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ON $\alpha\omega$-OPEN SETS AND α-LINDELÖF SPACES

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Abstract. In this paper, we introduce and investigate a new class of sets called $\alpha\omega$-open sets which are weaker than both ω-open sets and α-open sets. Moreover, we obtain a characterization and preserving theorems of α-Lindelöf spaces and decompositions of continuity.

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1. Introduction

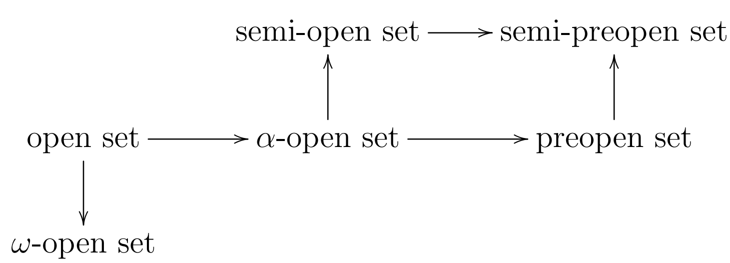
Throughout this paper, (X,τ) and (Y,σ) stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. Many topologists are focusing their recearch to introduce and investigate a weak form of open sets in topological spaces. Let (X,τ) be a space and S a subset of X. A subset S is said to be α-open [7] (resp. semi-open [6], preopen [10], semi-preopen [2]) if $S \subseteq Int(Cl(Int(S)))$ (resp. $S \subseteq Cl(Int(S))$, $S \subseteq Int(Cl(S))$, $S \subseteq Cl(Int(Cl(S)))$). Since the advent of these notions, several research papers with interesting results in different respects came to existence see [8, 9]. The family of all α-open sets in a space X is denoted by τ^α. It is shown in [7] that τ^α is a topology on X and that $\tau \subseteq \tau^\alpha$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets of X containing A is called the α-closure of S and is denoted by $\alpha Cl(S)$. The union of all α-open sets of X contained in S is called the α-interior of A and is denoted by $\alpha Int(A)$. The family of all α-open (resp. α-closed, regular open) subsets of a space X is denoted by $\alpha O(X)$ (resp. $\alpha C(X)$, $RO(X)$) and the collection of all α-open subsets of X containing a fixed point x is denoted by $\alpha O(X,x)$.

A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [3] if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X, τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, that can be defined in the same way as $Cl(A)$ and $Int(A)$, respectively, will be denoted by $\omega Cl(A)$ and $\omega Int(A)$, respectively. Several characterizations of ω-closed subsets were provided in [1, 5].

The fundamental relationships between the various types of sets considered above can be summarized in the following diagram.

The following implications hold:

DIAGRAM I



We observe that none of the implications in the above diagram can be reversed in general.

In this paper, we introduce a new class of sets called $\alpha\omega$-open sets which is a new generalization of both ω-open sets and α-open sets and investigate some properties of this set. Moreover, by using $\alpha\omega$-open sets we obtain a characterization and preserving theorems of α-Lindelöf spaces and decompositions of continuity.

2. $\alpha\omega$-open sets

In this section we introduce the following notion:

Definition 1. *A subset A of a space X is said to be $\alpha\omega$-open if for every $x \in A$, there exists an α-open subset $U_x \subseteq X$ containing x such that $U_x - A$ is countable. The complement of an $\alpha\omega$-open subset is said to be $\alpha\omega$-closed.*

The family of all $\alpha\omega$-open subsets of a space (X, τ) is denoted by $\alpha\omega O(X)$ or $\tau^{\alpha\omega}$.

Lemma 1. *For a subset A of a topological space (X, τ) both ω-openness and α-openness imply $\alpha\omega$-openness.*

Proof. (1) Assume A is ω-open. Then for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. Since every open set is α-open, A is $\alpha\omega$-open.
(2) Let A be α-open. For each $x \in A$, there exists an α-open set $U_x = A$ such that $x \in U_x$ and $U_x - A = \phi$. Therefore, A is $\alpha\omega$-open.

