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GENERALIZATION MEASURE FOR CONES AND SPHERES

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Abstract. Let S_2 be some the 2-dimensional figure and V_2 be the 2-dimensional measure of S_2. From this S_2, the n-dimensional cone S_n and the n-dimensional measure V_n of S_n are considered. Applying the fractional integrals $I_x^\alpha f(x)$ of $f(x)$, we introduce the α-dimensional cone S_α for $\alpha \geqq 3$ and the α-dimensional measure V_α of S_α. Furthremore, the α-dimensional sphere $S_\alpha(r)$ with the radius r and the α-dimensional measure $V_\alpha(r)$ of $S_\alpha(r)$ are considered.

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1. Introduction

Let S_2 be some the 2-dimensional figure with the 2-dimensional measure V_2. For such the 2-dimensional figure S_2, let S_3 be the 3-dimensional cone with the height h from the vertex and V_3 be the 3-dimensional measure of S_3. Similarly, we define the n-dimensional cone S_n with the height x from the vertex for S_{n-1} and the n-dimensional measure $V_n$of S_n. For the 3-dimensional cone S_3, we denote by $V_2(x)$ the area of the base $S_2(x)$ which has the height x from the vertex and $S_2 \backsim S_2(x)$. Then we have that

$$V_2(x) = \frac{V_2}{h^2}x^2. \tag{1.1}$$

Thus we get

$$V_3 = \int_0^h V_2(x)dx = \frac{V_2}{h^2}\int_0^h x^2 dx = \frac{2h}{3!}V_2. \tag{1.2}$$

Next, for the 3-dimensional cone S_3, we say that S_4 is the 4-dimensional cone which has the height h from the vertex for S_3. Denoting by V_4 the 4-dimensional measure of S_4, we consider the 3-dimensional measure $V_3(x)$ of $S_3(x)$ which is the height $x(0 < x < h)$ from the vertex. Then we know that

$$V_3(x) = \frac{V_3}{h^3}x^3, \tag{1.3}$$

and that

$$V_4 = \int_0^h V_3(x)dx = \frac{V_3}{h^3}\int_0^h x^3 dx = \frac{2h^2}{4!}V_2. \tag{1.4}$$

Similarly, we have that the n-dimensional measure V_n of the n-dimensional cone S_n is given by

$$V_n = \int_0^h V_{n-1}dx = \frac{V_{n-1}}{h^{n-1}}\int_0^h x^{n-1}dx = \frac{2h^{n-2}}{n!}V_2. \tag{1.5}$$

2. Generalization measure for cones

For a real number α such that $\alpha \geqq 3$, we consider the α-dimensional cone S_α. we also denote by V_α the α-dimensional measure of S_α. For a real number x such that $x \neq 0, -1, -2, \cdots$, the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt. \tag{2.1}$$

This gamma function $\Gamma(x)$ has the following properties

$$\Gamma(x+1) = x\Gamma(x) \quad (x \neq 0, -1, -2, \cdots) \tag{2.2}$$

$\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
Furthermore, for $x > 0$ and $y > 0$, Bate function $B(x, y)$ is given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{2.3}$$

Then, we know that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad (x > 0, y > 0). \tag{2.4}$$

Using $\Gamma(x)$, the fractional integral of order α $I_x^\alpha f(x)$ of $f(x)$ is defined by

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \qquad (\alpha > 0). \tag{2.5}$$

The fractional calculus (fractional integral and fractional derivative) were defined by Owa [1] and applied by Owa and Srivastava [2] and Srivastava and Owa [3].

For example, the fractional integral of order α for x^p $(p > 0)$ shows us that

$$I_x^\alpha x^p = \frac{1}{\Gamma(\alpha)} \int_0^x t^p (x-t)^{\alpha-1} dt$$

$$= \frac{x^{p+\alpha}}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1}(1-u)^p du \qquad (x - t = xu)$$

$$= \frac{x^{p+\alpha}}{\Gamma(\alpha)} B(\alpha, p+1) = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} x^{p+\alpha}. \tag{2.6}$$

From the above, if we consider the α-dimensional cone S_α with the height h for the α-dimension and the α-dimensional measure V_α of S_α, we have that

$$V_\alpha = \frac{2h^{\alpha-2}}{\Gamma(\alpha+1)} V_2 \qquad (\alpha \geqq 3). \tag{2.7}$$

If we take $\alpha = \dfrac{7}{2}$, then we obtain that

$$V_{\frac{7}{2}} = \left[I_x^{\frac{1}{2}} V_3(x) \right]_{x=0}^{x=h} = \frac{V_3}{h^3} \left[I_x^{\frac{1}{2}} x^3 \right]_{x=0}^{x=h} = \frac{V_3}{h^3} \left[\frac{\Gamma(4)}{\Gamma(\frac{9}{2})} x^{\frac{7}{2}} \right]_{x=0}^{x=h} = \frac{2h^{\frac{3}{2}}}{\Gamma(\frac{9}{2})} V_2, \tag{2.8}$$

