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ON ROTATIONAL SURFACES IN PSEUDO–EUCLIDEAN SPACE $\mathbb{E}^4_T$ WITH POINTWISE 1–TYPE GAUSS MAP

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Abstract. In this work, we study some classes of rotational surfaces in the pseudo–Euclidean space $\mathbb{E}^4_t$ with profile curves lying in 2–dimensional planes. First, we determine all such surfaces in the Minkowski 4–space $\mathbb{E}^4_1$ with pointwise 1–type Gauss map of the first kind and second kind. Then, we obtain rotational surfaces in $\mathbb{E}^4_2$ with zero mean curvature vector and having pointwise 1–type Gauss map of second kind.

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1. Introduction

In late 1970, B.-Y. Chen introduced the concept of finite type submanifolds of Euclidean space, [3]. Since then many works have been done to characterize or classify submanifolds of Euclidean space or pseudo–Euclidean space in terms of finite type. Then the notion of finite type was extended to differentiable maps, in particular Gauss map of submanifolds by B.-Y. Chen and P. Piccinni, [4]. A smooth map ϕ on a submanifold M of a Euclidean space or a pseudo Euclidean space is said to be *finite type* if ϕ has a finite spectral resolution, that is, $\phi = \phi_0 + \sum_{t=1}^{k} \phi_t$, where ϕ_0 is a constant vector and ϕ_t's are non–constant maps such that $\Delta\phi_t = \lambda_t\phi_t,\ \lambda_t \in \mathbb{R},\ t = 1, 2, \cdots, k$.

If a submanifold M of a Euclidean space or a pseudo–Euclidean space has 1–type Gauss map ν, then ν satisfies $\Delta\nu = \lambda(\nu + C)$ for some $\lambda \in \mathbb{R}$ and for some constant vector C. Also, it has been seen that the equation

$$\Delta\nu = f(\nu + C) \tag{1}$$

is satisfied for some smooth function f on M and some constant vector C by the Gauss map of some submanifolds such as helicoid, catenoid, right cones in $\mathbb{E}^3$ and

Enneper's hypersurfaces in $\mathbb{E}_1^{n+1}$, [7, 11]. A submanifold of a Euclidean or a pseudo–Euclidean space is said to have pointwise 1–type Gauss map if it satisfies (1). A submanifold with pointwise 1–type Gauss map is said to be of *the first kind* if C is the zero vector. Otherwise, it is said to be of *the second kind.*

Remark 1. *For an n–dimensional plane M in a pseudo–Euclidean space, the Gauss map ν is constant and $\Delta\nu = 0$. For $f = 0$ if we write $\Delta\nu = 0.\nu$, then M has pointwise 1–type Gauss map of the first kind. If we choose $C = -\nu$ for any nonzero smooth function f, then* (1) *holds. In this case, M has pointwise 1–type Gauss map of the second kind. Therefore, we say that an n–dimensional plane M in a pseudo–Euclidean space is a trivial pseudo–Riemannian submanifold with pointwise 1–type Gauss map of the first kind and the second kind.*

The classification of ruled surfaces and rational surfaces in $\mathbb{E}_1^3$ with pointwise 1–type Gauss map were studied in [5, 10]. Also, in [7] and [13], a characterization of rotational hypersurface and a complete classification of cylindrical and non–cylindrical surfaces in $\mathbb{E}_1^m$ were obtained, respectively.

The complete classification of Vranceanu rotational surfaces in the pseudo–Euclidean $\mathbb{E}_2^4$ with pointwise 1–type Gauss map was obtained in [12], and it was proved that a flat rotational surface in $\mathbb{E}_2^4$ with pointwise 1–type Gauss map is either the product of two plane hyperbolas or the product of a plane circle and a plane hyperbola.

Recently, a classification of flat spacelike and timelike rotational surfaces in $\mathbb{E}_1^4$ with pointwise 1–type Gauss map were given in [1, 8]. Also, in [6] Lorentzian surfaces in 4–dimensional Minkowski space $\mathbb{E}_1^4$ with pointwise 1–type Gauss map were studied and some classification theorems were obtained.

In this article, we present some results on rotational surfaces in the pseudo–Euclidean space $\mathbb{E}_t^4$ with profile curves lying in 2–dimensional planes and having pointwise 1–type Gauss map. First, we give classification of all such surfaces in the Minkowski space $\mathbb{E}_1^4$ defined by (10), called double rotational surface, with pointwise 1–type Gauss map of the first kind. Then, we show that there exists no a non–planar timelike double rotational surface in $\mathbb{E}_1^4$ with flat normal bundle and pointwise 1–type Gauss map of the second kind. Finally, we determine the rotational surfaces in the pseudo–Euclidean $\mathbb{E}_2^4$ defined by (22) and (23) with zero mean curvature vector and pointwise 1–type Gauss map of the second kind.

2. Preliminaries

Let $\mathbb{E}_t^m$ denote m–dimensional pseudo–Euclidean space with the canonical metric given by

$$g = \sum_{i=1}^{m-t} dx_i^2 - \sum_{j=m-t+1}^{m} dx_j^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}_t^m$.
We put

$$\mathbb{S}_t^{m-1}(x_0, r^2) = \left\{ x \in \mathbb{E}_t^m \mid \langle x - x_0, x - x_0 \rangle = r^{-2} \right\}, \tag{2}$$

$$\mathbb{H}_{t-1}^{m-1}(x_0, -r^2) = \left\{ x \in \mathbb{E}_t^m \mid \langle x - x_0, x - x_0 \rangle = -r^{-2} \right\}, \tag{3}$$

where $\langle , \rangle$ is the indefinite inner product associated to g. Then $\mathbb{S}_t^{m-1}(x_0, r^2)$ and $\mathbb{H}_{t-1}^{m-1}(x_0, -r^2)$ are complete pseudo–Riemannian manifolds of constant curvature r^2 and $-r^2$, respectively. We denote $\mathbb{S}_t^{m-1}(x_0, r^2)$ and $\mathbb{H}_{t-1}^{m-1}(x_0, -r^2)$ by $\mathbb{S}_t^{m-1}(r^2)$ and $\mathbb{H}_{t-1}^{m-1}(-r^2)$ when x_0 is the origin. In particular, $\mathbb{E}_1^m$, $\mathbb{S}_1^{m-1}(x_0, r^2)$ and $\mathbb{H}_1^{m-1}(x_0, -r^2)$ are known as *the Minkowski*, *de Sitter*, and *anti–de Sitter spaces*, respectively.

