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FEKETE-SZEGÖ TYPE COEFFICIENT INEQUALITIES FOR A NEW SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING THE Q-DERIVATIVE OPERATOR

S. Bulut

Abstract. We introduce a new subclass of analytic functions of complex order involving the q-derivative operator defined in the open unit disc. For this class, several Fekete-Szegö type coefficient inequalities are derived. Various known special cases of our results are also pointed out.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.

Fekete and Szegö [8] proved a noticeable result that the estimate

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} -4\mu + 3 & , \quad \mu \leq 0 \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) & , \quad 0 \leq \mu \leq 1 \\ 4\mu - 3 & , \quad \mu \geq 1 \end{cases} \tag{2}$$

holds for $f \in \mathcal{S}$. The result is sharp in the sense that for each μ there is a function in the class under consideration for which equality holds.

The coefficient functional

$$\phi_{\mu}(f) = a_3 - \mu a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\mu}{2}\left(f''(0)\right)^2\right)$$

on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_{\mu}\left(e^{-i\theta} f\left(e^{i\theta} z\right)\right) = e^{2i\theta}\phi_{\mu}(f) \quad (\theta \in \mathbb{R}).$$

In fact, other than the simplest case when

$$\phi_0(f) = a_3,$$

we have several important ones. For example,

$$\phi_1(f) = a_3 - a_2^2$$

represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Moreover, the first two non-trivial coefficients of the k-th root transform

$$\left(f(z^k)\right)^{\frac{1}{k}} = z + c_{k+1}z^{k+1} + c_{2k+1}z^{2k+1} + \cdots$$

of f with the power series (1), are written by

$$c_{k+1} = \frac{a_2}{k}$$

and

$$c_{2k+1} = \frac{a_3}{k} + \frac{(k-1)\,a_2^2}{2k^2},$$

so that

$$a_3 - \mu a_2^2 = k\left(c_{2k+1} - \delta c_{k+1}^2\right),$$

where

$$\delta = \mu k + \frac{k-1}{2}.$$

Thus it is quite natural to ask about inequalities for ϕ_{μ} corresponding to subclasses of $\mathcal{S}$. This is called Fekete-Szegö problem. Actually, many authors have considered this problem for typical classes of univalent functions (see, for instance [1, 2, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15]).

For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Quantum calculus is ordinary classical calculus without the notion of limits. It defines q-calculus and h-calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [9, 10]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. A comprehensive study on applications of q-calculus in operator theory may be found in [3].

For a function $f \in \mathcal{A}$ given by (1) and $0 < q < 1$, the q-derivative of function f is defined by (see [9, 10])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad (z \neq 0), \tag{3}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (3), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{4}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{5}$$

As $q \to 1^-$, $[k]_q \to k$. For a function $g(z) = z^k$, we get

$$D_q\left(z^k\right) = [k]_q\, z^{k-1},$$

$$\lim_{q \to 1^-}\left(D_q\left(z^k\right)\right) = kz^{k-1} = g'(z),$$

where g' is the ordinary derivative.

We denote by $\mathcal{P}$ the class of all functions φ which are analytic and univalent in $\mathbb{U}$ and for which $\varphi(\mathbb{U})$ is convex with

$$\varphi(0) = 1 \quad \text{and} \quad \Re\{\varphi(z)\} > 0 \quad (z \in \mathbb{U}).$$

By making use of the q-derivative of a function $f \in \mathcal{A}$ and the principle of subordination, we introduce the following subclass.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_{q,b}^{\lambda}(\varphi)$ $(0 \le \lambda \le 1, b \in \mathbb{C}\backslash\{0\}, \varphi \in \mathcal{P})$ if it satisfies the following subordination condition:*

$$1 + \frac{1}{b}\left(\frac{zD_q\mathcal{F}_\lambda(z)}{\mathcal{F}_\lambda(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathbb{U}),$$

where $\mathcal{F}_\lambda(z) = \lambda z D_q f(z) + (1-\lambda) f(z)$.

Remark 1. (**i**) *If we set $\lambda = 0$ in Definition 1, then we have the class*

$$\mathcal{M}_{q,b}^{0}(\varphi) = \mathcal{S}_{q,b}(\varphi)$$

which consists of functions satisfying

$$1 + \frac{1}{b}\left(\frac{zD_q f(z)}{f(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathbb{U}).$$

(**ii**) *If we set $\lambda = 1$ in Definition 1, then we have the class*

$$\mathcal{M}_{q,b}^{1}(\varphi) = \mathcal{C}_{q,b}(\varphi)$$

which consists of functions satisfying

$$1 + \frac{1}{b}\left(\frac{D_q(zD_q f(z))}{D_q f(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathbb{U}).$$

The classes $\mathcal{S}_{q,b}(\varphi)$ and $\mathcal{C}_{q,b}(\varphi)$ was introduced and studied by Seoudy and Aouf [16].