The following diagram shows the implications for properties of subsets

DIAGRAM II

$$\begin{array}{ccc} \text{open set} & \longrightarrow & \alpha\text{-open set} \\ \downarrow & & \downarrow \\ \omega\text{-open set} & \longrightarrow & \alpha\omega\text{-open set} \end{array}$$

The converses need not be true as shown by the following examples.

Example 1. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}, \{b\}, \{a, b\}\}$. Then $\{c\}$ is ω-open (since X is a countable set) and it is not α-open.*

Example 2. *Let X be an uncountable set and let A, B, C and D be subsets of X such that each of them is uncountable and the family $\{A, B, C, D\}$ is a partition of X. We defined the topology τ=$\{\phi, X, \{A\}, \{B\}, \{A, B\}, \{A, B, C\}\}$. Then $\{A, B, D\}$ is an α-open set which is not ω-open.*

Theorem 2. *Let (X, τ) be a topological space. Then $\tau^{\alpha\omega} = \alpha\omega O(X)$ is a topology for X such that $\tau \subseteq \tau^{\alpha} \subseteq \tau^{\alpha\omega}$ and $\tau \subseteq \tau_\omega \subseteq \tau^{\alpha\omega}$.*

Proof. (1): We have $\phi, X \in \alpha\omega O(X)$.
(2): Let $U, V \in \alpha\omega O(X)$ and $x \in U \cap V$. Then there exist α-open sets $G, H \in X$ containing x such that $G \setminus U$ and $H \setminus V$ are countable.

$$\begin{aligned}(G \cap H) \setminus (U \cap V) &= (G \cap H) \cap [X - (U \cap V)] \\ &= (G \cap H) \cap [(X - U) \cap (X - V)] \\ &= [(G \cap H) \cap (X - U)] \cap [G \cap H) \cap (X - V)] \\ &\subseteq [G \cap (X - U)] \cap [H \cap (X - V)] \\ &= (G - U) \cap (H - V).\end{aligned}$$

Thus $G \cap H \setminus U \cap V$ is countable and $G \cap H$ is an α-open set containing x. Hence $G \cap H \in \alpha\omega O(X)$.
(3): Let $\{U_i : i \in I\}$ be a family of $\alpha\omega$-open subsets of X and $x \in \cup_{i\in I} U_i$. Then $x \in U_j$ for some $j \in I$. This implies that there exists an α-open subset V of X containing x such that $V \setminus U_j$ is countable. Since $V \setminus \cup_{i\in I} U_i \subseteq V \setminus U_j$, then $V \setminus \cup_{i\in I} U_i$ is countable. Thus $\cup_{i\in I} U_i \in \alpha\omega O(X)$.

Theorem 3. *If (X, τ) is a locally countable space, then $\tau_\omega = \omega O(X)$ is the discrete topology.*

Proof. Let $A \subseteq X$ and $x \in A$. Then there exist a countable neighborhood U_x of x and an open set G_x containing x such that $G_x \subseteq U_x$. We have $G_x \setminus A \subseteq U_x \setminus A \subseteq U_x$. Thus $G_x \setminus A$ is countable and A is ω-open. Hence, $\omega O(X)$ is the discrete topology.

Corollary 4. *If (X, τ) is a locally countable space, then $\tau^{\alpha\omega} = \alpha\omega O(X)$ is the discrete topology.*

Corollary 5. *If (X, τ) is a countable space, then $\alpha\omega O(X)$ is the discrete topology.*

Proof. Since every countable space is locally countable, the proof is obvious.

Lemma 6. *A subset A of a space X is $\alpha\omega$-open if and only if for every $x \in A$, there exist an α-open subset U containing x and a countable subset C such that $U - C \subseteq A$.*

Proof. Let A be $\alpha\omega$-open and $x \in A$, then there exists an α-open subset U_x containing x such that $U_x - A$ is countable. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then there exist an α-open subset U_x containing x and a countable subset C such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is a countable set.