A vector $v \in \mathbb{E}_t^m$ is called spacelike (resp., timelike) if $\langle v, v \rangle > 0$ or $v = 0$ (resp., $\langle v, v \rangle < 0$). A vector v is called lightlike if $\langle v, v \rangle = 0$, and $v \neq 0$.

Let M be an oriented n–dimensional pseudo–Riemannian submanifold in an m–dimensional pseudo–Euclidean space $\mathbb{E}_t^m$. We choose an oriented local orthonormal frame $\{e_1, \ldots, e_m\}$ on M with $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$ such that $e_1, \ldots, e_n$ are tangent to M and $e_{n+1}, \ldots, e_m$ are normal to M. We use the following convention on the range of indices: $1 \leq i, j, k, \ldots \leq n,\ n+1 \leq r, s, t, \ldots \leq m$.

Let $\widetilde{\nabla}$ be the Levi–Civita connection of $\mathbb{E}_t^m$ and ∇ the induced connection on M. Denote by $\{\omega^1, \ldots, \omega^m\}$ the dual frame and by $\{\omega_{AB}\}, A, B = 1, \ldots, m$, the connection forms associated to $\{e_1, \ldots, e_m\}$. Then we have

$$\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} \varepsilon_r h_{ik}^r e_r,$$

$$\widetilde{\nabla}_{e_k} e_r = - A_r(e_k) + \sum_{s=n+1}^{m} \varepsilon_s \omega_{rs}(e_k) e_s, \quad D_{e_k} e_r = \sum_{s=n+1}^{m} \varepsilon_s \omega_{rs}(e_k) e_s,$$

where D is the normal connection, h_{ij}^r the coefficients of the second fundamental form h, and A_r the Weingarten map in the direction e_r.

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n} \sum_{i,r} \varepsilon_i \varepsilon_r h_{ii}^r e_r \tag{4}$$

and

$$\|h\|^2 = \sum_{i,j,r} \varepsilon_i \varepsilon_j \varepsilon_r h^r_{ij} h^r_{ji}. \tag{5}$$

A submanifold M is said to have parallel mean curvature vector H if $DH = 0$ identically.

The gradient of a smooth function f on M is defined by $\nabla f = \sum_{i=1}^{n} \varepsilon_i e_i(f) e_i$, and the Laplace operator acting on M is $\Delta = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} e_i - e_i e_i)$.

The Codazzi equation of M in $\mathbb{E}^m_t$ is given by

$$\begin{aligned} & h^r_{ij,k} = h^r_{jk,i}, \\ & h^r_{jk,i} = e_i(h^r_{jk}) - \sum_{\ell=1}^{n} \varepsilon_\ell \left(h^r_{\ell k} \omega_{j\ell}(e_i) + h^r_{\ell j} \omega_{k\ell}(e_i) \right) + \sum_{s=n+1}^{m} \varepsilon_s h^s_{jk} \omega_{sr}(e_i). \end{aligned} \tag{6}$$

Also, from the Ricci equation of M in $\mathbb{E}^m_t$, we have

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_r, A_s](e_j), e_k \rangle = \sum_{i=1}^{n} \varepsilon_i \left(h^r_{ik} h^s_{ij} - h^r_{ij} h^s_{ik} \right), \tag{7}$$

where R^D is the normal curvature tensor.

A submanifold M in $\mathbb{E}^m_t$ is said to have flat normal bundle if R^D vanishes identically.

Let $G(m-n, m)$ be the Grassmannian manifold consisting of all oriented $(m-n)$–planes through the origin of an m–dimensional pseudo–Euclidean space $\mathbb{E}^m_t$ with index t and $\bigwedge^{m-n} \mathbb{E}^m_t$ the vector space obtained by the exterior product of $m-n$ vectors in $\mathbb{E}^m_t$. Let $f_{i_1} \wedge \cdots \wedge f_{i_{m-n}}$ and $g_{i_1} \wedge \cdots \wedge g_{i_{m-n}}$ be two vectors in $\bigwedge^{m-n} \mathbb{E}^m_t$, where $\{f_1, f_2, \ldots, f_m\}$ and $\{g_1, g_2, \ldots, g_m\}$ are two orthonormal bases of $\mathbb{E}^m_t$. Define an indefinite inner product $\langle\langle , \rangle\rangle$ on $\bigwedge^{m-n} \mathbb{E}^m_t$ by

$$\langle\langle f_{i_1} \wedge \cdots \wedge f_{i_{m-n}}, g_{i_1} \wedge \cdots \wedge g_{i_{m-n}} \rangle\rangle = \det(\langle f_{i_\ell}, g_{j_k} \rangle). \tag{8}$$

Therefore, for some positive integer s, we may identify $\bigwedge^{m-n} \mathbb{E}^m_t$ with some pseudo–Euclidean space $\mathbb{E}^N_s$, where $N = \binom{m}{m-n}$. The map $\nu : M \to G(m-n, m) \subset \mathbb{E}^N_s$ from an oriented pseudo–Riemannian submanifold M into $G(m-n, m)$ defined by

$$\nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_m)(p) \tag{9}$$

is called the *Gauss map* of M which assigns to a point p in M the oriented $(m-n)$–plane through the origin of $\mathbb{E}^m_t$ and parallel to the normal space of M at p, [12].