Remark 2. *We also get*

$$\lim_{q\to 1^-} \mathcal{M}_{q,b}^{\lambda}(\varphi) = \mathcal{M}_{b}^{\lambda}(\varphi)$$

which consists of functions satisfying

$$1 + \frac{1}{b}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathbb{U}).$$

We shall require the following lemmas.

Lemma 1. *[17] Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Then for any complex number ν*

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1, |2\nu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 2. *[15] If $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & , \quad \nu \leq 0 \\ 2 & , \quad 0 \leq \nu \leq 1 \\ 4\nu - 2 & , \quad \nu \geq 1 \end{cases}.$$

When $\nu < 0$ or $\nu > 1$, equality holds true if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then equality holds true if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, then the equality holds true if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $\nu = 1$, then the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case when $\nu = 0$.

Although the above upper bound is sharp, in the case when $0 < \nu < 1$, it can be further improved as follows:

$$\left|c_2 - \nu c_1^2\right| + \nu |c_1|^2 \leq 2 \qquad \left(0 \leq \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)|c_1|^2 \leq 2 \qquad \left(\frac{1}{2} \leq \nu \leq 1\right).$$

2. Fekete-Szegö Problem for the Function Class $\mathcal{M}_{q,b}^{\lambda}(\varphi)$

Unless otherwise mentioned, we assume throughout this paper that the function $0 \leq \lambda \leq 1, 0 < q < 1, b \in \mathbb{C} \backslash \{0\}, \varphi \in \mathcal{P}, [k]_q$ is given by (5) and $z \in \mathbb{U}$.

Theorem 3. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ with $B_1 \neq 0$. If $f(z)$ given by (1) belongs to the function class $\mathcal{M}_{q,b}^{\lambda}(\varphi)$, then for any complex number μ*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1 b|}{\left([3]_q - 1\right)\left(1 - \lambda + [3]_q \lambda\right)} \times \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1 b}{[2]_q - 1}\left(1 - \frac{\left([3]_q - 1\right)\left(1 - \lambda + [3]_q \lambda\right)}{\left([2]_q - 1\right)\left(1 - \lambda + [2]_q \lambda\right)^2}\mu\right)\right|\right\}. \tag{6}$$

The result is sharp.

Proof. If $f \in \mathcal{M}_{q,b}^{\lambda}(\varphi)$, then we have

$$h(z) \prec \varphi(z),$$

where

$$h(z) = 1 + \frac{1}{b}\left(\frac{z D_q \mathcal{F}_\lambda(z)}{\mathcal{F}_\lambda(z)} - 1\right) = 1 + h_1 z + h_2 z^2 + \cdots \tag{7}$$

with $\mathcal{F}_\lambda(z) = \lambda z D_q f(z) + (1 - \lambda) f(z)$. From (7), we have

$$h_1 = \frac{1}{b}\left([2]_q - 1\right)\left(1 - \lambda + [2]_q \lambda\right) a_2, \tag{8}$$

$$h_2 = \frac{1}{b}\left([3]_q - 1\right)\left(1 - \lambda + [3]_q \lambda\right) a_3 - \left([2]_q - 1\right)\left(1 - \lambda + [2]_q \lambda\right)^2 a_2^2. \tag{9}$$

Since $\varphi(z)$ is univalent and $h(z) \prec \varphi(z)$, the function

$$p_1(z) = \frac{1 + \varphi^{-1}(h(z))}{1 - \varphi^{-1}(h(z))} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

is analytic and has a positive real part in $\mathbb{U}$. Also we have

$$h(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots. \tag{10}$$

Thus by (7) – (10) we get

$$a_2 = \frac{B_1 c_1 b}{2\left([2]_q - 1\right)\left(1 - \lambda + [2]_q \lambda\right)}, \tag{11}$$

$$a_3 = \frac{B_1 b}{2\left([3]_q - 1\right)\left(1 - \lambda + [3]_q \lambda\right)} \left[c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_q - 1}\right) c_1^2\right]. \tag{12}$$