Theorem 7. *Let X be a space and $C \subseteq X$. If C is $\alpha\omega$-closed, then $C \subseteq K \cup B$ for some α-closed subset K and a countable subset B.*

Proof. If C is $\alpha\omega$-closed, then $X - C$ is $\alpha\omega$-open and hence for every $x \in X - C$, there exists an α-open set U containing x and a countable set B such that $U - B \subseteq X - C$. Thus $C \subseteq X - (U - B) = X - (U \cap (X - B)) = (X - U) \cup B$. Let $K = X - U$. Then K is an α-closed set such that $C \subseteq K \cup B$.

The intersection of all $\alpha\omega$-closed sets of X containing A is called the $\alpha\omega$-closure of A and is denoted by $\alpha\omega Cl(A)$. And the union of all $\alpha\omega$-open sets of X contained in A is called the $\alpha\omega$-interior and is denoted by $\alpha\omega Int(A)$.

Lemma 8. *Let A be a subset of a space X. Then*

1. *A is $\alpha\omega$-closed in X if and only if $A = \alpha\omega Cl(A)$.*
2. *$\alpha\omega Cl(X \setminus A) = X \setminus \alpha\omega Int(A)$.*
3. *$\alpha\omega Cl(A)$ is $\alpha\omega$-closed in X.*
4. *$x \in \alpha\omega Cl(A)$ if and only $A \cap G \neq \phi$ for each $\alpha\omega$-open set G containing x.*

Definition 2. *[11] A function $f : X \to Y$ is said to be quasi α-open if the image of each α-open set in X is open in Y.*

Proposition 1. *If $f : X \to Y$ is quasi α-open, then the image of an $\alpha\omega$-open set of X is ω-open in Y.*

Proof. Let $f : X \to Y$ be quasi α-open and W an $\alpha\omega$-open subset of X. Let $y \in f(W)$, there exists $x \in W$ such that $f(x) = y$. Since W is $\alpha\omega$-open, there exists an α-open set U such that $x \in U$ and $U - W = C$ is countable. Since f is quasi α-open, $f(U)$ is open in Y such that $y = f(x) \in f(U)$ and $f(U) - f(W) \subseteq f(U - W) = f(C)$ is countable. Therefore, $f(W)$ is ω-open in Y.

3. α-Lindelöf spaces

Definition 3. *[4] (1) A space X is said to be α-Lindelöf if every α-open cover of X has a countable subcover.*
(2) A subset A of a space X is said to be α-Lindelöf relative to X if every cover of A by α-open sets of X has a countable subcover.

Theorem 9. *If X is a space such that every α-open subset of X is α-Lindelöf relative to X, then every subset is α-Lindelöf relative to X.*

Proof. Let B be an arbitrary subset of X and let $\{U_i : i \in I\}$ be a cover of B by α-open sets of X. Then the family $\{U_i : i \in I\}$ is an α-open cover of the α-open set $\cup\{U_i : i \in I\}$. Hence by hypothesis there is a countable subfamily $\{U_{i_j} : j \in \mathbb{N}\}$ which covers $\cup\{U_i : i \in I\}$. This subfamily is also a cover of the set B.

Theorem 10. *For any space X, the following properties are equivalent:*

1. *X is α-Lindelöf;*
2. *Every $\alpha\omega$-open cover of X has a countable subcover.*

Proof. (1)⇒ (2): Let $\{U_\alpha : \alpha \in \Lambda\}$ be any $\alpha\omega$-open cover of X. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is $\alpha\omega$-open, there exists an α-open set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)}$ and $V_{\alpha(x)} \backslash U_{\alpha(x)}$ is countable. The family $\{V_{\alpha(x)} | x \in X\}$ is an α-open cover of X and X is α-Lindelöf. There exists a countable subset, says $\alpha(x_1), \alpha(x_2), \cdots \alpha(x_n), \cdots$ such that $X = \cup\{V_{\alpha(x_i)} | i \in \mathbf{N}\}$. Now, we have

$$\begin{aligned} X &= \cup_{i\in\mathbf{N}} \{(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cup U_{\alpha(x_i)}\} \\ &= [\cup_{i\in\mathbf{N}}(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)})] \cup [\cup_{i\in\mathbf{N}} U_{\alpha(x_i)}]. \end{aligned}$$

