phi(U \cap A)). \end{aligned}$$

**Theorem 2.4.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. and $A \subseteq X$. Then the following properties hold:

1. $\Phi(A) = \cup\{U : U \text{is cozero set and } U - A \in \mathcal{I}\}$.
2. $\Phi(A) \supseteq \cup\{U : U \text{is cozero set and } (U - A) \cup (A - U) \in \mathcal{I}\}$.

*Proof.* (1) This follows immediately from the definition of $\Phi$-operator.
(2) Since $\mathcal{I}$ is heredity, it is obvious that $\cup\{U : U \text{is cozero set and } (U - A) \cup (A - U) \in \mathcal{I}\} \subseteq \cup\{U : U \text{is cozero set and } U - A \in \mathcal{I}\} = \Phi(A)$ for every $A \subseteq X$.

**Theorem 2.5.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\sigma = \{A \subseteq X : A \subseteq \Phi(A)\}$. Then $\sigma$ is a topology for $X$ and $\sigma = \tau_z^*$.

*Proof.* Let $\sigma = \{A \subseteq X : A \subseteq \Phi(A)\}$. First, we show that $\sigma$ is a topology. Observe that $\phi \subseteq \Phi(\phi)$ and $X \subseteq \Phi(X) = X$, and thus $\phi$ and $X \in \sigma$. Now if $A, B \in \sigma$, then $A \cap B \subseteq \Phi(A) \cap \Phi(B) = \Phi(A \cap B)$ which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \Phi(A_\alpha) \subseteq \Phi(\cup A_\alpha)$ for every $\alpha$ and hence $\cup A_\alpha \subseteq \Phi(\cup A_\alpha)$. This shows that $\sigma$ is a topology. Now if $U \in \tau_z^*$ and $x \in U$, then by Theorem 1.8 there exist a cozero set $V$ containing $x$ and $I \in \mathcal{I}$ such that $x \in V - I \subseteq U$. Clearly $V - U \subseteq I$ so that $V - U \in \mathcal{I}$ by heredity and hence $x \in \Phi(U)$. Thus $U \subseteq \Phi(U)$ and we have shown $\tau_z^* \subseteq \sigma$. Now let $A \in \sigma$, then we have $A \subseteq \Phi(A)$, that is, $A \subseteq X - (X - A)_z$ and $(X - A)_z \subseteq X - A$. This shows that $X - A$ is $\tau_z^*$-closed and hence $A \in \tau_z^*$. Thus $\sigma \subseteq \tau_z^*$ and hence $\sigma = \tau_z^*$.

**Definition 2.6.** [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say $\tau$ is $z$-compatible with the ideal $\mathcal{I}$, denoted $\tau \sim_z \mathcal{I}$, if the following holds for every $A \subseteq X$: For every $x \in A$ and cozero set $U$ with $x \in U$ and $U \cap A \in \mathcal{I}$, $A \in \mathcal{I}$.

**Theorem 2.7.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\tau \sim_z \mathcal{I}$ if and only if $\Phi(A) - A \in \mathcal{I}$ for every $A \subseteq X$.

*Proof. Necessity.* Assume $\tau \sim_z \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \Phi(A) - A \in \mathcal{I}$ if and only if $x \notin A$ and $x \notin (X - A)_z$ if and only if $x \notin A$ and there exists a cozero set $U_x$ containing $x$ such that $U_x - A \in \mathcal{I}$. Now, for each $x \in \Phi(A) - A$ and a cozero set $U_x$ containing $x$, $U_x \cap (\Phi(A) - A) \in \mathcal{I}$ by heredity and hence $\Phi(A) - A \in \mathcal{I}$ by assumption that $\tau \sim_z \mathcal{I}$.

*Sufficiency.* Let $A \subseteq X$ and assume that for each $x \in A$ there exists a cozero set $U_x$ containing $x$ such that $U_x \cap A \in \mathcal{I}$. Observe that $\Phi(X - A) - (X - A) = \{x :$ there exists a cozero set $U_x$ containing $x$ such that $x \in U_x \cap A \in \mathcal{I}\}$. Thus we have $A \subseteq \Phi(X - A) - (X - A) \in \mathcal{I}$ and hence $A \in \mathcal{I}$ by heredity of $\mathcal{I}$. Hence $\tau \sim_z \mathcal{I}$.

