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REAL HYPERSURFACES OF AN $(\epsilon_\alpha) - S-$MANIFOLD

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Abstract. In this paper, we introduced $(\epsilon_\alpha) - S-$manifold and give some examples about $(\epsilon_\alpha) - S-$manifold. Moreover, we studied real hypersurfaces of an $(\epsilon_\alpha) - S-$ manifold.

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1. Introduction

In 1963, Yano [10] introduced the notion of f-structure on a C^∞ m-dimensional manifold M, as a non-vanishing tensor field φ of type $(1,1)$ on M which satisfies $\varphi^3 + \varphi = 0$ and has constant rank r. It is known that r is even, say $r = 2n$. Moreover, TM splits into two complementary subbundles $\mathrm{Im}\varphi$ and $\ker\varphi$ and the restriction of φ to $\mathrm{Im}\varphi$ determines a complex structure on such subbundle. It is also known that the existence of an f-structure on M is equivalent to a reduction of the structure group to $U(n) \times O(s)$ [2], where $s = m - 2n$. In 1970, Goldberg and Yano [7] introduced globally frame f-manifolds. A wide class of globally frame f-manifolds was introduced in [2] by Blair according to the following definition: a metric f-structure is said to be a K-structure if the fundamental $2-$form Φ, defined usually as $\Phi(X,Y) = g(X,\varphi Y)$, for any vector fields X and Y on M, is closed and the normality condition holds, that is, $[\varphi,\varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ. In *[9],* let M a $(2n+s)-$dimensional metric $f-$manifold. If there exists 2-form Φ such that $\eta^1 \wedge ... \wedge \eta^s \wedge \Phi^n \neq 0$ on M then M is called an *almost $s-$contact metric manifold.* An almost $s-$contact metric manifold M is called almost $S-$manifold if $\Phi = d\eta^\alpha$, $1 \leq \alpha \leq s$. A normal almost $S-$manifold M is called $S-$manifold. The S-manifolds have been studied by several authors (see, for instance, [3, 4, 8]).

In [1] Bejancu and Duggal show the existence of (ϵ)-almost contact metric structures and give examples of (ϵ)-Sasakian manifold. They introduced (ϵ)-Sasakian manifolds which enclose the class of usual Sasakian manifolds. They investigated the induced structures on real hypersurfaces of an indefinite Kaehler manifold and studied some particular classes of such structures. In [6], Duggal introduced Lorentzian globally framed manifolds.

In this paper, we introduced $(\epsilon_\alpha) - S$-manifolds which enclose the class of usual $S-$manifolds. It has to be noted that in the definition of an $(\epsilon_\alpha) - S-$manifold it is essential that the causal characters of the all characteristic vector fields of the structure is preserved. We give some examples of $(\epsilon_\alpha) - S$-structures on R^{2n+s}. In the framework of Riemannian geometry, $(\epsilon_\alpha) - S$-manifolds represent a natural generalization of (ϵ_α)-Sasakian manifolds. In addition, we studied real hypersurfaces of $(\epsilon_\alpha) - S-$manifolds.

2. (ϵ_α)ALMOST **S**-MANIFOLDS

Let differentiable manifold M be a $(2n+s)-$dimensional manifold with an f-structure of rank $2n$. If there are s global vector fields $\xi_\alpha, \alpha \in \{1, ..., s\}$ and the 1-forms on M satisfying the following conditions

$$\varphi^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \xi_\alpha \, , \, \eta^\alpha (\xi_\beta) = \delta^\alpha_\beta, \, 1 \leq \alpha, \beta \leq s \tag{2.1}$$

then M is said to have a *framed $f-$structure* $(\varphi, \xi_\alpha, \eta_\alpha)$, and M is called a *framed manifold* or *framed $f-$manifold.* It follows that

$$\eta^\alpha \circ \varphi = 0, \quad \varphi (\xi_\alpha) = 0, \, 1 \leq \alpha \leq s. \tag{2.2}$$

Let $(M, \varphi, \xi_\alpha, \eta_\alpha, g)$ be a $(2n+s)$-dimensional framed manifold and a semi-Riemannian metric g on M with index $\upsilon, 0 < \upsilon < (2n+s)$. Then $(\varphi, \xi_\alpha, \eta_\alpha, g)$ is called an indefinite metric $f-$structure and $(M, \varphi, \xi_\alpha, \eta_\alpha, g)$ is called an indefinite metric $f-$manifold, if φ is skew-symmetric with respect to; that is;

$$g (\varphi (X) , Y) + g (X, \varphi (Y)) = 0$$

for any $X, Y \in \Gamma(TM)$.

We now, g semi-Riemannian metric on M with index , $0 < \upsilon < (2n+s)$. That satisfies

$$g(\varphi (X) , \varphi (Y)) = g(X, Y) - \sum_{\alpha=1}^{s} \epsilon_\alpha \eta^\alpha (X) \eta^\alpha (Y) \tag{2.3}$$

$$\eta^\alpha (X) = \epsilon_\alpha g (X, \xi_\alpha) \tag{2.4}$$

for each, $X, Y \in \Gamma(TM)$ and any $\alpha \in \{1, ..., s\}$ where $\epsilon_\alpha = \pm 1$ according to whether ξ_α are space-like or time-like and

$$\epsilon_\alpha = g(\xi_\alpha, \xi_\alpha) \tag{2.5}$$

[5]. This follows as a consequence of the fact that on M we many consider an orthonormal frame $\{E_1, ..., E_n, \varphi(E_1), ..., \varphi(E_n), \xi_1, ..., \xi_s\}$ with $E_i \in \Gamma(\mathcal{L})$ and such that $g(E_i, E_i) = g(\varphi(E_i), \varphi(E_i))$, where $\mathcal{L} = \{X \in \Gamma(TM), \eta^\alpha(X) = 0,\ 1 < \alpha < s\}$. We consider the fundamental 2-form Φ on the an indefinite metric $f-$structure defined by

$$\Phi(X, Y) = g(X, \varphi(Y)),\ \forall X, Y \in \Gamma(TM). \tag{2.6}$$

Let M a $(2n + s)-$dimensional indefinite metric $f-$manifold. If there exists 2-form Φ such that $\eta^1 \wedge ... \wedge \eta^s \wedge \Phi^n \neq 0$ on M then M is called an *almost s-contact indefinite metric manifold.*