For each $\alpha(x_i)$, $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}$ is a countable set and there exists a countable subset $\Lambda_{\alpha(x_i)}$ of Λ such that $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)} \subseteq \cup\{U_\alpha | \alpha \in \Lambda_{\alpha(x_i)}\}$. Therefore, we have $X \subseteq [\cup_{i\in\mathbf{N}}(\cup\{U_\alpha | \alpha \in \Lambda_{\alpha(x_i)}\})] \cup [\cup_{i\in\mathbf{N}} U_{\alpha(x_i)}]$.
(2)⇒ (1): Since every α-open is $\alpha\omega$-open, the proof is obvious.

Definition 4. *A function $f : X \to Y$ is said to be $\alpha\omega$-continuous if $f^{-1}(V)$ is $\alpha\omega$-open in X for each open set V in Y.*

Theorem 11. *Let f be an $\alpha\omega$-continuous function from a space X onto a space Y. If X is α-Lindelöf, then Y is Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an $\alpha\omega$-open cover of X. Since X is α-Lindelöf, by Theorem 10, X has a countable subcover, say $\{f^{-1}(V_{\alpha i})\}_{i=1}^{\infty}$ and $V_{\alpha i} \in \{V_\alpha : \alpha \in \Lambda\}$. Hence $\{V_{\alpha i}\}_{i=1}^{\infty}$ is a countable subcover of Y. Hence Y is Lindelöf.

Definition 5. *A function $f : X \to Y$ is said to be α-continuous [9] (resp. ω-continuous [5]) if $f^{-1}(V)$ is α-open (resp. ω-open) for each open set V in Y.*

Corollary 12. *Let f be an α-continuous (or ω-continuous) function from a space X onto a space Y. If X is α-Lindelöf, then Y is Lindelöf.*

Definition 6. *A function $f : X \to Y$ is said to be $\alpha^*\omega$-continuous if $f^{-1}(V)$ is $\alpha\omega$-open in X for each α-open set V in Y.*

Now we state the following theorem whose proof is similar to Theorem 11.

Theorem 13. *Let f be an $\alpha^*\omega$-continuous function from a space X onto a space Y. If X is α-Lindelöf, then Y is α-Lindelöf.*

Proposition 2. *An $\alpha\omega$-closed subset of an α-Lindelöf space X is α-Lindelöf relative to X.*

Proof. Let A be an $\alpha\omega$-closed subset of X. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by α-open sets of X. Now for each $x \in X - A$, there is an α-open set V_x such that $V_x \cap A$ is countable. Since $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X - A\}$ is an α-open cover of X and X is α-Lindelöf, there exists a countable subcover $\{U_{\alpha_i} : i \in \mathbb{N}\} \cup \{V_{x_i} : i \in \mathbb{N}\}$. Since $\cup_{i \in N}(V_{x_i} \cap A)$ is countable, so for each $x_j \in \cup(V_{x_i} \cap A)$, there is $U_{\alpha(x_j)} \in \{U_\alpha : \alpha \in \Lambda\}$ such that $x_j \in U_{\alpha(x_j)}$ and $j \in \mathbb{N}$. Hence $\{U_{\alpha_i} : i \in \mathbb{N}\} \cup \{U_{\alpha(x_j)} : j \in \mathbb{N}\}$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$ and it covers A. Therefore, A is α-Lindelöf relative to X.

Corollary 14. *If a space X is α-Lindelöf and A is ω-closed (or α-closed), then A is α-Lindelöf relative to X.*

Definition 7. *A function $f : X \to Y$ is said to be $\alpha\omega$-closed if $f(A)$ is $\alpha\omega$-closed in Y for each α-closed set A of X.*