**Proposition 2.8.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_z \mathcal{I}$, $A \subseteq X$. If $N$ is a nonempty cozero subset of $A_z \cap \Phi(A)$, then $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.

*Proof.* If $N \subseteq A_z \cap \Phi(A)$, then $N - A \subseteq \Phi(A) - A \in \mathcal{I}$ by Theorem 2.7 and hence $N - A \in \mathcal{I}$ by heredity. Since $N \in \mathcal{CZ} - \{\phi\}$ and $N \subseteq A_z$, we have $N \cap A \notin \mathcal{I}$ by the definition of $A_z$.

As a consequence of the above theorem, we have the following.

**Corollary 2.9.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_z \mathcal{I}$. Then $\Phi(\Phi(A)) = \Phi(A)$ for every $A \subseteq X$.

*Proof.* $\Phi(A) \subseteq \Phi(\Phi(A))$ follows from Theorem 2.2 (5). Since $\tau \sim_z \mathcal{I}$, it follows from Theorem 2.7 that $\Phi(A) \subseteq A \cup I$ for some $I \in \mathcal{I}$ and hence $\Phi(\Phi(A)) = \Phi(A)$ by Theorem 2.2 (10).

**Theorem 2.10.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_z \mathcal{I}$. Then $\Phi(A) = \cup\{\Phi(U) : U$ is a cozero set and $\Phi(U) - A \in \mathcal{I}\}$.

*Proof.* Let $\Psi(A) = \cup\{\Phi(U) : U$ is a cozero set and $\Phi(U) - A \in \mathcal{I}\}$. Clearly, $\Psi(A) \subseteq \Phi(A)$. Now let $x \in \Phi(A)$. Then there exists a cozero set $U$ containing $x$ such that $U - A \in \mathcal{I}$. By Corollary 2.3, $U \subseteq \Phi(U)$ and $\Phi(U) - A \subseteq [\Phi(U) - U] \cup [U - A]$. By Theorem 2.7, $\Phi(U) - U \in \mathcal{I}$ and hence $\Phi(U) - A \in \mathcal{I}$. Hence $x \in \Psi(A)$ and $\Psi(A) \supseteq \Phi(A)$. Consequently, we obtain $\Psi(A) = \Phi(A)$.

In [8], Newcomb defines $A = B$ [mod $\mathcal{I}$] if $(A - B) \cup (B - A) \in \mathcal{I}$ and observes that $=$ [mod $\mathcal{I}$] is an equivalence relation. By Theorem 2.2 (11), we have that if $A = B$ [mod $\mathcal{I}$], then $\Phi(A) = \Phi(B)$.

**Definition 2.11.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset $A$ of $X$ is called a $z$-Baire set with respect to $\tau$ and $\mathcal{I}$, denoted $A \in \mathcal{B}_z(X, \tau, \mathcal{I})$, if there exists a cozero set $U$ such that $A = U$ [mod $\mathcal{I}$].

**Lemma 2.12.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. with $\tau \sim_z \mathcal{I}$. If $U$, $V$ are cozero set and $\Phi(U) = \Phi(V)$, then $U = V$ [mod $\mathcal{I}$].

*Proof.* Since $U$ is a cozero set, we have $U \subseteq \Phi(U)$ and hence $U - V \subseteq \Phi(U) - V = \Phi(V) - V \in \mathcal{I}$ by Theorem 2.7. Similarly $V - U \in \mathcal{I}$. Now $(U - V) \cup (V - U) \in \mathcal{I}$ by additivity. Hence $U = V$ [mod $\mathcal{I}$].

**Theorem 2.13.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_z \mathcal{I}$. If $A$, $B \in \mathcal{B}_z(X, \tau, \mathcal{I})$ and $\Phi(A) = \Phi(B)$, then $A = B$ [mod $\mathcal{I}$].

