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NON-ARCHIMEDIAN STABILITY OF GENERALIZED JENSEN'S AND QUADRATIC EQUATIONS

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Abstract. We use the operatorial approach to provide a proof of the Hyers-Ulam stability for the equations

$$\begin{aligned} \sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) &= Nf(x),\ x,y\in E, \\ \sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) &= Nf(x)+Nf(y),\ x,y\in E, \end{aligned}$$

where E is a normed space, F is a non-Archimedean Banach space, Φ is a finite group of automorphisms of E, $N=|\Phi|$ designates the number of its elements, and $\{a_\lambda, \lambda\in\Phi\}$ are arbitrary elements of E. These equations provides a common generalization of many functional equations such as Cauchy's, Φ- Jensens's, Φ-quadratic, Łukasik's equation. Some applications of our results will be illustrated.

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1. Introduction

In [50], Ulam posed the question of the stability of Cauchy's equation: If a function f approximately satisfies Cauchy's functional equation $f(x+y)=f(x)+f(y)$ when does it has an exact solution which f approximates. The problem has been considered for various equations, also for mappings with many different types of domains and ranges by a number of authors including Hyers [22, 23], Aoki [2], T. M. Rassias [41], J.M. Rassias [39, 40], Gajda [19] Gàvrutà [20] and others. For definitions, approaches, and results on Hyers-Ulam-Rassias stability we refer the reader to, e.g., ([18],[24],[29],[31],[43],[44],[51]-[53]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1}$$

is called a quadratic functional equation. The first stability theorem for the Eq. (1) was proved by Skof [46] for mappings f from a normed space X into a Banach space Y. Cholewa [12] extended Skof's theorem by replacing X by an abelian group G. Skof's result was later generalized by Czerwik [14] in the spirit of Hyers-Ulam-Rassias. Since then, a number of stability results have been obtained for quadratic functional equations and Jensen's functional equation ([1],[4],[6]-[10],[26]-[28],[33],[38]). Informations and applications about the Eq. (1) and its further generalizations can be found e.g. in ([13],[14],[17],[32],[42],[45],[47]-[49]).

The stability problem for the functional equation

$$\frac{1}{|\Phi|}\sum_{\lambda\in\Lambda} f(x+\lambda y)=f(x)+g(y),\ x,y\in X, \tag{2}$$

where X is an abelian group, Φ is is a finite subgroup of the automorphism group of X and $f,g:X\to\mathbb{C}$ was posed and solved by Badora in [4]. Equation (2) is a joint generalization of Cauchy's functional equation ($\Phi=\{id\}$, $g=f$), Jensen's equation ($\Phi=\{id,-id\}$, $g=0$) and the quadratic equation ($\Phi=\{id,-id\}$, $g=f$). This result was published (with a different proof and $h=f$) by Ait Sibaha et al. in [1] and generalized by Charifi et al. in ([6],[7]).

In [10], the authors gave an explicit description of the solutions $f:S\to H$ each of the following generalized equations

$$\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda)=Nf(x),\ x,y\in S, \tag{3}$$

$$\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda)=Nf(x)+Nf(y),\ x,y\in S, \tag{4}$$

where S is an abelian monoid, H is an abelian group and Φ is a finite subgroup of authomorphisms of S, and $f,g:X\to H$, which covers the functional equations

$$f(x+y+a)=f(x)+f(y),\ x,y\in S, \qquad \Phi=\{id\} \tag{5}$$

$$f(x+y+a)+f(x+\sigma(y)+b)=2f(x),\ x,y\in S, \qquad \Phi=\{id,\sigma\} \tag{6}$$

$$f(x+y+a)+f(x+\sigma(y)+b)=2f(x)+2f(y),\ x,y\in S, \quad \Phi=\{id,\sigma\} \tag{7}$$

where a, b are fixed elements of S and σ is an involution of S i. e. $\sigma(x+y) = \sigma(y) + \sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x \in S$.