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an almost $s-$contact indefinite metric manifold. If for all $\epsilon_\alpha = g(\xi_\alpha, \xi_\alpha)$, $\alpha = 1, ..., s$ are the same causel character then M is called an $(\epsilon_\alpha)-$*almost* $s-$*contact manifold* (the causal character of the all characteristic vector fields of the structure are preserved).

Thus we have the following new classes of manifolds.

1) $\epsilon_\alpha = 1$ for all $\alpha \in \{1, ..., s\}$ and $\nu = 2r$, M is called a *space-like almost* $s-$*contact metric manifold.*

2) $\epsilon_\alpha = -1$ for all $\alpha \in \{1, ..., s\}$ and $\nu = 2r + s$, M is called a *time-like almost* $s-$*contact metric manifold.*

A space-like or time-like almost s-contact metric manifold is called an $(\epsilon_\alpha)-$*almost* S*-contact metric manifold.* If an $(\epsilon_\alpha)-$almost s-contact metric manifold is normal then M is called an $(\epsilon_\alpha)-$ $S-$*contact metric manifold.*

Theorem 1. *Let $(\varphi, \xi_\alpha, \eta^\alpha, g)$ be an almost $S-$contact structure and g_0 a metric on semi-Riemannian manifold such that all characteristic vector fields ξ_α $(\alpha = 1, ..., s)$ are non-null and the same causal character. Then there exist on M a $(1, 2)$ type symmetric tensor field g satisfying* (2.3).

Proof. Two semi-Riemannian metrics are defined $\tilde{g}_1 = -\frac{\epsilon_\alpha}{\gamma_\alpha}\tilde{g}_0$ where $\gamma_\alpha = \tilde{g}_0(\xi_\alpha, \xi_\alpha)$, $\alpha = 1, ..., s$ and $\tilde{g}$ such that

$$\tilde{g}(X, Y) = \tilde{g}_1(\varphi^2(X), \varphi^2(Y)) + \sum_{\alpha=1}^{s} \epsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y), \quad \forall X, Y \in \Gamma(TM).$$

Thus, we have $\eta^\alpha(X) = \epsilon_\alpha \tilde{g}(X, \xi_\alpha)$ and $\epsilon_\alpha = \tilde{g}(\xi_\alpha, \xi_\alpha)$, $\alpha = 1, ..., s$. In addition, we denote by $\mathcal{M}$ the distribution spanned by the structure vector fields $\xi_1, \ldots, \xi_s$ and by

$\mathcal{L}$ its orthogonal complementary distribution with respect to $\tilde{g}_1$.Since $\tilde{g}_1(X, \xi_\alpha) = 0$ and $\tilde{g}_1(\xi_\alpha, \xi_\alpha) = -\epsilon_\alpha$ $\alpha = 1, ..., s$ we get

$$\tilde{g}(X, X) = \tilde{g}_1(X, X) \quad \forall X \in \Gamma(\mathcal{L}).$$

Hence, $\tilde{g}$ is a semi-Riemannian metric on M of the same index as $\tilde{g}_1$ is on $\mathcal{L}$. A symmetric tensor field g is defined by

$$g(X, Y) = \frac{1}{2}\{\tilde{g}(X, Y) + \tilde{g}(\varphi X, \varphi Y) + \sum_{\alpha=1}^{s} \epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y)\} \quad \forall X, Y \in \Gamma(TM).$$

Therefore, we get $g(\xi_\alpha, \xi_\alpha) = \epsilon_\alpha$, $\alpha = 1, ..., s$ and

$$\begin{aligned} g(\varphi X, \varphi Y) &= \frac{1}{2}\{\tilde{g}(\varphi X, \varphi Y) + \tilde{g}(\varphi^2 X, \varphi^2 Y) + \sum_{\alpha=1}^{s} \epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y)\} \\ &= g(X, Y) - \sum_{\alpha=1}^{s} \epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y). \end{aligned}$$

An $(\epsilon_\alpha)-$almost $s-$contact manifold M is called an (ϵ_α) *−almost S−manifold* if

$$\Phi(X, Y) = d\eta^\alpha(X, Y), 1 \leq \alpha \leq s.$$

We recall that an $(\epsilon_\alpha)-$almost $S-$manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ is normal if

$$[\varphi, \varphi] + 2\sum_{\alpha=1}^{s} d\eta^\alpha \otimes \xi_\alpha = 0$$

where $[\varphi, \varphi]$ is the Nijenhuis tensor field associated to φ.

An $(\epsilon_\alpha)-$almost $S-$structure which is normal is called an $(\epsilon_\alpha) -$ *S−structure.* A manifold endowed with an $(\epsilon_\alpha) - S-$structure is an $(\epsilon_\alpha) - S-$manifold.

If we have $s = 1, \upsilon = 0$ then M is Riemannian Sasakian manifold. If $s = 1$ then $(\epsilon_\alpha) - S-$manifold M is $(\epsilon)-$Sasaki manifold, resent studied an important subclass by Bejancu and Duggal of the second class is Lorentzian Sasakian manifold ($s = 1$, $\epsilon = -1$, $\upsilon = 1$).