We put $\varepsilon = \langle\langle \nu, \nu \rangle\rangle = \varepsilon_{n+1}\varepsilon_{n+2}\cdots\varepsilon_m = \pm 1$ and

$$\widetilde{M}_s^{N-1}(\varepsilon) = \begin{cases} \mathbb{S}_s^{N-1}(1) & \text{in } \mathbb{E}_s^N \quad \text{if } \varepsilon = 1 \\ \mathbb{H}_{s-1}^{N-1}(-1) & \text{in } \mathbb{E}_s^N \quad \text{if } \varepsilon = -1. \end{cases}$$

Then the Gauss image $\nu(M)$ can be viewed as $\nu(M) \subset \widetilde{M}_s^{N-1}(\varepsilon)$.

2.1. Rotational surfaces in $\mathbb{E}_1^4$ with profile curves lying in 2–planes

We consider timelike rotational surfaces in the Minkowski space $\mathbb{E}_1^4$ whose profile curves lie in timelike 2–planes. By choosing a profile curve $\gamma(s) = (x(s), 0, 0, w(s))$ in the xw–plane defined on an open interval I in $\mathbb{R}$. We can parametrize a timelike rotational surface in $\mathbb{E}_1^4$ as follows

$$M : r(s,t) = (x(s)\cos at, x(s)\sin at, w(s)\sinh bt, w(s)\cosh bt), \tag{10}$$

where s is the arc lenght parameter of γ, $s \in I$ and $t \in (0, 2\pi)$. The rotational surface M is called a double rotational surface in $\mathbb{E}_1^4$. Then, $x'^2(s) - w'^2(s) = -1$ and the curvature κ of γ is given by $\kappa(s) = w'(s)x''(s) - x'(s)w''(s)$.

We form the following orthonormal moving frame field $\{e_1, e_2, e_3, e_4\}$ on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M:

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{q}\frac{\partial}{\partial t}, \tag{11}$$
$$e_3 = (w'(s)\cos at, w'(s)\sin at, x'(s)\sinh bt, x'(s)\cosh bt), \tag{12}$$
$$e_4 = \frac{1}{q}(bw(s)\sin at, -bw(s)\cos at, ax(s)\cosh bt, ax(s)\sinh bt), \tag{13}$$

where $q = \sqrt{a^2x^2(s) + b^2w^2(s)}$ and $\varepsilon_1 = -1,\ \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$.

By a direct computation, we have the components of the second fundamental form and the connection forms as follows

$$h_{11}^3 = \kappa(s), \qquad h_{22}^3 = -\frac{a^2x(s)w'(s) + b^2w(s)x'(s)}{a^2x^2(s) + b^2w^2(s)}, \tag{14}$$

$$h_{12}^3 = h_{11}^4 = h_{22}^4 = 0, \qquad h_{12}^4 = \frac{ab(x(s)w'(s) - w(s)x'(s))}{a^2x^2(s) + b^2w^2(s)}, \tag{15}$$

$$\omega_{12}(e_1) = 0, \qquad \omega_{12}(e_2) = \frac{a^2x(s)x'(s) + b^2w(s)w'(s)}{a^2x^2(s) + b^2w^2(s)}, \tag{16}$$

$$\omega_{34}(e_1) = 0, \qquad \omega_{34}(e_2) = \frac{ab(x(s)x'(s) - w(s)w'(s))}{a^2x^2(s) + b^2w^2(s)}. \tag{17}$$

Hence we obtain the mean curvature vector and the normal curvature of M from (4) and (7), respectively, as

$$H = \frac{1}{2}(h^3_{22} - h^3_{11})e_3, \tag{18}$$

$$R^D(e_1, e_2; e_3, e_4) = h^4_{12}(h^3_{11} + h^3_{22}). \tag{19}$$

On the other hand, from the Codazzi equation (6) we have

$$e_1(h^3_{22}) = -\omega_{12}(e_2)\left(h^3_{11} + h^3_{22}\right) - h^4_{12}\omega_{34}(e_2), \tag{20}$$

$$e_1(h^4_{12}) = -2h^4_{12}\omega_{12}(e_2) + h^3_{11}\omega_{34}(e_2). \tag{21}$$

2.2. Rotational surfaces in $\mathbb{E}^4_2$ with profile curves lying in 2–planes

In the pseudo–Euclidean space $\mathbb{E}^4_2$, we consider two rotational surfaces whose profile curves lie in 2–planes.

First, we choose a profile curve α in the yw–plane as $\alpha(s) = (0, y(s), 0, w(s))$ defined on an open interval $I \subset \mathbb{R}$. Then the parametrization of the rotational surface $M_1(b)$ in $\mathbb{E}^4_2$ is given by

$$M_1(b) : r_1(s,t) = (w(s)\sinh t, y(s)\cosh(bt), y(s)\sinh(bt), w(s)\cosh t), \tag{22}$$

for some constant $b > 0$, where $s \in I$ and $t \in \mathbb{R}$.

Secondly, we choose a profile curve β in the xz–plane as $\beta(s) = (x(s), 0, z(s), 0)$ defined on an open interval $I \subset \mathbb{R}$. Then the parametrization of the rotational surface $M_2(b)$ in $\mathbb{E}^4_2$ is given by

$$M_2(b) : r_2(s,t) = (x(s)\cos t, x(s)\sin t, z(s)\cos(bt), z(s)\sin(bt)), \tag{23}$$

for some constant $b > 0$, where $s \in I$ and $t \in (0, 2\pi)$.