In 1897, Hensel [21] has introduced a normed space which does not have the Archimedean property. Let p be a fixed prime number and x be a non-zero rational number, there exists a unique integer $v_p(x) \in \mathbb{Z}$ such that $x = p^{v_p(x)}\frac{a}{b}$ where a and b are integers co-prime to p. The function defined in $\mathbb{Q}$ by $|x|_p = p^{-v_p(x)}, x \in \mathbb{Q}$ is called a p-adic, a ultrametric or simply a non-Archimedean absolute value on $\mathbb{Q}$. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|.| : \mathbb{K} \to [0, +\infty)$, called a non-Archimedean absolute value on $\mathbb{K}$ and satisfying the following conditions:

(i) $|x| = 0 \Leftrightarrow x = 0$, $x \in \mathbb{K}$,

(ii) $|xy| = |x| |y|$, $x, y \in \mathbb{K}$,

(iii) $|x+y| \leq \max(|x|, |y|)$, $x, y \in \mathbb{K}$.

We assume, throughout this paper that this value absolute is non-trivial i.e., there exists an element k of $\mathbb{K}$ such that, $|k| \neq 0, 1$.

Definition 1. *By a non-Archimedean vector space, we mean a vector space E over a non-Archimedean field $\mathbb{K}$ equipped with a function $\|.\| : E \to [0, +\infty)$ called a non-Archimedean norm on E and satisfying the following properties:*

(i) $\|x\| = 0 \Leftrightarrow x = 0$, $x \in E$,

(ii) $\|kx\| = |k| \|x\|$, $(k, x) \in \mathbb{K} \times E$,

(iii) $\|x+y\| \leq \max(\|x\|, \|y\|)$, $x, y \in E$.

Due to the fact that

$$\|x_m - x_n\| \leq \max\{\|x_j - x_{j-1}\|\}, n+1 \leq j \leq m \quad , m > n,$$

a sequence $(x_n)_n$ is Cauchy if and only if $(x_{n+1} - x_n)_n$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all x and $y > 0$, there exists an integer n such that $x < ny$.

In [3], Arriola and Beyer initiated the stability of Cauchy's functional equation over p-adic fields. Moslehian and T.M. Rassias [37] proved the Hyers Ulam Rassias stability of Cauchy's functional and the quadratic functional equations in non-Archimedean normed space. For various aspects of the theory of stability in non-Archimedean normed space we can refer to ([8],[9],[16],[36],[37]).

Let $\mathbb{K}$ be an ultrametric field of characteristic zero, E be a $\mathbb{K}$-vector space and F be a complete ultrametric $\mathbb{K}$-vector space (in particular in the field of p-adic numbers).

As continuation of some previous works, the purpose of the present paper is to prove the Hyers–Ulam stability of the functional equations (3) and (4) for mappings f from a normed space E into a non-Archimedean Banach space F.

2. Preliminaries

To formulate our results we introduce the following notation and assumptions that will be used throughout the paper:

Let $\mathbb{K}$ be an ultrametric field of characteristic zero (in particular in the field of p-adic numbers), E be a $\mathbb{K}$-vector space, F be a complete ultrametric $\mathbb{K}$-vector space and let F^E denotes the vector space consisting of all maps from E into F. We let Φ denotes a finite group of automorphisms of E, N designates the number of its elements and $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of E.

We now recall the definition and some necessary notions of multi-additive mappings, using the sequel.

A function $\mathcal{A} : E \to F$ is additive if $\mathcal{A}(x+y) = \mathcal{A}(x) + \mathcal{A}(y)$ for all $x, y \in E$.

Let $k \in \mathbb{N}$, be a function $\mathcal{A}_k : E^k \to F$ is k-additive if it is additive in each variable, in addition we say that $\mathcal{A}_k$ is symmetric if it satisfies $\mathcal{A}_k(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(k)}) = \mathcal{A}_k(x_1, x_2, ..., x_k)$ for all $(x_1, x_2, ..., x_k) \in E^k$ and all permutations π of k elements. Some informations concerning on such mappings can be found for instance in [31].