1) $\epsilon_\alpha = 1$ for all $\alpha \in \{1, ..., s\}$ and $\nu = 2r$, M is called a *space-like almost S−manifold.*

2) $\epsilon_\alpha = -1$ for all $\alpha \in \{1, ..., s\}$ and $\nu = 2r + s$, M is called a *time-like almost S−manifold.*

Theorem 2. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an $(\epsilon_\alpha)-$ almost $s-$contact manifold. Then we have*

$$\begin{aligned} 2g((\nabla_X\varphi)Y, Z) &= 3d\Phi(X, \varphi(Y), \varphi(Z)) - 3d\Phi(X, Y, Z) + g(N^1(Y, Z), \varphi(X)) \\ &+\sum_{\alpha=1}^{s} \epsilon_\alpha\{N^2(Y, Z)\eta^\alpha(X) + 2d\eta^\alpha(\varphi(Y), X)\eta^\alpha(X) \\ &-2d\eta^\alpha(\varphi(Z), X)\eta^\alpha(Y)\} \end{aligned}$$

Proof. We know that Kozsul formula is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$.By direct calculations using Kozsul formula, (2.3) and (2.6), we get the desired result.

Theorem 3. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an $(\epsilon_\alpha)-$ $s-$contact manifold. M is an $(\epsilon_\alpha)-$ $S-$manifold if and only if*

$$(\nabla_X\varphi)Y = \sum_{\alpha=1}^{s} \left\{g(\varphi X, \varphi Y)\xi_\alpha + \epsilon_\alpha\eta^\alpha(Y)\varphi^2(X)\right\}, \forall X, Y \in \Gamma(TM)$$

where ∇ is the Levi-Civita connection with respect to g.

Proof. Assume that M is an $(\epsilon_\alpha) - S-$manifold. From the Theorem 2, we have

$$\begin{aligned} g((\nabla_X\varphi)Y, Z) &= \sum_{\alpha=1}^{s} \epsilon_\alpha \{d\eta^\alpha(\varphi(Y), X)\eta^\alpha(X) - d\eta^\alpha(\varphi(Z), X)\eta^\alpha(Y)\} \\ &= \sum^{s} \epsilon_\alpha \{g(\varphi(Y), \varphi(X))\eta^\alpha(X) - g(\varphi(Z), \varphi(X))\eta^\alpha(Y)\} \\ &= g\left(\sum_{\alpha=1}^{s} (g(\varphi(Y), \varphi(X))\xi_\alpha + \epsilon_\alpha\eta^\alpha(Y)\varphi^2(X)), Z\right). \end{aligned}$$

Conversely, if we take $Y = \xi_\gamma$ in the hypothesis then we get

$$(\nabla_X\varphi)\xi_\gamma = \sum_{\gamma=1}^{s} \epsilon_\gamma\eta^\alpha(\xi_\gamma)\varphi^2(X).$$

Thus, we have

$$-\varphi\nabla_X\xi_\gamma = \epsilon_\gamma\varphi^2(X)$$

Using (2.1), we get

$$\nabla_X \xi_\gamma = -\epsilon_\gamma \varphi X.$$

On the other hand, we get

$$(L_{\xi_\gamma} g)(X,Y) = g(\nabla_X \xi_\gamma, Y) + g(X, \nabla_Y \xi_\gamma) = 0.$$

That is; for all $(\gamma = 1,...,s)$ ξ_γ are killing vector fields. Therefore, we have

$$d\eta^\gamma(X,Y) = -\epsilon_\gamma g(X, \nabla_Y \xi_\gamma) + \epsilon_\gamma g(Y, \nabla_X \xi_\gamma) - g(X, \varphi Y) = \Phi(X,Y), \ \gamma = 1,2,...,s.$$

for all $X, Y \in \Gamma(TM)$. In addition, the Nijenhuis torsion of φ is obtained

$$N_\varphi(X,Y) = -2\sum_{\gamma=1}^{s} g(X, \varphi Y)\xi_\gamma$$

Hence, we have

$$[\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0.$$

The proof is completed.

Corollary 4. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an $(\epsilon_\alpha) - S-$manifold. Then we have*

$$\nabla_X \xi_\alpha = -\epsilon_\alpha \varphi(X), \ \forall X \in \Gamma(TM). \tag{2.7}$$

Corollary 5. *Characteristic vector fields ξ_α for all $1 \leq \alpha \leq s$ on an $(\epsilon_\alpha) - S-$manifold are Killing vector fields .*

We consider $s = 1$ then we have following the corollaries.

Corollary 6. *[1] An $(\epsilon)-$ almost contact metric manifold M is an $(\epsilon)-$ Sasakian manifold if and only if*

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \epsilon\eta(Y)X, \quad \forall X, Y \in \Gamma(TM).$$

Corollary 7. *[1] Let M be an $(\epsilon)-$ Sasakian manifold. Then we have*

$$\nabla_X \xi = -\epsilon\varphi X, \quad \forall X \in \Gamma(TM)$$

and ξ is a killing vector field.

Now, we give an example.

Example 1. *The first example of an $(\epsilon_\alpha) - S-$manifold $\left(R^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g\right)$ where for any $\alpha \in \{1, ..., s\}$ we consider $r \neq s$ and $r \leq n$. Thus, we defined*

$$\epsilon^i = \begin{cases} -1, & 0 \leq i \leq r \\ 1, & (r+1) \leq i \leq n \end{cases}$$

where $\epsilon_\alpha = \pm 1$, for all $\alpha \in \{1, ..., s\}$.