Now, for the rotational surface $M_1(b)$ defined by (22), we consider the following orthonormal moving frame field $\{e_1, e_2, e_3, e_4\}$ on $M_1(b)$ such that e_1, e_2 are tangent to $M_1(b)$, and e_3, e_4 are normal to $M_1(b)$:

$$e_1 = \frac{1}{q}\frac{\partial}{\partial t}, \quad e_2 = \frac{1}{A}\frac{\partial}{\partial s}, \tag{24}$$

$$e_3 = \frac{1}{A}(y'(s)\sinh t, w'(s)\cosh(bt), w'(s)\sinh(bt), y'(s)\cosh t), \tag{25}$$

$$e_4 = -\frac{\varepsilon\varepsilon^*}{q}(by(s)\cosh t, w(s)\sinh(bt), w(s)\cosh(bt), by(s)\sinh t), \tag{26}$$

where $A = \sqrt{\varepsilon(y'^2(s) - w'^2(s))} \neq 0$, $q = \sqrt{\varepsilon^*(w^2(s) - b^2y^2(s))} \neq 0$, and $\varepsilon = \operatorname{sgn}(y'^2(s) - w'^2(s))$, $\varepsilon^* = \operatorname{sgn}(w^2(s) - b^2y^2(s))$. Then, $\varepsilon_1 = -\varepsilon_4 = \varepsilon^*$ and $\varepsilon_2 = -\varepsilon_3 = \varepsilon$.

By a direct calculation, we have the components of the second fundamental form and the connection forms as follows

$$h^3_{11} = \frac{1}{Aq^2}(b^2y(s)w'(s) - w(s)y'(s)), \quad h^3_{22} = \frac{1}{A^3}(w'(s)y''(s) - y'(s)w''(s)), \tag{27}$$

$$h^4_{12} = \frac{\varepsilon\varepsilon^* b}{Aq^2}(w(s)y'(s) - y(s)w'(s)), \quad h^3_{12} = h^4_{11} = h^4_{22} = 0, \tag{28}$$

$$\omega_{12}(e_1) = \frac{1}{Aq^2}(b^2y(s)y'(s) - w(s)w'(s)), \quad \omega_{12}(e_2) = 0, \tag{29}$$

$$\omega_{34}(e_1) = \frac{\varepsilon\varepsilon^* b}{Aq^2}(w(s)w'(s) - y(s)y'(s)), \quad \omega_{34}(e_2) = 0. \tag{30}$$

Similarly, for the rotational surface $M_2(b)$ defined by (23), we consider the following orthonormal moving frame field $\{e_1, e_2, e_3, e_4\}$ on $M_2(b)$ such that e_1, e_2 are tangent to $M_2(b)$, and e_3, e_4 are normal to $M_2(b)$:

$$e_1 = \frac{1}{\bar{q}}\frac{\partial}{\partial t}, \quad e_2 = \frac{1}{\bar{A}}\frac{\partial}{\partial s}, \tag{31}$$

$$e_3 = \frac{1}{\bar{A}}(z'(s)\cos t, z'(s)\sin t, x'(s)\cos(bt), x'(s)\sin(bt)), \tag{32}$$

$$e_4 = -\frac{\varepsilon\varepsilon^*}{\bar{q}}(bz(s)\sin t, -bz(s)\cos t, x(s)\sin(bt), -x(s)\cos(bt)), \tag{33}$$

where $\bar{A} = \sqrt{\varepsilon(x'^2(s) - z'^2(s))} \neq 0$, $\bar{q} = \sqrt{\varepsilon^*(x^2(s) - b^2z^2(s))} \neq 0$, $\varepsilon = \operatorname{sgn}(x'^2(s) - z'^2(s))$, and $\varepsilon^* = \operatorname{sgn}(x^2(s) - b^2z^2(s))$. Then, $\varepsilon_1 = -\varepsilon_4 = \varepsilon^*$ and $\varepsilon_2 = -\varepsilon_3 = \varepsilon$.

By a direct computation, we have the components of the second fundamental form and the connection forms as follows

$$h^3_{11} = \frac{1}{\bar{A}\bar{q}^2}(b^2z(s)x'(s) - x(s)z'(s)), \quad h^3_{22} = \frac{1}{\bar{A}^3}(z'(s)x''(s) - x'(s)z''(s)), \tag{34}$$

$$h^4_{12} = \frac{\varepsilon\varepsilon^* b}{\bar{A}\bar{q}^2}(z(s)x'(s) - x(s)z'(s)), \quad h^3_{12} = h^4_{11} = h^4_{22} = 0, \tag{35}$$

$$\omega_{12}(e_1) = \frac{1}{\bar{A}\bar{q}^2}(b^2z(s)z'(s) - x(s)x'(s)), \quad \omega_{12}(e_2) = 0, \tag{36}$$

$$\omega_{34}(e_1) = \frac{\varepsilon\varepsilon^* b}{\bar{A}\bar{q}^2}(z(s)z'(s) - x(s)x'(s)), \quad \omega_{34}(e_2) = 0. \tag{37}$$

Therefore, we have the mean curvature vector and normal curvature for the rotational surfaces $M_1(b)$ and $M_2(b)$ as follows

$$H = -\frac{1}{2}(\varepsilon\varepsilon^* h^3_{11} + h^3_{22})e_3, \tag{38}$$

$$R^D(e_1, e_2; e_3, e_4) = h^4_{12}(\varepsilon h^3_{22} - \varepsilon^* h^3_{11}). \tag{39}$$

On the other hand, by using the Codazzi equation (6) we obtain

$$e_2(h^3_{11}) = \varepsilon^* h^4_{12}\omega_{34}(e_1) + \omega_{12}(e_1)(\varepsilon^* h^3_{11} - \varepsilon h^3_{22}), \tag{40}$$

$$e_2(h^4_{12}) = -\varepsilon h^3_{22}\omega_{34}(e_1) + 2\varepsilon^* h^4_{12}\omega_{12}(e_1). \tag{41}$$

The rotational surfaces $M_1(b)$ and $M_2(b)$ defined by (22) and (23) for $b = 1$, $x(s) = y(s) = f(s)\sinh s$ and $z(s) = w(s) = f(s)\cosh s$ are also known as Vranceanu rotational surface, where $f(s)$ is a smooth function, [9].

3. Rotational surfaces in $\mathbb{E}^4_1$ with pointwise 1–type Gauss map

In this section, we study rotational surfaces in the Minkowski space $\mathbb{E}^4_1$ defined by (10) with pointwise 1–type Gauss map.