Taking into account (11) and (12) , we obtain

$$a_3 - \mu a_2^2 = \frac{B_1 b}{2\left([3]_q - 1\right)\left(1 - \lambda + [3]_q \lambda\right)} \left(c_2 - \delta c_1^2\right), \tag{13}$$

where

$$\delta = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_q - 1}\left(1 - \frac{\left([3]_q - 1\right)\left(1 - \lambda + [3]_q \lambda\right)}{\left([2]_q - 1\right)\left(1 - \lambda + [2]_q \lambda\right)^2}\mu\right)\right]. \tag{14}$$

Our result now follows by an application of Lemma 1. The result is sharp for the functions

$$1 + \frac{1}{b}\left(\frac{z D_q \mathcal{F}_\lambda(z)}{\mathcal{F}_\lambda(z)} - 1\right) = \varphi\left(z^2\right) \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{z D_q \mathcal{F}_\lambda(z)}{\mathcal{F}_\lambda(z)} - 1\right) = \varphi(z).$$

This completes the proof of Theorem 3.

Corollary 4. *Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 3, we get [16, Theorem 1] and [16, Theorem 2], respectively.*

Taking $q \to 1^-$ in Theorem 3, we obtain the following result for the functions belonging to the class $\mathcal{M}_b^\lambda(\varphi)$.

Corollary 5. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ with $B_1 \neq 0$. If $f(z)$ given by (1) belongs to the function class $\mathcal{M}_b^\lambda(\varphi)$, then for any complex number μ*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1 b|}{2(1 + 2\lambda)} \times \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{2(1 + 2\lambda)}{(1 + \lambda)^2}\mu\right) B_1 b\right|\right\}.$$

The result is sharp.

Corollary 6. *Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 5, we get [16, Corollary 1] and [16, Corollary 2], respectively.*

Theorem 7. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1) belongs to the function class $\mathcal{M}_{q,b}^{\lambda}(\varphi)$ with $b > 0$, then*

$$\left|a_3 - \mu a_2^2\right| \leq$$

$$\begin{cases} \frac{B_2 b}{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)} + \frac{B_1^2 b^2}{\left([2]_q-1\right)}\left[\frac{1}{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)} - \frac{\mu}{\left([2]_q-1\right)\left(1-\lambda+[2]_q\lambda\right)^2}\right] & , \quad \mu \leq \sigma_1 \\ \frac{B_1 b}{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)} & , \quad \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{B_2 b}{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)} - \frac{B_1^2 b^2}{\left([2]_q-1\right)}\left[\frac{1}{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)} - \frac{\mu}{\left([2]_q-1\right)\left(1-\lambda+[2]_q\lambda\right)^2}\right] & , \quad \mu \geq \sigma_2 \end{cases},$$

where

$$\sigma_1 = \frac{\left([2]_q - 1\right)\left(1-\lambda+[2]_q\lambda\right)^2\left[B_1^2 b + \left([2]_q - 1\right)(B_2 - B_1)\right]}{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)B_1^2 b}, \tag{15}$$

$$\sigma_2 = \frac{\left([2]_q - 1\right)\left(1-\lambda+[2]_q\lambda\right)^2\left[B_1^2 b + \left([2]_q - 1\right)(B_2 + B_1)\right]}{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)B_1^2 b}, \tag{16}$$

$$\sigma_3 = \frac{\left([2]_q - 1\right)\left(1-\lambda+[2]_q\lambda\right)^2\left[B_1^2 b + \left([2]_q - 1\right)B_2\right]}{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)B_1^2 b}. \tag{17}$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} & \left|a_3 - \mu a_2^2\right| \\ & + \frac{\left([2]_q - 1\right)^2\left(1-\lambda+[2]_q\lambda\right)^2}{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)B_1^2 b} \\ & \times \left\{B_1 - B_2 - \frac{B_1^2 b}{\left([2]_q - 1\right)}\left(1 - \frac{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)}{\left([2]_q - 1\right)\left(1-\lambda+[2]_q\lambda\right)^2}\mu\right)\right\}|a_2|^2 \\ \leq \; & \frac{B_1 b}{\left([3]_q - 1\right)\left(1-\lambda+[3]_q\lambda\right)}. \end{aligned}$$

Furthermore, if $\sigma_3 \leq \mu \leq \sigma_2$*, then*

$$\begin{aligned} & \left|a_3-\mu a_2^2\right| \\ & +\frac{\left([2]_q-1\right)^2\left(1-\lambda+[2]_q\lambda\right)^2}{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)B_1^2b} \\ & \times\left\{B_1+B_2+\frac{B_1^2b}{\left([2]_q-1\right)}\left(1-\frac{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}{\left([2]_q-1\right)\left(1-\lambda+[2]_q\lambda\right)^2}\mu\right)\right\}|a_2|^2 \\ \leq\ & \frac{B_1b}{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}. \end{aligned}$$

Each of these results is sharp.