Now we consider another case $r = 0$. In this case, we have $\epsilon^i = +1$ for all $i \in \{1, ..., n\}$. Then, we consider $\left(x^i, y^i, z^\alpha\right)$, $i = 1, ..., n$ and $\alpha = 1, ..., s$ as cartesian coordinates on R^{2n+s} and define with respect a tensor field of frames $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^\alpha}\right\}$ a tensor field of type $(1,1)$ by its matrix.

$$[\varphi] = \begin{bmatrix} 0_{n,n} & I_n & 0_{n,s} \\ -I_n & 0_{n,n} & 0_{n,s} \\ 0_{s,n} & \epsilon^i y^i & 0_{s,s} \end{bmatrix}_{(2n+s)\times(2n+s)} \tag{2.8}$$

where

$$\left[\epsilon^i y^i\right] = \begin{bmatrix} \epsilon^1 y^1 & . & . & . & . & \epsilon^n y^n \\ . & . & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ \epsilon^1 y^1 & . & . & . & . & \epsilon^n y^n \end{bmatrix}_{(s\times n)} = \begin{bmatrix} -y^1 & . & -y^r & y^{r+1} & . & y^n \\ . & . & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ -y^1 & . & -y^r & y^{r+1} & . & y^n \end{bmatrix}_{(s\times n)}$$

The differential 1-form η^α are defined by

$$\eta^\alpha = \frac{\epsilon_\alpha}{2}\left\{dz_\alpha - \sum_{i=1}^{r} \epsilon^i y^i dx_i - \sum_{i^*=r+1}^{s} \epsilon^{i^*} y^{i^*} dx_{i^*}\right\} \tag{2.9}$$

$$\eta^\alpha = \frac{\epsilon_\alpha}{2}\left\{dz_\alpha + \sum_{i=1}^{r} y^i dx_i - \sum_{i^*=r+1}^{s} y^{i^*} dx_{i^*}\right\}$$

if $r \neq 0$ and

$$\eta^\alpha = \frac{\epsilon_\alpha}{2}\left\{dz_\alpha - \sum_{i=1}^{n} \epsilon^i y^i dx_i\right\} = \eta^\alpha = \frac{\epsilon_\alpha}{2}\left\{dz_\alpha - \sum_{i=1}^{n} y^i dx_i\right\} \tag{2.10}$$

if $r = 0$.

The vector fields ξ_α are defined by

$$\xi_\alpha = 2\epsilon_{\alpha\frac{\partial}{\partial z_\alpha}}, \qquad 1 \leq \alpha \leq s. \tag{2.11}$$

It is easy to check (2.1) and thus $(\varphi, \xi_\alpha, \eta^\alpha)$ *is a space-like almost* S*-structure on* R^{2n+s} *for each* $i \in \{1, ..., n\}$. *Finally, we define the semi-Riemannian metric* g *by the matrix*

$$[g] = \frac{\epsilon_\alpha}{2}\begin{bmatrix} \epsilon^i\delta^{ij} + \epsilon^i\epsilon^j y^i y^j & \epsilon^i y^i \epsilon^{j^*} y^{j^*} & 0_{r,r} & 0_{r,n-r} & A^T \\ \epsilon^i y^i \epsilon^{j^*} y^{j^*} & \epsilon^{i^*}\delta^{i^*j^*} + \epsilon^{i^*}\epsilon^{j^*} y^{i^*} y^{j^*} & 0_{n-r,r} & 0_{n-r,n-r} & B^T \\ 0_{r,r} & 0_{r,n-r} & \epsilon^i\delta^{ij} & 0_{r,n-r} & 0_{r,s} \\ 0_{n-r,r} & 0_{n-r,n-r} & 0_{n-r,r} & \epsilon^{i^*}\delta^{i^*j^*} & 0_{n-r,s} \\ A & B & 0_{s,r} & 0_{s,n-r} & \delta_{\alpha\beta} \end{bmatrix} \tag{2.12}$$

where

$$A = \begin{bmatrix} -\epsilon^1 y^1 & . & . & -\epsilon^r y^r \\ . & . & . & . \\ -\epsilon^1 y^1 & . & . & -\epsilon^r y^r \end{bmatrix}_{s\times r} = \begin{bmatrix} y^1 & . & . & y^r \\ . & . & . & . \\ y^1 & . & . & y^r \end{bmatrix}_{s\times r}$$

and

$$B = \begin{bmatrix} -\epsilon^{r+1} y^{r+1} & . & . & -\epsilon^n y^n \\ . & . & . & . \\ -\epsilon^{r+1} y^{r+1} & . & . & -\epsilon^n y^n \end{bmatrix}_{s\times(n-r)} = \begin{bmatrix} -y^{r+1} & . & . & -y^n \\ . & . & . & . \\ -y^{r+1} & . & . & -y^n \end{bmatrix}_{s\times(n-r)}$$

Then, we defined the semi-Riemannian metric g *by the matrix*

$$[g] = \frac{\epsilon_\alpha}{2}\begin{bmatrix} -\delta^{ij} + y^i y^j & -y^i y^{j^*} & 0_{n,r} & 0_{r,n-r} & y^i \\ -y^i y^{j^*} & \delta^{i^*j^*} + y^{i^*} y^{j^*} & 0_{n-r,r} & 0_{n-r,n-r} & -y^{j^*} \\ 0_{r,r} & 0_{r,n-r} & -I_r & 0_{r,n-r} & 0_{r,s} \\ 0_{n-r,r} & 0_{n-r,n-r} & 0_{n-r,r} & I_{n-r} & 0_{n-r,s} \\ y^i & -y^{j^*} & 0_{s,n} & 0_{s,n-r} & I_s \end{bmatrix}_{(2n+s)\times(2n+s)}$$

for $r \neq 0$*, and*

$$[g] = \frac{\epsilon_\alpha}{2}\begin{bmatrix} \epsilon^i\delta^{ij} + \epsilon^i\epsilon^j y^i y^j & 0_{n,n} & y^i \\ 0_{n,n} & \delta^{ij} & 0_{n,s} \\ y^i & 0_{s,n} & I_s \end{bmatrix} = \frac{\epsilon_\alpha}{2}\begin{bmatrix} \delta_{ij} + y^i y^j & 0_{n,n} & y^i \\ 0_{n,n} & \delta^{ij} & 0_{n,s} \\ y^i & 0_{s,n} & I_s \end{bmatrix}_{(2n+s)\times(2n+s)}$$

for $r = 0$.