By a direct calculation, the Laplacian of the Gauss map ν for an n–dimensional submanifold M in a pseudo–Euclidean space $\mathbb{E}^{n+2}_t$ is obtained as follows:

Lemma 1. *Let M be an n–dimensional submanifold of a pseudo–Euclidean space $\mathbb{E}^{n+2}_t$. Then, the Laplacian of the Gauss map $\nu = e_{n+1} \wedge e_{n+2}$ is given by*

$$\begin{aligned}\Delta\nu =&||h||^2\nu + 2\sum_{j<k}\varepsilon_j\varepsilon_k R^D(e_j, e_k; e_{n+1}, e_{n+2})e_j \wedge e_k\\ &+ \nabla(\mathrm{tr}A_{n+1}) \wedge e_{n+2} + e_{n+1} \wedge \nabla(\mathrm{tr}A_{n+2})\\ &+ n\sum_{j=1}^{n}\varepsilon_j\omega_{(n+1)(n+2)}(e_j)H \wedge e_j,\end{aligned} \tag{42}$$

where $||h||^2$ is the squared length of the second fundamental form, R^D the normal curvature tensor, and $\nabla(\mathrm{tr}A_r)$ the gradient of $\mathrm{tr}A_r$.

Let M be a surface in the pseudo–Euclidean space $\mathbb{E}^4_t$. We choose a local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M. Let C be a vector field in $\Lambda^2\mathbb{E}^4_t \equiv \mathbb{E}^6_s$. Since the set $\{e_A \wedge e_B | 1 \le A < B \le 4\}$ is an orthonormal basis for $\mathbb{E}^6_s$, the vector C can be expressed as

$$C = \sum_{1\le A<B\le 4}\varepsilon_A\varepsilon_B C_{AB}\, e_A \wedge e_B, \tag{43}$$

where $C_{AB} = \langle C, e_A \wedge e_B \rangle$.

Lemma 2. *A vector C in $\Lambda^2\mathbb{E}^4_t \equiv \mathbb{E}^6_s$ written by (43) is constant if and only if the following equations are satisfied for $i = 1, 2$*

$$\begin{aligned} e_i\,(C_{12}) =& \varepsilon_3 h^3_{i2} C_{13} + \varepsilon_4 h^4_{i2} C_{14} - \varepsilon_3 h^3_{i1} C_{23} - \varepsilon_4 h^4_{i1} C_{24}, && (44)\\ e_i\,(C_{13}) =& -\varepsilon_2 h^3_{i2} C_{12} + \varepsilon_4 \omega_{34}(e_i) C_{14} + \varepsilon_2 \omega_{12}(e_i) C_{23} - \varepsilon_4 h^4_{i1} C_{34}, && (45)\\ e_i\,(C_{14}) =& -\varepsilon_2 h^4_{i2} C_{12} - \varepsilon_3 \omega_{34}(e_i) C_{13} + \varepsilon_2 \omega_{12}(e_i) C_{24} + \varepsilon_3 h^3_{i1} C_{34}, && (46)\\ e_i\,(C_{23}) =& \varepsilon_1 h^3_{i1} C_{12} - \varepsilon_1 \omega_{12}(e_i) C_{13} + \varepsilon_4 \omega_{34}(e_i) C_{24} - \varepsilon_4 h^4_{i2} C_{34}, && (47)\\ e_i\,(C_{24}) =& \varepsilon_1 h^4_{i1} C_{12} - \varepsilon_1 \omega_{12}(e_i) C_{14} - \varepsilon_3 \omega_{34}(e_i) C_{23} + \varepsilon_3 h^3_{i2} C_{34}, && (48)\\ e_i\,(C_{34}) =& \varepsilon_1 h^4_{i1} C_{13} - \varepsilon_1 h^3_{i1} C_{14} + \varepsilon_2 h^4_{i2} C_{23} - \varepsilon_2 h^3_{i2} C_{24}. && (49) \end{aligned}$$

Using (42) the following results can be stated for the characterization of timelike surfaces in $\mathbb{E}^4_1$ with pointwise 1–type Gauss map of the first kind.

Theorem 3. *Let M be an oriented timelike surface with zero mean curvature in $\mathbb{E}^4_1$. Then M has pointwise 1–type Gauss map of the first kind if and only if M has flat normal bundle. Hence, the Gauss map ν satisfies (1) for $f = \|h\|^2$ and $C = 0$.*

Theorem 4. *Let M be an oriented timelike surface with non–zero mean curvature in $\mathbb{E}^4_1$. Then M has pointwise 1–type Gauss map of the first kind if and only if M has parallel mean curvature vector.*

We will classify timelike rotational surface in $\mathbb{E}^4_1$ defined by (10) with pointwise 1–type Gauss map of the first kind by using the above theorems.

Theorem 5. *Let M be a timelike rotational surface in $\mathbb{E}^4_1$ defined by (10). Then M has zero mean curvature vector, and its normal bundle is flat if and only if M is an open part of a timelike plane in $\mathbb{E}^4_1$.*

Proof. Let M be a timelike rotational surface in $\mathbb{E}^4_1$ given by (10). Then, there exists a frame field $\{e_1, e_2, e_3, e_4\}$ defined on M given by (11)–(13), and the components of the second fundamental form are given by (14) and (15). Since M has zero mean curvature, and its normal bundle is flat, then (18) and (19) imply, respectively,

$$\begin{aligned} h^3_{22} - \kappa &= 0, && (50)\\ h^4_{12}(\kappa + h^3_{22}) &= 0 && (51) \end{aligned}$$

as $h^3_{11} = \kappa$, where κ is the curvature of the profile curve of M. By using (50) and (51) we obtain $h^4_{12}\kappa = 0$ which implies either $\kappa = 0$ or $h^4_{12} = 0$.