Proof. Applying Lemma 2 to (13) and (14), we can get our results. On the other hand, using (13) for the values of $\sigma_1 \leq \mu \leq \sigma_3$, we have

$$\begin{aligned} \left|a_3-\mu a_2^2\right|+(\mu-\sigma_1)|a_2|^2 &= \frac{B_1b}{2\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}\left|c_2-\delta c_1^2\right| \\ &\quad +(\mu-\sigma_1)\frac{B_1^2b^2|c_1|^2}{4\left([2]_q-1\right)^2\left(1-\lambda+[2]_q\lambda\right)^2} \\ &= \frac{B_1b}{2\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}\left\{\left|c_2-\delta c_1^2\right|+\delta|c_1|^2\right\} \\ &\leq \frac{B_1b}{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}. \end{aligned}$$

Similarly, for the values of $\sigma_3 \leq \mu \leq \sigma_2$, we get

$$\begin{aligned}\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)\left|a_2\right|^2 &= \frac{B_1 b}{2\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}\left|c_2-\delta c_1^2\right| \\ &\quad +(\sigma_2-\mu)\frac{B_1^2 b^2\left|c_1\right|^2}{4\left([2]_q-1\right)^2\left(1-\lambda+[2]_q\lambda\right)^2} \\ &= \frac{B_1 b}{2\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}\left\{\left|c_2-\delta c_1^2\right|+(1-\delta)\left|c_1\right|^2\right\} \\ &\leq \frac{B_1 b}{\left([3]_q-1\right)\left(1-\lambda+[3]_q\lambda\right)}.\end{aligned}$$

To show that the bounds asserted by Theorem 7 are sharp, we define the following functions:

$$K_{\varphi_n}(z) \qquad (n=2,3,\ldots),$$

with

$$K_{\varphi_n}(0)=0=K'_{\varphi_n}(0)-1,$$

by

$$1+\frac{1}{b}\left(\frac{zD_qK_{\varphi_n}(z)}{K_{\varphi_n}(z)}-1\right)=\varphi\left(z^{n-1}\right),$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0\leq\eta\leq 1)$, with

$$F_\eta(0)=0=F'_\eta(0)-1 \quad \text{and} \quad G_\eta(0)=0=G'_\eta(0)-1,$$

by

$$1+\frac{1}{b}\left(\frac{zD_qF_\eta(z)}{F_\eta(z)}-1\right)=\varphi\left(\frac{z(z+\eta)}{1+\eta z}\right)$$

and

$$1+\frac{1}{b}\left(\frac{zD_qG_\eta(z)}{G_\eta(z)}-1\right)=\varphi\left(-\frac{z(z+\eta)}{1+\eta z}\right),$$

respectively. Then, clearly, the functions $K_{\varphi_n}, F_\eta, G_\eta \in \mathcal{M}_{q,b}^{\lambda}(\varphi)$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality in Theorem 7 holds true if and only if f is K_{φ_2} or one of its rotations. When $\sigma_1<\mu<\sigma_2$, then the equality holds true if and only if f is K_{φ_3} or one of its rotations. If $\mu=\sigma_1$, then the equality holds true if and only if f is F_η or one of its rotations. If $\mu=\sigma_2$, then the equality holds true if and only if f is G_η or one of its rotations.

Corollary 8. *Taking* $\lambda = 0$ *and* $\lambda = 1$ *in Theorem* 7*, we get [16, Theorem 3] and [16, Theorem 4], respectively.*

Taking $q \to 1^-$ in Theorem 7, we obtain the following result for the functions belonging to the class $\mathcal{M}_b^\lambda(\varphi)$.

Corollary 9. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ *with* $B_1 > 0$ *and* $B_2 \geq 0$*. If* $f(z)$ *given by* (1) *belongs to the function class* $\mathcal{M}_b^\lambda(\varphi)$ *with* $b > 0$*, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_2 b}{2(1+2\lambda)} + \left[\frac{1}{2(1+2\lambda)} - \frac{\mu}{(1+\lambda)^2}\right] B_1^2 b^2 & , \quad \mu \leq \sigma_1 \\ \frac{B_1 b}{2(1+2\lambda)} & , \quad \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{B_2 b}{2(1+2\lambda)} - \left[\frac{1}{2(1+2\lambda)} - \frac{\mu}{(1+\lambda)^2}\right] B_1^2 b^2 & , \quad \mu \geq \sigma_2 \end{cases},$$

where

$$\begin{aligned} \sigma_1 &= \frac{(1+\lambda)^2 \left[B_1^2 b + B_2 - B_1\right]}{2(1+2\lambda) B_1^2 b}, \\ \sigma_2 &= \frac{(1+\lambda)^2 \left[B_1^2 b + B_2 + B_1\right]}{2(1+2\lambda) B_1^2 b}, \\ \sigma_3 &= \frac{(1+\lambda)^2 \left[B_1^2 b + B_2\right]}{2(1+2\lambda) B_1^2 b}. \end{aligned}$$