*Proof.* Let $U, V$ be a cozero set such that $A = U$ [mod $\mathcal{I}$] and $B = V$ [mod $\mathcal{I}$]. Now $\Phi(A) = \Phi(U)$ and $\Phi(B) = \Phi(V)$ by Theorem 2.2(11). Since $\Phi(A) = \Phi(U)$ implies that $\Phi(U) = \Phi(V)$ and hence $U = V$ [mod $\mathcal{I}$] by Lemma 2.12. Hence $A = B$ [mod $\mathcal{I}$] by transitivity.

**Proposition 2.14.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space.

1. If $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$, then there exists a nonempty cozero set $A$ such that $B = A$ [mod $\mathcal{I}$].

2. If $\mathcal{CZ} \cap \mathcal{I} = \phi$, then $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$ if and only if there exists a nonempty cozero set $A$ such that $B = A$ [mod $\mathcal{I}$].

*Proof.* (1) Assume $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$, then $B \in \mathcal{B}_z(X, \tau, \mathcal{I})$. Now if there does not exist a nonempty cozero set $A$ such that $B = A$ [mod $\mathcal{I}$], we have $B = \phi$ [mod $\mathcal{I}$]. This implies that $B \in \mathcal{I}$ which is a contradiction.
(2) Assume there exists a nonempty cozero set $A$ such that $B = A$ [mod $\mathcal{I}$]. Then $A = (B - J) \cup I$, where $J = B - A, I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$, then $A \in \mathcal{I}$ by heredity and additivity, which contradicts that $\mathcal{CZ} \cap \mathcal{I} = \phi$.

**Proposition 2.15.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\mathcal{CZ} \cap \mathcal{I} = \phi$. If $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$, then $\Phi(B) \cap Int_z(B_z) \neq \phi$.

*Proof.* Assume $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$, then by Proposition 2.14(1), there exists a nonempty cozero set $A$ such that $B = A$ [mod $\mathcal{I}$]. This implies that $\phi \neq A \subseteq A_z = ((B - J) \cup I)_z = B_z$, where $J = B - A, I = A - B \in \mathcal{I}$ by Theorem 1.9 and Corollary 1.6. Also $\phi \neq A \subseteq \Phi(A) = \Phi(B)$ by Theorem 2.2 (11), so that $A \subseteq \Phi(B) \cap Int_z(B_z)$.

Given an ideal topological space $(X, \tau, \mathcal{I})$, let $\mathcal{Z}(X, \tau, \mathcal{I})$ denote $\{A \subseteq X :$ there exists $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

**Proposition 2.16.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\mathcal{CZ} \cap \mathcal{I} = \phi$. The following properties are equivalent:

1. $A \in \mathcal{Z}(X, \tau, \mathcal{I})$;
2. $\Phi(A) \cap Int_z(A_z) \neq \phi$;
3. $\Phi(A) \cap A_z \neq \phi$;
4. $\Phi(A) \neq \phi$;
5. $Int_z(A) \neq \phi$;
6. There exists a nonempty cozero set $N$ such that $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.

*Proof.* (1) $\Rightarrow$ (2): Let $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A$. Then $Int_z(B_z) \subseteq Int_z(A_z)$ and $\Phi(B) \subseteq \Phi(A)$ and hence $Int_z(B_z) \cap \Phi(B) \subseteq Int_z(A_z) \cap \Phi(A)$. By Proposition 2.15, we have $\Phi(A) \cap Int_z(A_z) \neq \phi$.
(2) $\Rightarrow$ (3): The proof is obvious.
(3) $\Rightarrow$ (4): The proof is obvious.
(4) $\Rightarrow$ (5): If $\Phi(A) \neq \phi$, then there exists a nonempty cozero set $U$ such that $U - A \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - A) \cup (U \cap A)$, we have $U \cap A \notin \mathcal{I}$. By Theorem 2.2, $\phi \neq (U \cap A) \subseteq \Phi(U) \cap A = \Phi((U - A) \cup (U \cap A)) \cap A = \Phi(U \cap A) \cap A \subseteq \Phi(A) \cap A = Int_z(A)$. Hence $Int_z(A) \neq \phi$.
(5) $\Rightarrow$ (6): If $Int_z(A) \neq \phi$, then by Theorem 1.8 there exists a nonempty cozero set $N$ and $I \in \mathcal{I}$ such that $\phi \neq N - I \subseteq A$. We have $N - A \in \mathcal{I}$, $N = (N - A) \cup (N \cap A)$ and $N \notin \mathcal{I}$. This implies that $N \cap A \notin \mathcal{I}$.
(6) $\Rightarrow$ (1): Let $B = N \cap A \notin \mathcal{I}$ with a nonempty cozero set $N$ and $N - A \in \mathcal{I}$. Then $B \in \mathcal{B}_z(X, \tau, \mathcal{I}) - \mathcal{I}$ since $B \notin \mathcal{I}$ and $(B - N) \cup (N - B) = N - A \in \mathcal{I}$.