Case 1. $\kappa = 0$. Then, the profile curve of M is a line. We can parametrize the line as

$$x(s) = x_0 s + x_1, \quad w(s) = w_0 s + w_1 \tag{52}$$

for some constants $x_0,\ x_1,\ w_0,\ w_1 \in \mathbb{R}$ with $x_0^2 - w_0^2 = -1$. From (50), we also have $h_{22}^3 = 0$. By using the second equation in (14) and (52) we obtain

$$h_{22}^3 = -\frac{(a^2+b^2)x_0 w_0 s + a^2 x_1 w_0 + b^2 x_0 w_1}{a^2(x_0 s + x_1)^2 + b^2(w_0 s + w_1)^2} = 0$$

which gives

$$(a^2+b^2)x_0 w_0 = 0, \tag{53}$$

$$a^2 x_1 w_0 + b^2 w_1 x_0 = 0. \tag{54}$$

From (53) if $w_0 = 0$, then $x_0^2 = -1$ which is inconsistent equation. Hence, $w_0 \neq 0$ and $x_0 = 0$, and thus $w_0 = \pm 1$. Also, from (54) we get $x_1 = 0$. Thus, $x = 0$ which implies that M is an open part of the timelike zw–plane.
Case 2. $h_{12}^4 = 0$. From the first equation in (15) we have the differential equation $xw' - wx' = 0$ that gives $x = c_0 w$ where c_0 is a constant. Therefore, the profile curve of M is an open part of a line passing through the origin. Since the curvature κ is zero, we have $h_{11}^3 = 0$, and thus $h_{22}^3 = 0$ because of (50). From the second equation in (14) we get $c_0(a^2+b^2)ww' = 0$ which implies that $c_0 = 0$, i.e., $x = 0$. Therefore M is an open part of the timelike zw–plane.

In view of Remark 1, the converse of the proof is trivial.

By Theorem 3 and Theorem 5, we state

Corollary 6. *There exists no non–planar timelike surface with zero mean curvature in* $\mathbb{E}_1^4$ *defined by* (10) *with pointwise 1–type Gauss map of the first kind.*

Now, we focus on timelike rotational surfaces in $\mathbb{E}_1^4$ with parallel non–zero mean curvature vector to obtain surfaces in $\mathbb{E}_1^4$ defined by (10) with pointwise 1–type Gauss map of the first kind.

Theorem 7. *A timelike rotational surface in* $\mathbb{E}_1^4$ *defined by* (10) *has parallel non–zero mean curvature if and only if it is an open part of the timelike surface defined by*

$$\begin{aligned} r(s,t) = (&r_0 \cosh(\frac{s}{r_0}) \cos at,\ r_0 \cosh(\frac{s}{r_0}) \sin at,\ r_0 \sinh(\frac{s}{r_0}) \sinh bt, \\ &r_0 \sinh(\frac{s}{r_0}) \cosh bt) \end{aligned} \tag{55}$$

which has zero mean curvature in the de Sitter space $\mathbb{S}_1^3\left(\frac{1}{r_0^2}\right) \subset \mathbb{E}_1^4$.

Proof. Let M be a timelike rotational surface in $\mathbb{E}^4_1$ defined by (10). Then, we have an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M in $\mathbb{E}^4_1$ given by (11)–(13), and the components of the second fundamental form are given by (14) and (15). Suppose that the mean curvature vector H is parallel, i.e., $D_{e_i}H = 0$ for $i = 1, 2$. By considering (17) and (18) we have

$$D_{e_2}H = \frac{ab(h^3_{22} - h^3_{11})(xx' - ww')}{2(a^2x^2 + b^2w^2)} e_4 = 0.$$

Since M has nonzero mean curvature, this equation reduces $xx' - ww' = 0$ that implies $x^2 - w^2 = \mu_0$, where μ_0 is a real number. Since γ is a timelike curve with parametrized by arc length parameter s, we can choose $\mu_0 = r_0^2$ and the components of γ as

$$x(s) = r_0 \cosh \frac{s}{r_0}, \quad w(s) = r_0 \sinh \frac{s}{r_0}.$$

Therefore, M is an open part of the timelike surface given by (55) which is minimal in the de Sitter space $\mathbb{S}^3_1\left(\frac{1}{r_0^2}\right) \subset \mathbb{E}^4_1$.

The converse of the proof follows from a direct calculation.

Considering Theorem 4 and Theorem 7 we state the following:

Corollary 8. *A timelike rotational surface M with non–zero mean curvature in $\mathbb{E}^4_1$ defined by* (10) *has pointwise 1–type Gauss map of the first kind if and only if it is an open part of the surface given by* (55).

By combining (5) and (7) we obtain the following classification theorem:

Theorem 9. *Let M be a timelike rotational surface in $\mathbb{E}^4_1$ defined by* (10). *Then M has pointwise 1–type Gauss map of the first kind if and only if M is an open part of a timelike plane or the surface given by* (55). *Moreover, the Gauss map $\nu = e_3 \wedge e_4$ of the surface* (55) *satisfies* (1) *for $C = 0$ and the function*

$$f = \|h\|^2 = \frac{2}{r_0^2}\left(1 - \frac{a^2b^2}{(a^2\cosh^2(\frac{s}{r_0}) + b^2\sinh^2(\frac{s}{r_0}))^2}\right).$$

Note that there is no non–planar timelike rotational surface in $\mathbb{E}^4_1$ defined by (10) with global 1–type Gauss map of the first kind.

Now, we investigate timelike rotational surfaces in $\mathbb{E}^4_1$ defined by (10) with pointwise 1–type Gauss map of the second kind.

Theorem 10. *A timelike rotational surface M in $\mathbb{E}^4_1$ defined by* (10) *with flat normal bundle has pointwise 1–type Gauss map of the second kind if and only if M is an open part of a timelike plane in $\mathbb{E}^4_1$.*

Proof. Let M be a timelike rotational surface with flat normal bundle in $\mathbb{E}^4_1$ defined by (10). Thus, we have $R^D(e_1, e_2; e_3, e_4) = h^4_{12}(h^3_{11} + h^3_{22}) = 0$ which implies that $h^4_{12} = 0$ or $h^3_{11} = -h^3_{22} \neq 0$.