With respect to the natural field on frames. In order to help the reader to see the right form of $[g]$ *we write it down for* $n = 4$, $r = 1$ *and* $s = 3$:

$$[g] = \frac{\epsilon_\alpha}{2}\begin{bmatrix} -1+\left(y^1\right)^2 & -y^1y^2 & -y^1y^3 & -y^1y^4 & 0 & 0 & 0 & 0 & y^1 & y^1 & y^1 \\ -y^1y^2 & -1+(y^2)^2 & y^2y^3 & y^2y^4 & 0 & 0 & 0 & 0 & -y_2 & -y_2 & -y_2 \\ -y^1y^3 & y^2y^3 & -1+(y^3)^2 & y^3y^4 & 0 & 0 & 0 & 0 & -y^3 & -y^3 & -y^3 \\ -y^1y^4 & y^2y^4 & y^3y^4 & -1+(y^4)^2 & 0 & 0 & 0 & 0 & -y^4 & -y^4 & -y^4 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & o & o & o \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & o & o & o \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & o & o & o \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ y^1 & -y^2 & -y^3 & -y^4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ y^1 & -y^2 & -y^3 & -y^4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ y^1 & -y^2 & -y^3 & -y^4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{11\times 11}$$

An orthonormal field of frames with respect to the semi-Riemannian metric (2.12) is

$$E_i = 2\frac{\partial}{\partial y_i}, \quad E_{i^*} = 2\frac{\partial}{\partial y_{i^*}}, \quad \varphi E_i = 2\left\{\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{s} \epsilon^i y_i \frac{\partial}{\partial z_\alpha}\right\} = 2\left\{\frac{\partial}{\partial x_i} - \sum_{\alpha=1}^{s} y_i \frac{\partial}{\partial z_\alpha}\right\},$$

$$\varphi E_{i^*} = 2\left\{\frac{\partial}{\partial x_{i^*}} + \sum_{\alpha=1}^{s} \epsilon^{i^*} y_{i^*} \frac{\partial}{\partial z_\alpha}\right\} = 2\left\{\frac{\partial}{\partial x_{i^*}} + \sum_{\alpha=1}^{s} y_{i^*} \frac{\partial}{\partial z_\alpha}\right\}, \quad \xi_\alpha = 2\epsilon_\alpha \frac{\partial}{\partial z_\alpha}$$

where $1 \leq i \leq r$, $r+1 \leq i^* \leq n$ and $1 \leq \alpha \leq s$.

It is easy to check that $(\varphi, \xi_\alpha, \eta^\alpha, g)$ given by (2.8)−(2.12) is an $(\epsilon_\alpha)-S-$structure on R^{2n+s} for any $i \in \{1, ..., n\}$. In case $r = 0$ and $\epsilon_\alpha = 1$ for all $\alpha \in \{1, ..., s\}$ we obtain the classical $S-$structure on R^{2n+s}.In other cases, we get either a space-like $S-$structure on R^{2n+s}_{2r} ($\epsilon_\alpha = 1$ for all $\alpha \in \{1, ..., s\}$, $r = s$) or time-like $S-$structure on $R^{2n+s}_{2(n-s)+1}$ ($\epsilon_\alpha = -1$ for all $\alpha \in \{1, ..., s\}$, $r \neq 0$) .The Lorentz $S-$structure is obtain from the latter for $r = n$ and $\nu = 1$.

3. Real hypersurfaces of $(\epsilon_\alpha) - \mathbf{S}-$manifolds

3.1. The Induced (ϵ_α) −S−Structure of The Different Rank as the Ambient Manifolds

Let $\overline{M}$ be a real $(2n+s)-$dimensional $(\epsilon_\alpha) - S-$manifold. M is an orientable non-degenerate real hypersurface of $\overline{M}$. For this, consider are orthonormal basis $\{E_1, ..., E_n, \varphi E_1, ..., \varphi E_n, \xi_1, ..., \xi_s\}$ of $T\overline{M}$. Let N be the normal vector field of M. We denoted $\xi_{s+1} = \varphi N$.Thus we have

$$\varphi X = fX + w(X)N \tag{3.1}$$

where $w(X) = \overline{g}(X, N)$. In addition, we have

$$\eta^\alpha(N) = \epsilon_\alpha(N, \xi_\alpha) = 0, \tag{3.2}$$

$$\overline{g}(\xi_{s+1}, \xi_{s+1}) = \overline{g}(\varphi N, \varphi N) = \overline{g}(N, N) - \sum_{\alpha=1}^{s} \epsilon_\alpha \eta^\alpha(N)\eta^\alpha(N) = \overline{g}(N, N)$$

$$\overline{g}(N, N) = \epsilon_N = \epsilon_{s+1} \tag{3.3}$$

and $\overline{g}(\xi_{s+1}, N) = \overline{g}(\varphi N, N) = 0$. In addition, we have

$$\overline{g}(\xi_{s+1}, \xi_\alpha) = \overline{g}(\varphi N, \xi_\alpha) = -\overline{g}(N, \varphi\xi_\alpha) = 0, \quad \xi_{s+1} \in \Gamma(TM)$$

$$\overline{g}(\varphi X, \xi_{s+1}) = \overline{g}(\varphi X, \varphi N) = \overline{g}(X, N) - \sum_{\alpha=1}^{s} \epsilon_\alpha \eta^\alpha(X)\eta^\alpha(N) = 0$$

and

$$\overline{g}(\varphi X, N) = \overline{g}(fX + w(X)N, N) = \overline{g}(fX, N) + w(X)\overline{g}(N, N).$$