Theorem 15. *If $f : X \to Y$ is an $\alpha\omega$-closed surjection such that $f^{-1}(y)$ is α-Lindelöf relative to X and Y is α-Lindelöf, then X is α-Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any α-open cover of X. For each $y \in Y$, $f^{-1}(y)$ is α-Lindelöf relative to X and there exists a countable subset $\Lambda_1(y)$ of Λ such that $f^{-1}(y) \subset \cup\{U_\alpha : \alpha \in \Lambda_1(y)\}$. Now we put $U(y) = \cup\{U_\alpha : \alpha \in \Lambda_1(y)\}$ and $V(y) = Y - f(X - U(y))$. Then, since f is $\alpha\omega$-closed, $V(y)$ is an $\alpha\omega$-open set in Y containing y such that $f^{-1}(V(y)) \subset U(y)$. Since $V(y)$ is $\alpha\omega$-open, there exists an α-open set $W(y)$ containing y such that $W(y) - V(y)$ is a countable set. For each $y \in Y$, we have $W(y) \subset (W(y) - V(y)) \cup V(y)$ and hence

$$\begin{aligned} f^{-1}(W(y)) &\subset f^{-1}(W(y) - V(y)) \cup f^{-1}(V(y)) \\ &\subset f^{-1}(W(y) - V(y)) \cup U(y). \end{aligned}$$

Since $W(y) - V(y)$ is a countable set and $f^{-1}(y)$ is α-Lindelöf relative to X, there exists a countable set $\Lambda_2(y)$ of Λ such that

$$f^{-1}(W(y) - V(y)) \subset \cup\{U_\alpha : \alpha \in \Lambda_2(y)\}$$

and hence

$$f^{-1}(W(y)) \subset [\cup\{U_\alpha : \alpha \in \Lambda_2(y)\}] \cup [U(y)].$$

Since $\{W(y) : y \in Y\}$ is an α-open cover of the α-Lindelöf space Y, there exist countable points of Y, say, $y_1, y_2, ..., y_n$, ... such that
$Y = \cup\{W(y_i) : i \in N\}$. Therefore, we obtain

$$\begin{aligned} X &= \cup_{i\in N} f^{-1}(W(y_i)) = \cup_{i\in N}[\cup_{\alpha\in\Lambda_2(y_i)} U_\alpha) \cup (\cup_{\alpha\in\Lambda_1(y_i)} U_\alpha)] \\ &= \cup\{U_\alpha : \alpha \in \Lambda_1(y_i) \cup \Lambda_2(y_i), i \in N\}. \end{aligned}$$

This shows that X is α-Lindelöf.

4. Decompositions of continuity

A topological space X is said to be anti-locally countable (see [1]) if every non-empty open set is uncountable. Note that $\mathbb{R}$ with the usual topology is anti-locally countable.

Lemma 16. *A topological space (X, τ^α) is anti locally countable space if and only if $(X, \alpha\omega O(X))$ is anti locally countable.*

Proof. Let $A \in \alpha\omega O(X)$ and $x \in A$. Then by Lemma 6 , there exist an α-open subset $U \subseteq X$ containing x and a countable set C such that $U \setminus C \subseteq A$. Thus A is uncountable and $(X, \alpha\omega O(X))$ is anti locally countable.

Theorem 17. *Let (X, τ^α) be an anti locally countable space. If A is $\alpha\omega$-open, then $\alpha\omega Cl(A) = \alpha Cl(A) = Cl(A)$.*

Proof. Clearly $\alpha\omega Cl(A) \subseteq Cl(A)$. Let $x \in Cl(A)$ and B be an $\alpha\omega$-open subset containing x. Then by Lemma 6, there exists an α-open subset V containing x and a countable set C such that $V \setminus C \subseteq B$. Thus $(V \setminus C) \cap A \subseteq B \cap A$ and so $(V \cap A) \setminus C \subseteq B \cap A$. Since $x \in V$ and $x \in Cl(A)$, $V \cap A \neq \phi$. Since V and A are $\alpha\omega$-open, $V \cap A$ is $\alpha\omega$-open and as (X, τ^α) is an anti locally countable space, by Lemma 16, $V \cap A$ is uncountable and so is $(V \cap A) \setminus C$. Then

$B \cap A$ is uncountable. Therefore, $B \cap A \neq \phi$ and hence $x \in \alpha\omega Cl(A)$. Hence, $\alpha\omega Cl(A) = \alpha Cl(A) = Cl(A)$.