Case 1. $h^4_{12} = 0$. Now considering the second equation in (15) the general solution of $xw' - wx' = 0$ is $x = c_0 w$, where c_0 is constant. Hence, M is a timelike regular cone in the Minkowski space $\mathbb{E}^4_1$. For $c_0 = 0$, it can be easily seen that M is an open part of the timelike zw–plane. We suppose that $c_0 \neq 0$. If we parametrize the line $x = c_0 w$ with respect to arc length parameter s, we then have $w(s) = \pm\frac{1}{\sqrt{1-c_0^2}}s + w_0$ and $x(s) = \pm\frac{c_0}{\sqrt{1-c_0^2}}s + c_0 w_0,\ w_0, c_0 \in \mathbb{R}$ with $c_0^2 < 1$. Thus, from (14)–(17) we obtain that

$$\begin{aligned}
h^3_{11} &= 0, & h^3_{22} &= \mp\frac{c_0(a^2+b^2)}{\sqrt{1-c_0^2}(a^2c_0^2+b^2)w},\\
h^3_{12} &= 0, & h^4_{ij} &= 0,\ i,j = 1,2,\\
\omega_{12}(e_1) &= 0, & \omega_{12}(e_2) &= \pm\frac{1}{\sqrt{1-c_0^2}w},\\
\omega_{34}(e_1) &= 0, & \omega_{34}(e_2) &= \mp\frac{ab\sqrt{1-c_0^2}}{(a^2c_0^2+b^2)w}.
\end{aligned} \tag{56}$$

Therefore, using equations (20) and (42) the Laplacian of the Gauss map $\nu = e_3 \wedge e_4$ is given by

$$\Delta\nu = ||h||^2\nu + h^3_{22}\omega_{12}(e_2)e_1 \wedge e_4 - h^3_{22}\omega_{34}(e_2)e_2 \wedge e_3. \tag{57}$$

Assume that M has pointwise 1–type Gauss map of the second kind. Then, there exists a smooth function f and non–zero constant vector C such that (1) is satisfied. Therefore, from (1) and (57) we get

$$f(1 + C_{34}) = \|h\|^2 = (h^3_{22})^2, \tag{58}$$

$$fC_{14} = -\,h^3_{22}\omega_{12}(e_2), \tag{59}$$

$$fC_{23} = -\,h^3_{22}\omega_{34}(e_2), \tag{60}$$

$$C_{12} = C_{13} = C_{24} = 0. \tag{61}$$

It follows from (56), (59) and (60) that $C_{14} \neq 0$ and $C_{23} \neq 0$.

Now, from (59) and (60) we have

$$\omega_{34}(e_2)C_{14} - \omega_{12}(e_2)C_{23} = 0. \tag{62}$$

On the other hand, for $i = 2$ equation (45) implies

$$\omega_{34}(e_2)C_{14} + \omega_{12}(e_2)C_{23} = 0. \tag{63}$$

Thus, considering (56) the solution of equations (62) and (63) gives $C_{14} = C_{23} = 0$ which is a contradiction. That is, $c_0 = 0$, and thus $x = 0$. Therefore M is an open part of a timelike zw–plane.

Case 2. $h^3_{22} = -h^3_{11} \neq 0$, that is, M is a pseudo–umbilical timelike surface in $\mathbb{E}^4_1$. Now we will show that M has no pointwise 1–type Gauss map of the second kind. Note that for this case $h^4_{12} \neq 0$. If it were zero, then M would be a cone obtained in Case 1 which is not pseudo–umbilical. Similarly, considering (42) and using the Codazzi equation (20) we obtain the Laplacian of the Gauss map ν as

$$\Delta\nu = ||h||^2\nu + 2h^4_{12}\omega_{34}(e_2)e_1 \wedge e_4 + 2h^3_{11}\omega_{34}(e_2)e_2 \wedge e_3. \tag{64}$$

Suppose that M has pointwise 1–type Gauss map of the second kind. Thus, (1) is satisfied for some function $f \neq 0$ and nonzero constant vector C. From (1), (43) and (64) we have

$$f(1 + C_{34}) = \|h\|^2, \tag{65}$$

$$fC_{14} = -\,2h^4_{12}\omega_{34}(e_2), \tag{66}$$

$$fC_{23} = 2h^3_{11}\omega_{34}(e_2), \tag{67}$$

$$C_{12} = C_{13} = C_{24} = 0. \tag{68}$$

From (66) and (67) it is seen that $C_{14} \neq 0$ and $C_{23} \neq 0$. Equations (66) and (67) imply that

$$h^3_{11}C_{14} + h^4_{12}C_{23} = 0. \tag{69}$$

From (44) for $i = 1$, we also obtain that

$$h^4_{12}C_{14} - h^3_{11}C_{23} = 0. \tag{70}$$

Hence, equations (69) and (70) give that $h^4_{12} = h^3_{11} = 0$, $(h^3_{22} = 0)$, that is, M is an open part of the timelike zw–plane.

From Remark 1, the converse of the proof is trivial.

Corollary 11. *There exists no a non–planar timelike rotational surface in* $\mathbb{E}^4_1$ *defined by* (10) *with flat normal bundle and pointwise 1–type Gauss map of the second kind.*

Using Proposition 3.2 in [6], we get the following:

Corollary 12. *A timelike rotational surface with zero mean curvature and non–flat normal bundle in* $\mathbb{E}^4_1$ *defined by* (10) *has no pointwise 1–type Gauss map of the second kind.*

4. Rotational surfaces in $\mathbb{E}^4_2$ with pointwise 1–type Gauss map

In this section, we determine the rotational surfaces in the pseudo–Euclidean space $\mathbb{E}^4_2$ defined by (22) and (23) with pointwise 1–type Gauss map.