Thus, we get

$$w(X) = \overline{g}(\varphi X, N) = -\overline{g}(X, \varphi N) = -\overline{g}(X, \xi_{s+1}) = -\epsilon_{s+1}\eta^{s+1}(X).$$

$$\varphi X = fX - \epsilon_{s+1}\eta^{s+1}(N) \tag{3.4}$$

where

$$\eta^{s+1}(X) = \epsilon_{s+1}\overline{g}(X, \xi_{s+1}) = \epsilon_{s+1}\overline{g}(X, \varphi N).$$

Lemma 8. *$\overline{M}$ be a real $(2n+s)-$dimensional $(\epsilon_\alpha) - S-$manifold. The tensor field of type (1.1), f on a real hypersurface M of $\overline{M}$ proved as follow*

$$f^2 = -I + \sum_{\alpha=1}^{s+1} \eta^\alpha \otimes \xi_\alpha. \tag{3.5}$$

Proof. From (3.5), we have

$$\varphi X = fX - \eta^{s+1}(X)N. \tag{3.6}$$

Thus, we obtain

$$\begin{aligned}
\varphi^2(X) &= \varphi(fX) - \eta^{s+1}(X)\varphi N \\
&= f^{\,2}(X) - \eta^{s+1}(fX)N - \eta^{s+1}(X)(fN - \eta^{s+1}(N)N) \\
&= f^{\,2}(X) - \eta^{s+1}(X)fN - \epsilon_{s+1}\overline{g}(fX, \xi_{s+1})N \\
&= f^{\,2}(X) - \eta^{s+1}(X)\xi_{s+1} - \epsilon_{s+1}\overline{g}(\varphi X, \xi_{s+1}) + \epsilon_{s+1}\eta^{s+1}(X)\overline{g}(N, \xi_{s+1})N \\
&= f^{\,2}(X) - \eta^{s+1}(X)\xi_{s+1}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
f^{\,2}(X) &= \varphi^2(X) + \eta^{s+1}(X)\xi_{s+1} \\
&= -X + \sum_{\alpha=1}^{s} \eta^\alpha(X)\xi_\alpha + \eta^{s+1}(X)\xi_{s+1} \\
&= -X + \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\xi_\alpha.
\end{aligned}$$

Lemma 9. *Let $\overline{M}$ be a real $(2n+s)-$dimensional $(\epsilon_\alpha) - S-$manifold. Then f is the tensor field of type (1.1), f is $f-$structure on the real hypersurface M of $\overline{M}$ that is,*

$$f^3 + f = 0.$$

Proof. Using (3.6) we have

$$f^{\,3}(X) = -fX + \sum_{\alpha=1}^{s+1} \eta^\alpha(X) f(\xi_\alpha) = -fX$$

for all $X \in \Gamma(TM)$.

Lemma 10. *Let $\overline{M}$ be a real $(2n+s)$-dimensional $(\epsilon_\alpha)-S-$manifold and M be a is real hypersurface of $\overline{M}$. Then f is skew-symmetric with respect to semi-Riemannian metric g induced by $\overline{g}$ on $\overline{M}$; that is, for any $X, Y \in \Gamma(TM)$*

$$g(fX, Y) + g(X, fY) = 0.$$

Proof. From (3.5) we get

$$g(fX, Y) = g(\varphi X + \eta^{s+1}(X)N, Y) = -g(X, \varphi Y).$$

In similar, we have

$$\begin{aligned} g(X, fY) &= g(X, \varphi Y + \eta^{s+1}(Y)N) \\ &= g(X, \varphi Y) + \eta^{s+1}(Y) g(X, N). \end{aligned}$$

It is completed the proof.

We now denote by $\{\xi_\alpha\}$ for any $\alpha \in \{1, ..., s\}$ the distribution spanned by ξ_α on M and by $\mathcal{L}$ complementary orthogonal distribution to ξ_α in $\Gamma(TM)$. The projection morphism of $\Gamma(TM)$ to $\mathcal{L}$ is denote by P. Hence any vector field X on M is written as follow

$$X = PX - \sum_{\alpha=1}^{s+1} \eta^\alpha(X) \xi_\alpha$$

Where η^α are $1-$forms on M defined by $\eta^\alpha(X) = \epsilon_\alpha g(X, \xi_\alpha)$. So, we get

$$\eta^\alpha(\xi_\beta) = \delta_{\alpha\beta}, \qquad f^{\,2}(X) = -X + \eta^{s+1}(X) \xi_{s+1}$$

Moreover, we get

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s+1} \epsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y)$$

by using (3.3).

Theorem 11. *Let M be a real hypersurface of $(\epsilon_\alpha) - S-$manifold, and A be a symmetric tensor field of type $(1,1)$. The $(\epsilon_\alpha) - S-$structure on M immersed in an $(\epsilon_\alpha) - S-$manifold M satisfies*

$$(\nabla_X f)Y = \sum_{\alpha=1}^{s+1} \{\epsilon_\alpha g(AX, Y)\xi_\alpha - \eta^\alpha(Y)AX\} \tag{3.7}$$

$$(\nabla_X \eta^\alpha)Y = \epsilon_\alpha g(fAX, Y) \tag{3.8}$$

for any $X, Y \in \Gamma(TM)$.