**Theorem 2.17.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\mathcal{CZ} \cap \mathcal{I} = \phi$. Then for $A \subseteq X$, $\Phi(A) \subseteq A_z$.

*Proof.* Suppose $x \in \Phi(A)$ and $x \notin A_z$. Then there exists a cozero set $U_x$ containing $x$ such that $U_x \cap A \in \mathcal{I}$. Since $x \in \Phi(A)$, by Theorem 2.4 $x \in \cup\{U : U \text{is a cozero set and } U - A \in \mathcal{I}\}$ and there exists a cozero set $V$ such that $x \in V$ and $V - A \in \mathcal{I}$. Now we have $U_x \cap V$ is a cozero set, $U_x \cap V \cap A \in \mathcal{I}$ and $(U_x \cap V) - A \in \mathcal{I}$ by heredity. Hence by finite additivity we have $(U_x \cap V \cap A) \cup (U_x \cap V - A) = (U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V)$ is a cozero set , this is contrary to $\mathcal{CZ} \cap \mathcal{I} = \phi$. Therefore, $x \in A_z$. This implies that $\Phi(A) \subseteq A_z$.

**Corollary 2.18.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\mathcal{CZ} \cap \mathcal{I} = \phi$. Then for $A \subseteq X$, $\Phi(A) \subseteq Cl_z(A_z)$.

**Theorem 2.19.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:

1. $\mathcal{CZ} \cap \mathcal{I} = \phi$;
2. $\Phi(\phi) = \phi$;
3. If $A \subseteq X$ is zero set, then $\Phi(A) - A = \phi$;
4. If $I \in \mathcal{I}$, then $\Phi(I) = \phi$.

*Proof.* (1) $\Rightarrow$ (2) Since $\mathcal{CZ} \cap \mathcal{I} = \phi$, by Theorem 2.4 we have $\Phi(\phi) = \cup\{U : U \text{is cozero set and } U \in \mathcal{I}\} = \phi$.
(2) $\Rightarrow$ (3) Suppose $x \in \Phi(A) - A$, then there exists a cozero set $U_x$ containing $x$ such that $x \in U_x - A \in \mathcal{I}$ and $U_x - A \in \mathcal{CZ}$. But $U_x - A \in \{U \in \mathcal{CZ} : U \in \mathcal{I}\} = \Phi(\phi)$ which implies that $\Phi(\phi) \neq \phi$. Hence $\Phi(A) - A = \phi$.
(3) $\Rightarrow$ (4) Let $I \in \mathcal{I}$ and since $\phi$ is zero set, then $\Phi(I) = \Phi(I \cup \phi) = \Phi(\phi) = \phi$.
(4) $\Rightarrow$ (1) Suppose $A \in \mathcal{CZ} \cap \mathcal{I}$, then $A \in \mathcal{I}$ and by (4) $\Phi(A) = \phi$. Since $A \in \mathcal{CZ}$, by Corollary 2.3 we have $A \subseteq \Phi(A) = \phi$. Hence $\mathcal{CZ} \cap \mathcal{I} = \phi$.