Theorem 13. *Let $M_1(b)$ be a non–planar regular rotational surface with zero mean curvature in $\mathbb{E}^4_2$ defined by (22). Then,*

i. for some constants $\lambda_0 \neq 0$ and μ_0, the regular surface $M_1(1)$ with the profile curve α whose components satisfy

$$(w(s) + y(s))^2 + \lambda_0(w(s) - y(s))^2 = \mu_0 \tag{71}$$

has pointwise 1–type Gauss map of the second kind.

ii. for $b \neq 1$, the timelike surface $M_1(b)$ has pointwise 1–type Gauss map of the second kind if and only if the profile curve α is given by $y(s) = b_0(w(s))^{\pm b}$ for some constant $b_0 \neq 0$.

Proof. Assume that $M_1(b)$ is a non–planar regular rotational surface with zero mean curvature in $\mathbb{E}^4_2$ defined by (22). From equation (42), the Laplacian of the Gauss map of the rotational surface $M_1(b)$ is given by

$$\begin{aligned}\Delta\nu =&||h||^2\nu + 2h^4_{12}(\varepsilon^* h^3_{22} - \varepsilon h^3_{11})e_1 \wedge e_2 \\ &+ \omega_{34}(e_1)(\varepsilon h^3_{11} + \varepsilon^* h^3_{22})e_1 \wedge e_3 + (\varepsilon\varepsilon^* e_2(h^3_{11}) + e_2(h^3_{22}))e_2 \wedge e_4.\end{aligned} \tag{72}$$

Since the mean curvature of $M_1(b)$ is zero, equation (72) becomes

$$\Delta\nu = ||h||^2\nu - 4\varepsilon h^3_{11}h^4_{12}e_1 \wedge e_2. \tag{73}$$

Suppose that $M_1(b)$ has pointwise 1–type Gauss map of second kind. Comparing (1) and (73), we get

$$f(1 + \varepsilon\varepsilon^* C_{34}) = ||h||^2, \tag{74}$$

$$fC_{12} = -4\varepsilon^* h^3_{11}h^4_{12}, \tag{75}$$

$$C_{13} = C_{14} = C_{23} = C_{24} = 0. \tag{76}$$

For $i = 1, 2$, from (45) and (46), we have

$$h^3_{11}C_{12} + h^4_{12}C_{34} = 0, \tag{77}$$

$$h^4_{12}C_{12} + h^3_{11}C_{34} = 0. \tag{78}$$

Since the Gauss map ν is of the second kind, equations (77) and (78) must have non–zero solution which implies $(h^3_{11})^2 - (h^4_{12})^2 = 0$. Considering the first equations in (27) and (28) we have $(b^2-1)(b^2y^2(s)w'^2(s)-w^2(s)y'^2(s))=0$, that is, $b=1$ or $b^2y^2(s)w'^2(s)-w^2(s)y'^2(s)=0$.

If $b=1$, it was shown that the components of the profile curve α of the surface $M_1(1)$ with zero mean curvature satisfy equation (71), [2]. In this case, from (27) and (28) it can be seen easily that $h^4_{12}=-\varepsilon\varepsilon^* h^3_{11}$. Hence, by using equations (74), (75) and (77), we find $C_{12}=-\frac{1}{2}$, $C_{34}=-\frac{\varepsilon\varepsilon^*}{2}$ and $f=-8\varepsilon(h^3_{22})^2$. Since α is a plane curve, $h^3_{22}=\kappa$, where κ is a curvature of the curve α. Thus, the Gauss map ν of $M_1(1)$ satisfies (1) for the function $f=-8\varepsilon\kappa^2$ and the constant vector $C=-\frac{\varepsilon\varepsilon^*}{2}e_1\wedge e_2-\frac{1}{2}e_3\wedge e_4$. This completes the proof of (a).

If $b^2y^2(s)w'^2(s)-w^2(s)y'^2(s)=0$ and $b\neq 1$, then we have $y(s)=b_0(w(s))^{\pm b}$, where b_0 is non–zero constant. Also, the rotational surface $M_1(b)$ with this profile curve α is timelike, i.e., $\varepsilon\varepsilon^*=-1$. Hence, from the first equations in (27) and (28), we get $h^4_{12}=\pm h^3_{11}$. By using equations (74), (75) and (77), we get the function $f=-8\varepsilon\kappa^2$ and the constant vector $C=\pm\frac{1}{2}e_1\wedge e_2-\frac{1}{2}e_3\wedge e_4$.

The converse of the proof is followed from a direct calculation. This completes the proof of (b).

Similarly, we can state the following theorem for the rotational surface $M_2(b)$ defined by (23) in the pseudo–Euclidean space $\mathbb{E}^4_2$.

Theorem 14. *Let $M_2(b)$ be a non–planar regular rotational surface with zero mean curvature in $\mathbb{E}^4_2$ defined by* (23)*. Then,*

i. for some constants $\lambda_0\neq 0$ and μ_0, the regular surface $M_2(1)$ with the profile curve β whose components satisfy

$$(x(s)+z(s))^2+\lambda_0(x(s)-z(s))^2=\mu_0 \tag{79}$$

has pointwise 1–type Gauss map of the second kind.

ii. for $b\neq 1$, the spacelike surface $M_2(b)$ has pointwise 1–type Gauss map of the second kind if and only if the profile curve β is given by $z(s)=\bar{b}_0(x(s))^{\pm b}$ for some constant $\bar{b}_0\neq 0$.

Note that considering equation (73), if the Gauss map ν of the rotational surface $M_1(b)$ and $M_2(b)$ were of the first kind which implies that $h^3_{11}=0$ or $h^4_{12}=0$, then $M_1(b)$ and $M_2(b)$ would be lying in 3–dimensional pseudo–Euclidean space.

Corollary 15. *A rotational surface in the pseudo–Euclidean space $\mathbb{E}^4_2$ defined by* (22) *or* (23) *with zero mean curvature has no pointwise 1–type Gauss map of the first kind.*

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