(2.9)

$$V_4 = \left[I_x^{\frac{1}{2}} V_{\frac{7}{2}}(x) \right]_{x=0}^{x=h} = \left[I_x^{\frac{1}{2}} \left(\frac{\Gamma(4) V_3}{\Gamma(\frac{9}{2}) h^3} x^{\frac{7}{2}} \right) \right]_{x=0}^{x=h} = \frac{2V_2}{\Gamma(\frac{9}{2}) h^2} \left[\frac{\Gamma(\frac{9}{2})}{\Gamma(5)} x^4 \right]_{x=0}^{x=h} = \frac{2h^2}{4!} V_2,$$

and

(2.10)

$$V_{\frac{9}{2}} = \left[I_x^{\frac{1}{2}} V_4(x) \right]_{x=0}^{x=h} = \left[I_x^{\frac{1}{2}} \left(\frac{2V_2}{4! h^2} x^4 \right) \right]_{x=0}^{x=h} = \frac{2V_2}{4! h^2} \left[\frac{\Gamma(5)}{\Gamma(\frac{11}{2})} x^{\frac{9}{2}} \right]_{x=0}^{x=h} = \frac{2h^{\frac{5}{2}}}{\Gamma(\frac{11}{2})} V_2.$$

Further, if we consider $\alpha = \frac{10}{3}$, then we have that

$$V_{\frac{10}{3}} = \left[I_x^{\frac{1}{3}} V_3(x) \right]_{x=0}^{x=h} = \frac{V_3}{h^3} \left[I_x^{\frac{1}{3}} x^3 \right]_{x=0}^{x=h} = \frac{V_3}{h^3} \left[\frac{\Gamma(4)}{\Gamma(\frac{13}{3})} x^{\frac{10}{3}} \right]_{x=0}^{x=h} = \frac{2h^{\frac{4}{3}}}{\Gamma(\frac{13}{3})} V_2 \tag{2.11}$$

and
(2.12)

$$V_4 = \left[I_x^{\frac{2}{3}} V_{\frac{10}{3}}(x) \right]_{x=0}^{x=h} = \frac{\Gamma(4)V_3}{h^3\Gamma(\frac{13}{3})} \left[I_x^{\frac{2}{3}} x^{\frac{10}{3}} \right]_{x=0}^{x=h} = \frac{2V_2}{\Gamma(\frac{13}{3})h^2} \left[\frac{\Gamma(\frac{13}{3})}{\Gamma(5)} x^4 \right]_{x=0}^{x=h} = \frac{2h^2}{4!} V_2.$$

Consequently, if we consider the α-dimensional cone S_α for $\alpha \geqq 3$ and the α-dimensional measure V_α of S_α, then we say that V_α is given by

$$V_\alpha = \frac{2h^{\alpha-2}}{\Gamma(\alpha+1)} V_2. \tag{2.13}$$

3. Generalization measure for spheres

Let $S_2(r)$ be the 2-dimensional disk with the radius r. Then $S_2(r)$ is given by $x^2+y^2=r^2$. If we write the 2-dimensional measure of $S_2(r)$ by $V_2(r)$, then we know that $V_2(r)=\pi r^2$.
Let $S_3(r)$ be the 3-dimensional sphere with the radius r and $V_3(r)$ be the 3-dimensional measure of $S_3(r)$. Then, using $x^2+y^2+z^2=r^2$, we have that

$$V_3(r) = 2\int_0^r \pi(x^2+y^2)dz = 2\int_0^r \pi(r^2-z^2)dz = \frac{4}{3}\pi r^3. \tag{3.1}$$

Further, let $S_4(r)$ be the 4-dimensional sphere with the radius r and $V_4(r)$ be the 4-dimensional measure of $S_4(r)$. Then using $x^2+y^2+z^2+w^2=r^2$, we obtain that

$$V_4(r) = 2\int_0^r \frac{4}{3}\pi \left(\sqrt{x^2+y^2+z^2}\right)^3 dw = \frac{8\pi}{3}\int_0^r \left(\sqrt{r^2-w^2}\right)^3 dw \tag{3.2}$$

$$= \frac{8\pi r^4}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = \frac{1}{2}\pi^2 r^4$$

with $w = r\sin\theta$.

Similarly, we have that

$$V_5(r) = \frac{8}{15}\pi^2 r^5, \quad V_6(r) = \frac{1}{6}\pi^3 r^6, \quad V_7(r) = \frac{16}{105}\pi^3 r^7, \quad V_8(r) = \frac{1}{24}\pi^4 r^8, \quad V_9(r) = \frac{32}{945}\pi^4 r^9, \cdots.$$

Therefore, if we consider the n-dimensional sphere $S_n(r)$ with the radius r and the n-dimensional measure $V_n(r)$ of $S_n(r)$, then we obtain that

$$V_n(r) = \frac{1}{m!}\pi^m r^{2m} \qquad (n=2m) \tag{3.2}$$

and

$$V_n(r) = \frac{2^{2m-1}(m-1)!}{(2m-1)!}\pi^{m-1}r^{2m-1} \qquad (n = 2m-1). \tag{3.3}$$

Therefore, using the gamma function, we can write that

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}r^n \qquad (n = 1, 2, 3, \cdots) \tag{3.4}$$

Consequently, if we consider that the α-dimensional sphere $S_\alpha(r)$ and the α-dimensional measure $V_\alpha(r)$ of $S_\alpha(r)$ for $\alpha \geqq 2$, then we can write that

$$V_\alpha(r) = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(1+\frac{\alpha}{2})}r^\alpha. \tag{3.5}$$

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