Corollary 18. *Let (X, τ^{α}) be an anti locally countable space. If A is $\alpha\omega$-closed, then $\alpha\omega Int(A) = \alpha Int(A) = Int(A)$.*

Theorem 19. *Let (X, τ^{α}) be an anti locally countable space. Then $RO(X, \tau^{\alpha}) = RO(X, \alpha\omega O(X))$.*

Proof. If $A \in RO(X, \tau^{\alpha})$, then $A = \alpha Int(\alpha Cl(A))$. Since A is $\alpha\omega$-open, and by Theorem 17, we have $A = \alpha Int(\alpha\omega Cl(A))$ and as $\alpha\omega Cl(A)$ is $\alpha\omega$-closed, Then $A = \alpha\omega Int(\alpha\omega Cl(A))$ and hence $A \in RO(X, \alpha\omega O(X))$.
Conversely, let $A \in RO(X, \alpha\omega O(X))$. We have $A = \alpha\omega Int(\alpha\omega Cl(A))$. Since A is $\alpha\omega$-open, by Theorem 17, $A = \alpha\omega Int(\alpha Cl(A))$. Since $\alpha Cl(A)$ is $\alpha\omega$-closed, then $A = \alpha Int(\alpha Cl(A))$. Thus $A \in RO(X, \tau^{\alpha})$.

Theorem 20. *Let (X, τ) be a topological space. Then $RO(X, \tau^{\alpha}) = RO(X, \tau)$.*

Proof. Let $A \in RO(X, \tau)$, then $A = Int(Cl(A))$,

$$\begin{aligned} \alpha Cl(A) &= A \cup Cl(Int(Cl(A))) = A \cup Cl(A) = Cl(A) \text{ and} \\ \alpha Int(\alpha Cl(A)) &= \alpha Int(Cl(A)) = Cl(A) \cap Int(Cl(Int(Cl(A)))) \\ &= Cl(A) \cap Int(Cl(A)) = Int(Cl(A)) = A \end{aligned}$$

Thus $A \in RO(X, \tau^{\alpha})$.
Conversely, Let $A = \alpha Int(\alpha Cl(A))$ i.e. $A \in RO(X, \tau^{\alpha})$. Since A is α-open,

$$\begin{aligned} A \subseteq Int(Cl(Int(A))) &\subseteq Int(Cl(A)) ....(1). \text{ On the other hand,} \\ A = \alpha Int(\alpha Cl(A)) &= \alpha Int[A \cup Cl(Int(Cl(A)))] \\ &\subseteq \alpha Int[Cl(Int(Cl(A)))] \\ &\subseteq Int(Cl(Int(Cl(A)))) = Int(Cl(A)) ....(2) \end{aligned}$$

By (1) and (2), we have $Int(Cl(A)) = A$.

Theorem 21. *If A is an $\alpha\omega$-open subset of (X, τ), then $(\tau^{\alpha\omega})_{|A} \subseteq (\tau_{|A})^{\alpha\omega}$*

Proof. Let $G \in (\tau^{\alpha\omega})_{|A}$. Then $G = H \cap A$ for some $\alpha\omega$-open subset H. For every $x \in G$, there exist $V_H, V_A \in \tau^{\alpha}$ containing x and countable sets C_H and C_A such that $V_H \backslash C_H \subseteq H$ and $V_A \backslash C_A \subseteq A$. Therefore $x \in A \cap (V_H \cap V_A) \in \tau^{\alpha}_{|A}$, $C_H \cup C_A$ is countable and $A \cap (V_H \cap V_A) \setminus (C_H \cup C_A) \subseteq (V_H \cap V_A) \cap (X \setminus C_H) \cap (X \setminus C_A) = (V_H \setminus C_H) \cap (V_A \setminus C_A) \subseteq H \cap A = G$. Therefore, $G \in (\tau_{|A})^{\alpha\omega}$

Definition 8. *A subset A of a topological space $(X.\tau)$ is said to be:*

1. *an $(\alpha\omega, \omega)$-set if $\alpha\omega Int(A) = \omega Int(A)$.*
2. *an $(\alpha\omega, \alpha)$-set if $\alpha\omega Int(A) = \alpha Int(A)$.*
3. *an $(\alpha\omega, O)$-set if $\alpha\omega Int(A) = Int(A)$.*

Remark 1.