If $\sigma_1 \leq \mu \leq \sigma_3$*, then*

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| + \frac{(1+\lambda)^2}{2(1+2\lambda) B_1^2 b} \left\{B_1 - B_2 - \left(1 - \frac{2(1+2\lambda)}{(1+\lambda)^2}\mu\right) B_1^2 b\right\} |a_2|^2 \\ \leq\ & \frac{B_1 b}{2(1+2\lambda)}. \end{aligned}$$

Furthermore, if $\sigma_3 \leq \mu \leq \sigma_2$*, then*

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| + \frac{(1+\lambda)^2}{2(1+2\lambda) B_1^2 b} \left\{B_1 + B_2 + \left(1 - \frac{2(1+2\lambda)}{(1+\lambda)^2}\mu\right) B_1^2 b\right\} |a_2|^2 \\ \leq\ & \frac{B_1 b}{2(1+2\lambda)}. \end{aligned}$$

Each of these results is sharp.

Corollary 10. *Taking* $\lambda = 0$ *and* $\lambda = 1$ *in Theorem* 9*, we get [16, Corollary 3] and [16, Corollary 4], respectively.*

References

[1] H.R. Abdel-Gawad, D.K. Thomas, *The Fekete-Szegö problem for strongly close-to-convex functions*, Proc. Amer. Math. Soc. 114 (1992), 345-349.

[2] H.S. Al-Amiri, *Certain generalization of prestarlike functions*, J. Aust. Math. Soc. 28 (1979), 325-334.

[3] A. Aral, V. Gupta, R.P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer, New York, USA, 2013.

[4] J.H. Choi, Y.Ch. Kim, T. Sugawa, *A general approach to the Fekete-Szegö problem*, J. Math. Soc. Japan 59, 3 (2007), 707-727.

[5] A. Chonweerayoot, D.K. Thomas, W. Upakarnitikaset, *On the Fekete-Szegö theorem for close-to-convex functions*, Publ. Inst. Math. (Beograd) (N.S.) 66 (1992), 18-26.

[6] M. Darus, D.K. Thomas, *On the Fekete-Szegö theorem for close-to-convex functions*, Math. Japonica 44 (1996), 507-511.

[7] M. Darus, D.K. Thomas, *On the Fekete-Szegö theorem for close-to-convex functions*, Math. Japonica 47 (1998), 125-132.

[8] M. Fekete, G. Szegö, *Eine bemerkung über ungerade schlichte funktionen*, J. Lond. Math. Soc. 8 (1933), 85-89.

[9] F.H. Jackson, *On q-definite integrals*, Quarterly J. Pure Appl. Math. 41 (1910), 193-203.

[10] F.H. Jackson, *On q-functions and a certain difference operator*, Trans. R. Soc. Edinb. 46 (1908), 253-281.

[11] S. Kanas, A. Lecko, *On the Fekete-Szegö problem and the domain convexity for a certain class of univalent functions*, Folia Sci. Univ. Tech. Resov. 73 (1990), 49-58.

[12] F.R. Keogh, E.P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. 20 (1969), 8-12.

[13] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions*, Proc. Amer. Math. Soc. 101 (1987), 89-95.

[14] R.R. London, *Fekete-Szegö inequalities for close-to-convex functions*, Proc. Amer. Math. Soc. 117 (1993), 947-950.

[15] W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157-169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

[16] T.M. Seoudy, M.K. Aouf, *Coefficient estimates of new classes of q-starlike and q-convex functions of complex order*, J. Math. Inequal. 10, 1 (2016), 135-145.

[17] V. Ravichandran, A. Gangadharan, M. Darus, *Fekete-Szegö inequality for certain class of Bazilevic functions*, Far East J. Math. Sci. 15 (2004), 171-180.

Serap Bulut
Kocaeli University,
Faculty of Aviation and Space Sciences,
Arslanbey Campus,
Kocaeli, Turkey
email: *serap.bulut@kocaeli.edu.tr*