**Definition 2.20.** A subset $A$ in an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $z$-dense if $A_z = X$.

The collection of all $z$-dense in $(X, \tau, \mathcal{I})$ is denoted by $z$-$D(X, \tau)$. The collection of all dense sets in $(X, \tau)$ is denoted by $D(X, \tau)$. Now we show that the collection of dense sets in a topological space $(X, \tau_z^*)$ and the collection of $z$-dense sets in ideal topological spac $(X, \tau, \mathcal{I})$ are equal if $\mathcal{CZ} \cap \mathcal{I} = \phi$.

**Theorem 2.21.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{CZ} \cap \mathcal{I} = \phi$, then $z$-$D(X, \tau) = D(X, \tau_z^*)$.

*Proof.* Let $D \in z$-$D(X, \tau)$. Then $Cl_z(D) = D \cup D_z = X$, i.e. $D \in D(X, \tau_z^*)$. Therefore, $z$-$D(X, \tau) \subseteq D(X, \tau_z^*)$.

Conversely, let $D \in D(X, \tau_z^*)$. Then $Cl_z(D) = D \cup D_z = X$. We prove that $D_z = X$. Let $x \in X$ such that $x \notin D_z$. Therefore there exists a cozero set $U \neq \phi$ such that $U \cap D \in \mathcal{I}$. Since $U \notin \mathcal{I}$, $U \cap (X - D) \notin \mathcal{I}$ and hence $U \cap (X - D) \neq \phi$. Let $x_0 \in U \cap (X - D)$. Then $x_0 \notin D$ and also $x_0 \notin D_z$. Because $x_0 \in D_z$ implies that $U \cap D \notin \mathcal{I}$ which is contrary to $U \cap D \in \mathcal{I}$. Thus $x_0 \notin D \cup D_z = Cl_z(D) = X$. This is a contradiction. Therefore, we obtain $D \in z$-$D(X, \tau)$. Therefore, $D(X, \tau_z^*) \subseteq z$-$D(X, \tau)$. Hence $z$-$D(X, \tau) = D(X, \tau_z^*)$.

**Theorem 2.22.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for $x \in X$, $X - \{x\}$ is $z$-dense if and only if $\Phi(\{x\}) = \phi$.

*Proof.* The proof follows from the definition of $z$-dense sets, since $\Phi(\{x\}) = X - (X - \{x\})_z = \phi$ if and only if $X = (X - \{x\})_z$.

**Proposition 2.23.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. with $\mathcal{CZ} \cap \mathcal{I} = \phi$. Then $\Phi(A) \neq \phi$ if and only if $A$ contains a nonempty $\tau_z^*$-interior.

*Proof.* Let $\Phi(A) \neq \phi$. By Theorem 2.4 (1),
$\Phi(A) = \cup\{U : U$ is cozero set and $U - A \in \mathcal{I}\}$ and there exists a nonempty set a cozero set $U$ such that $U - A \in \mathcal{I}$. Let $U - A = P$, where $P \in \mathcal{I}$. Now $U - P \subseteq A$. By Theorem 1.8, $U - P \in \tau_z^*$ and $A$ contains a nonempty $\tau_z^*$-interior.

Conversely, suppose that $A$ contains a nonempty $\tau_z^*$-interior. Hence there exists a cozero set $U$ and $P \in \mathcal{I}$ such that $U - P \subseteq A$. So $U - A \subseteq P$. Let $H = U - A \subseteq P$, then $H \in \mathcal{I}$. Hence $\cup\{U : U$ is cozero set and $U - A \in \mathcal{I}\} = \Phi(A) \neq \phi$.

**Proposition 2.24.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Let $x \in X$. Then $\{x\} \subseteq Cl_z(\Phi(\{x\}))$ if and only if $\{x\}$ is $\tau_z^*$-open in $X$.

*Proof.* Let $\{x\} \subseteq Cl_z(\Phi(\{x\}))$ then $\Phi(\{x\} \neq \emptyset$. By Proposition 2.23, $\{x\}$ contains a nonempty $\tau_z^*$-interior. Therefore $\{x\}$ is $\tau_z^*$-open in $X$. Conversely suppose that $\{x\}$ is $\tau_z^*$-open in $X$, implies that by Theorem 2.2 $\{x\} \subseteq \Phi(\{x\})$ and hence $\{x\} \subseteq Cl_z(\Phi(\{x\}))$.

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