1. *Every ω-open set is an $(\alpha\omega, \omega)$-set.*
2. *Every α-open set is an $(\alpha\omega, \alpha)$-set.*
3. *Every open set is an $(\alpha\omega, O)$-set.*

The above implications are not reversible as shown in the following examples.

Example 3. *In Example 2, if $H = \{A, B, D\}$ then, $Int(H) = \omega Int(H) = \{A, B\}$, $\alpha\omega Int(H) = \alpha Int(H) = H$. Thus H is $(\alpha\omega, \alpha)$-set but it is not $(\alpha\omega, \omega)$-set and $(\alpha\omega, O)$-set.*

Example 4. *In Example 2, if $H = \{C\}$ then, $Int(H) = \alpha Int(H) = \omega Int(H) = \alpha\omega Int(H) = \phi$. Thus H is $(\alpha\omega, \omega)$-set, $(\alpha\omega, \alpha)$-set and $(\alpha\omega, O)$-set. But it is not ω-open, α-open and open.*

Example 5. *In Example 1, if $A = \{b\}$ then, $Int(A) = \alpha Int(A) = \phi$, $\omega Int(A) = \alpha\omega Int(A) = A$. Thus A is $(\alpha\omega, \omega)$-set but it is not $(\alpha\omega, \alpha)$-set and $(\alpha\omega, O)$-set.*

Proposition 3. *Let A be a subset of a space X. The following are equivalent:*

1. *A is ω-open;*
2. *A is $\alpha\omega$-open and an $(\alpha\omega, \omega)$-set.*

Proof. (1) $\Rightarrow$ (2): It follows form the fact that every ω-open set is $\alpha\omega$-open.
(2) $\Rightarrow$ (1): Let A be $\alpha\omega$-open and an $(\alpha\omega, \omega)$-set. Then $A = \alpha\omega Int(A) = \omega Int(A)$. This shows that A is ω-open.

Proposition 4. *Let A be a subset of a space X. The following are equivalent:*

1. *A is α-open;*
2. *A is $\alpha\omega$-open and an $(\alpha\omega, \alpha)$-set.*

Proof. (1) $\Rightarrow$ (2): It follows form the fact that every α-open set is $\alpha\omega$-open.
(2) $\Rightarrow$ (1): Let A be $\alpha\omega$-open and an $(\alpha\omega, \alpha)$-set. Then $A = \alpha\omega Int(A) = \alpha Int(A)$. This shows that A is α-open.

Proposition 5. *Let A be a subset of a space X. The following are equivalent:*

1. *A is open;*
2. *A is $\alpha\omega$-open and an $(\alpha\omega, O)$-set.*

Proof. (1) $\Rightarrow$ (2): It follows form the fact that every open set is $\alpha\omega$-open.
(2) $\Rightarrow$ (1): Let A be $\alpha\omega$-open and an $(\alpha\omega, O)$-set. Then $A = \alpha\omega Int(A) = Int(A)$. This shows that A is open.

Definition 9. *A function $f : X \to Y$ is said to be $(\alpha\omega, \omega)$-continuous (resp. $(\alpha\omega, \alpha)$-continuous, $(\alpha\omega, O)$-continuous) if $f^{-1}(V)$ is $(\alpha\omega, \omega)$-set (resp. $(\alpha\omega, \alpha)$-set, $(\alpha\omega, O)$-set) for each open set V in Y.*

Theorem 22. *A function $f : X \to Y$ is continuous (resp. α-continuous, ω-continuous) if and only if f is $\alpha\omega$-continuous and $(\alpha\omega, O)$-continuous (resp. $(\alpha\omega, \alpha)$-continuous, $(\alpha\omega, \omega)$-continuous)*

Proof. This is an immediate consequence of Propositions 3, 4 and 5.

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