Let $\mathcal{A}_k : E^k \to F$ be a k-additive and symmetric function and let $\mathcal{A}_k^* : E \to F$ defined by $\mathcal{A}_k^*(x) = \mathcal{A}(x, x, ..., x)$ for all $x \in E$. Such a function $\mathcal{A}_k^*$ will be called a monomial function of degree k (if $\mathcal{A}_k^* \neq 0$). We note that it is easily seen that $\mathcal{A}_k^*(rx) = r^k \mathcal{A}_k^*(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$.

A function $P : E \to F$ is called a GP function (generalized polynomial function) of degree $m \in \mathbb{N}$ iff there exist $\mathcal{A}_0 \in E$ and symmetric k-additive functions $\mathcal{A}_k : E^k \to F$ (for $1 \leq k \leq m$) such that

$$\mathcal{A}_m^* \neq 0 \text{ and } P(x) = \mathcal{A}_0 + \sum_{k=1}^{m} \mathcal{A}_k^*(x) \text{ for all } x \in E.$$

For $h \in E$ we define the linear difference operator Δ_h on F^E by

$$\Delta_h(f)(x) = f(x+h) - f(x),$$

for all $f \in F^E$ and $x \in E$. Notice that these difference operators commute ($\Delta_h \Delta_{h'} = \Delta_{h'} \Delta_h$ for all $h, h' \in E$) and if $h \in E$, $n \in \mathbb{N}$ then Δ_h^n the n-th iterate of Δ_h satisfies

$$\Delta_h^n(f)(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh), \text{ for all } x, h \in E \text{ and } f \in F^E.$$

Now we note some results for later use.

Theorem 1. *[5] Let $n \in \mathbb{N}$, $f \in F^E$ and $\delta \in \mathbb{R}^+$. Then the following statements are equivalent.*

i) $\|\Delta_h^n f(x)\| \leq \delta$ *for all* $x, h \in E$.

ii) There is, up to a constant, a unique GP function P of degree at most $n-1$ such that $\|f(x) - f(0) - P(x)\| \leq \delta$ *for all* $x \in E$.

Theorem 2. *[9] Let $(S, +)$ be an abelian monoid, Φ be a finite subgroup of the group of automorphisms of S, $N = card(\Phi)$, $(H, +)$ be an abelian group uniquely divisible by $(N+1)!$ and $a_\lambda \in S$ $(\lambda \in \Phi)$. Then the function $f : S \to G$ is a solution of equation*

$$\sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \; x, y \in S, \tag{8}$$

if and only if f has the following form

$$f(x) = \mathcal{A}_0 + \sum_{i=1}^{N} \mathcal{A}_i^*(x), \; x \in S, \tag{9}$$

where $\mathcal{A}_0 \in G$ and $\mathcal{A}_k : S^k \to G$, $k \in \{1, 2, ..., N\}$ are symmetric and k-additive functions satisfying the two conditions:

i) $\sum_{i=max}^{N} \binom{i}{j} \binom{i-j}{k} \sum_{\lambda \in \Phi} \mathcal{A}_i(\underbrace{x, x, ..., x}_{k}, a_\lambda, ..., a_\lambda, \underbrace{\lambda y, \lambda y, .., \lambda y}_{j}) = 0, \; x, y \in S,$

$0 \leq k \leq N-1$, $0 \leq j \leq N-k$, $2 \leq max = max\{j+1, k+1, k+j\}$ *and*

ii) $\sum_{\lambda \in \Phi} \sum_{i=1}^{N} \mathcal{A}_i^*(a_\lambda) = N \mathcal{A}_0$.

Theorem 3. *[8] Let Φ be a finite subgroup of the group of automorphisms of E, $N = card(\Phi)$, $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of E and $f : E \to F$ satisfying the inequality*

$$\left\| \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) - N f(x) - \sum_{\lambda \in \Phi} f(\lambda y) \right\| \leq \delta,$$

for all $x, y \in E$. Then there exists a unique GP function $P : E \to F$ of degree at most N solution of the equation

$$\sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = N f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \; x, y \in E, \tag{10}$$

such that

$$\|f(x) - P(x)\| \leq \frac{\delta}{|N|} \; \text{for all } x \in E.$$

Lemma 4. *[8] Let Φ be a finite automorphism group of E, $N = card\Phi$, $\delta, \delta' \in \mathbb{R}^+$, $a_\lambda \in E$ $(\lambda \in \Phi)$, and $f \in F^E$ such that*

$$\left\| \sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) - Nf(x) - \sum_{\lambda\in\Phi} f(\lambda y) \right\| \leq \delta, \;\; x, y \in E. \tag{11}$$