Proof. (i)This suggest to put, for any $\alpha \in \{1, ..., s\}$, $A = f \circ \nabla\xi_\alpha$, i.e., for any vector field X on M

$$AX = f\left(\nabla_X \xi_\alpha\right) \tag{3.9}$$

where A is the of type $(1,1)$ tensor field. Since

$$fAX = -\nabla_X \xi_\alpha \tag{3.10}$$

we have

$$f\left(\nabla_X \xi_\alpha\right) = -AX + \sum_{\alpha=1}^{s+1} \eta^\alpha\left(AX\right)\xi_\alpha, \tag{3.11}$$

$$g(\left(\nabla_X f\right)Y, Z) = \sum_{\alpha=1}^{s+1} \left\{d\eta^\alpha\left(fY, X\right)\eta^\alpha\left(Z\right) - d\eta^\alpha\left(fZ, X\right)\eta^\alpha\left(Y\right)\right\}, \tag{3.12}$$

$$g(\left(\nabla_X f\right)Y, Z) = g(\nabla_X\left(fY\right) - f\left(\nabla_X Y\right), Z). \tag{3.13}$$

If we replace Y by ξ_α in (3.13) we get

$$g(\left(\nabla_X f\right)\xi_\alpha, Z) = -g(f\left(\nabla_X \xi_\alpha\right), Z).$$

If we replace Y by ξ_α in (3.12) we get

$$g(\left(\nabla_X f\right)\xi_\alpha, Z) = -\sum_{\alpha=1}^{s+1} d\eta^\alpha(fZ, X). \tag{3.14}$$

In similar, we have

$$g(\left(\nabla_X f\right)\xi_\alpha, Y) = -g(f\left(\nabla_X \xi_\alpha\right), Y). \tag{3.15}$$

We put (3.14) and (3.15) at (3.12) then

$$\begin{aligned} g(\left(\nabla_X f\right)Y, Z) &= -\sum_{\alpha=1}^{s+1} \{g(f(\nabla_X \xi_\alpha), Y)\eta^\alpha(Z) - g(f(\nabla_X \xi_\alpha), Z)\eta^\alpha(Y)\} \\ &= g(\sum_{\alpha=1}^{s+1} \{\epsilon_\alpha g(f(\nabla_X \xi_\alpha), Y)\xi_\alpha - f(\nabla_X \xi_\alpha)\eta^\alpha(Y)\}, Z). \end{aligned}$$

Since g is non-degenerate metric, and from (3.11) then

$$\begin{aligned}(\nabla_X f)Y &= \sum_{\alpha=1}^{s+1}\{\epsilon_\alpha g(f(\nabla_X \xi_\alpha), Y)\xi_\alpha - f(\nabla_X \xi_\alpha)\eta^\alpha(Y)\} \\ &= \sum_{\alpha=1}^{s+1}\{\epsilon_\alpha g(AX, Y)\xi_\alpha - \eta^\alpha(Y)(AX)\}.\end{aligned}$$

ii) We take covariant derivative of $\eta^\alpha(Y)$ then we get

$$\nabla_X(\eta^\alpha(Y)) = \nabla_X(\epsilon_\alpha g(Y, \xi_\alpha)),$$

$$(\nabla_X \eta^\alpha)(Y) + \eta^\alpha(\nabla_X Y) = \epsilon_\alpha g(\nabla_X Y, \xi_\alpha) + \epsilon_\alpha g(\nabla_X \xi_\alpha, Y).$$

Therefore, we have

$$(\nabla_X \eta^\alpha)(Y) = \epsilon_\alpha g(\nabla_X \xi_\alpha, Y) = \epsilon_\alpha g(fAX, Y).$$

Theorem 12. *Let M be a real hypersurface of $(\epsilon_\alpha) - S-$manifold. Then the following assertions are equivalent*

(**i**) *f is parallel on M*
(**ii**) *η^α is parallel on M*
(**iii**) *ξ_α is parallel on M*
(**iv**) *A is a symmetric tensor field A of type $(1,1)$ satisfies*

$$AX = \sum_{\alpha=1}^{s+1}\{\eta^\alpha(AX)\xi_\alpha\}, \ \forall X \in \Gamma(TM).$$

Proof. $(i) \Longrightarrow (ii)$Let f be parallel on M. Then by using (3.8) we get

$$0 = (\nabla_X f)Y = \sum_{\alpha=1}^{s+1}\{\eta^\alpha(Y)AX - \epsilon_\alpha g(AX, Y)\xi_\alpha\}.$$

If we replace Y by ξ_α, we get $AX = \sum_{\alpha=1}^{s+1}\epsilon_\alpha g(AX, Y)\xi_\alpha$. Thus $fAX = \sum_{\alpha=1}^{s+1}\epsilon_\alpha g(AX, Y)f(\xi_\alpha) = 0$. Therefore, we have

$$(\nabla_X \eta^\alpha)Y = \epsilon_\alpha g(fAX, Y) = 0;$$

i.e. η^α are parallel on M.

$(ii) \Longrightarrow (iii)$Let η^α be parallel on M. Then by using (3.8) and (3.9) we get

$$(\nabla_X \eta^\alpha) Y = \epsilon_\alpha g(fAX, Y) = 0, \quad \forall Y \in \Gamma(TM).$$

Since g is non-degenerate metric then

$$fAX = 0 = -\nabla_X \xi_\alpha;$$

i.e. ξ_α are parallel on M.

$(iii) \Longrightarrow (iv)$By using (3.10), we get $\nabla_X \xi_\alpha = -fAX = 0$.Thus $f^2 X = 0$.Hence $AX = \sum_{\alpha=1}^{s+1} \eta^\alpha (AX) \xi_\alpha$.

$(iv) \Longrightarrow (i)$ If we replace Y by ξ_α in (3.7), we get

$$(\nabla_X f) \xi_\alpha = 0.$$

Theorem 13. *Let M be a real hpersurface of $(\epsilon_\alpha) - S-$manifold. Then the following assertions are equivalent*

(i) M is an $(\epsilon_\alpha) - S$ manifold

(ii) The ξ_α-characteristic vector field satisfies (2.7)

(iii) A is the symmetric tensor field of type $(1,1)$ *satisfies*

$$AX = \epsilon_\alpha X + \sum_{\alpha=1}^{s+1} \{\eta^\alpha(A\xi_\alpha) - \epsilon_\alpha\} \eta^\alpha(X)\xi_\alpha.$$

Proof. $(i) \Longrightarrow (ii)$ If we replace Y by ξ_α then we get

$$(\nabla_X f)\xi_\alpha = \sum_{\alpha=1}^{s+1} \{g(X, \xi_\alpha) \xi_\alpha - \epsilon_\alpha \eta^\alpha (\xi_\alpha) X\}. \tag{3.16}$$

On the other hand, we know

$$(\nabla_X f)\xi_\alpha = \nabla_X (f\xi_\alpha) - f(\nabla_X \xi_\alpha).$$

From (3.16), we get

$$-f(\nabla_X \xi_\alpha) = \sum_{\alpha=1}^{s+1} \{\epsilon_\alpha \eta^\alpha (X) \xi_\alpha - \epsilon_\alpha X\},$$

$$\nabla_X \xi_\alpha = -\epsilon_\alpha f X. \tag{3.17}$$

$(ii) \Longrightarrow (iii)$ From (3.10) and (3.17), we have $\epsilon_\alpha fX = fAX$. Using (3.8), we get

$$AX = PAX - \sum_{\alpha=1}^{s+1} \eta^\alpha(AX)\xi_\alpha. \tag{3.18}$$

From (3.18) we have

$$\epsilon_\alpha f(PX + \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\xi_\alpha) = f(PAX + \sum_{\alpha=1}^{s+1} \eta^\alpha(AX)\xi_\alpha),$$

$$\epsilon_\alpha fPX = fPAX,$$

$$\epsilon_\alpha PX = PAX. \tag{3.19}$$

If we replace X by ξ_α in (3.18), we get

$$A\xi_\alpha = \sum_{\alpha=1}^{s+1} \eta^\alpha(A\xi_\alpha)\xi_\alpha.$$

By using (3.18) and (3.19) we have

$$\begin{aligned}
AX &= PAX + \sum_{\alpha=1}^{s+1} \epsilon_\alpha g(AX, \xi_\alpha)\xi_\alpha \\
&= \epsilon_\alpha PX + \sum_{\alpha=1}^{s+1} \epsilon_\alpha g(X, \sum_{\alpha=1}^{s+1} \eta^\alpha(A\xi_\alpha)\xi_\alpha)\xi_\alpha \\
&= \epsilon_\alpha PX + \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\eta^\alpha(A\xi_\alpha)\xi_\alpha \\
&= \epsilon_\alpha PX - \sum_{\alpha=1}^{s+1} \epsilon_\alpha \eta^\alpha(PX)\xi_\alpha + \sum_{\alpha=1}^{s+1} \epsilon_\alpha \eta^\alpha(A\xi_\alpha)\eta^\alpha(PX)\xi_\alpha + \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\eta^\alpha(A\xi_\alpha)\xi_\alpha \\
&= \epsilon_\alpha PX - \sum_{\alpha=1}^{s+1} \epsilon_\alpha \eta^\alpha(PX)\xi_\alpha + \sum_{\alpha=1}^{s+1} \epsilon_\alpha \eta^\alpha(A\xi_\alpha)\eta^\alpha(PX)\xi_\alpha + \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\eta^\alpha(A\xi_\alpha)\eta^\alpha(\xi_\alpha)\xi_\alpha \\
&\quad - \sum_{\alpha=1}^{s+1} \epsilon_\alpha \eta^\alpha(X)\xi_\alpha + \sum_{\alpha=1}^{s+1} \epsilon_\alpha \eta^\alpha(X)\eta^\alpha(\xi_\alpha)\xi_\alpha \\
&= \epsilon_\alpha(PX - \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\xi_\alpha) - \epsilon_\alpha \eta^\alpha(PX - \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\xi_\alpha)\xi_\alpha + \eta^\alpha(A\xi_\alpha)\eta^\alpha(PX - \sum_{\alpha=1}^{s+1} \eta^\alpha(X)\xi_\alpha)\xi_\alpha \\
&= \epsilon_\alpha X - \sum_{\alpha=1}^{s+1} (\eta^\alpha(A\xi_\alpha) - \epsilon_\alpha)\eta^\alpha(X)\xi_\alpha.
\end{aligned}$$

$(iii) \Longrightarrow (i)$ From (3.13), we have

$$\begin{aligned}
(\nabla_X f)Y &= \sum_{\alpha=1}^{s+1} \{\eta^\alpha(Y)AX - \epsilon_\alpha g(AX,Y)\xi_\alpha\} \\
&= \sum_{\alpha=1}^{s+1} \left\{\eta^\alpha(Y)\left(\epsilon_\alpha X - \sum_{\alpha=1}^{s+1}(\eta^\alpha(A\xi_\alpha) - \epsilon_\alpha)\eta^\alpha(X)\xi_\alpha\right)\right. \\
&\quad \left. -\epsilon_\alpha g\left(\left(\epsilon_\alpha X - \sum_{\alpha=1}^{s+1}(\eta^\alpha(A\xi_\alpha) - \epsilon_\alpha)\eta^\alpha(X)\xi_\alpha\right), Y\right)\xi_\alpha\right\} \\
&= \sum_{\alpha=1}^{s+1} \{g(X,Y)\xi_\alpha - \epsilon_\alpha\eta^\alpha(Y)X\}.
\end{aligned}$$

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