Then, there exists a mapping $h \in F^E$ which satisfies

$$\left\| \Delta_y^N f(x) - h(y) \right\| \leq \frac{\delta}{|N|}, \;\; x, y \in E,$$

and

$$\left\| \Delta_y^{N+1} f(x) \right\| \leq \frac{\delta}{|N|}, \;\; x, y \in E. \tag{12}$$

Furthermore, if $\left\| \sum_{\lambda\in\Phi} (\lambda y) \right\| \leq \delta'$, $y \in E$, then $\left\| \Delta_y^N f(x) \right\| \leq max(\frac{\delta}{|N|}, \frac{\delta'}{|N|})$, $x, y \in E$.

In the next two theorems the solutions of the functional equations (3) and (4), respectively, will be expressed in terms of GP functions.

Theorem 5. *[10] Let $(S, +)$ be an abelian monoid, Φ be a finite subgroup of the group of automorphisms of S, $N = card(\Phi)$, $(H, +)$ be an abelian group uniquely divisible by $N!$ and $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of S. Then the function $f : S \to H$ is a solution of the equation (3) if and only if f has the following form*

$$f(x) = \mathcal{A}_0 + \sum_{i=1}^{N-1} \mathcal{A}_i^*(x), \;\; x \in S, \tag{13}$$

where $A_0 \in H$ and $\mathcal{A}_k : S^k \to H$, $k \in \{1, 2, ..., N-1\}$ are k-additive and symmetric functions which satisfy the following conditions

$$\sum_{i=max(k+j,k+1)}^{N-1} \binom{i}{k}\binom{i-k}{j} \sum_{\lambda\in\Phi} \mathcal{A}_i(\underbrace{x, ...x}_{k}, a_\lambda, ..., a_\lambda, \underbrace{\lambda y, ..., \lambda y}_{j}) = 0 \text{ for } x, y \in S,$$

$0 \leq k \leq N-2$, $0 \leq j \leq N-k-1$.

Theorem 6. *[10] Let $(S, +)$ be an abelian semigroup, Φ be a finite subgroup of the group of automorphisms of S, $N = card(\Phi)$, $(H, +)$ be an abelian group uniquely divisible by $(N+1)!$ and $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of S. Then the function $f : S \to H$ is a solution of the equation (4) if and only if f has the following form*

$$f(x) = \mathcal{A}_0 + \sum_{i=1}^{N} \mathcal{A}_i^*(x), \;\; x \in S, \tag{14}$$

where $\mathcal{A}_0 \in H$ *and* $\mathcal{A}_k : S^k \to H,\ k \in \{1, 2, ..., N\}$ *are symmetric and k-additive functions satisfying the three conditions:*

i) $\sum_{\lambda\in\Phi} \sum_{k=1}^{N} \mathcal{A}_i^*(a_\lambda) = N\mathcal{A}_0,$

ii) $\sum_{2\le i=max(k+j,k+1)}^{N} \binom{i}{k}\binom{i-k}{j} \sum_{\lambda\in\Phi} \mathcal{A}_i(\underbrace{x, ...x}_{k}, a_\lambda, ..., a_\lambda, \underbrace{\lambda y, ..., \lambda y}_{j}) = 0,\ x, y \in S,$ $1 \le k \le N-1,\ 0 \le j \le N-k$ *and*

iii) $\sum_{k=i}^{N} \binom{i}{k} \sum_{\lambda\in\Phi} \mathcal{A}_k(\underbrace{\lambda x, ..., \lambda x}_{i}, a_\lambda, ..., a_\lambda) = N\mathcal{A}_i^*(x),\ x \in S,\ 1 \le i \le N.$

3. Main results

The following lemma will be used in the proof of our main results namely Theorems 8 and 11.

Lemma 7. *Let* $\mathbb{K}$ *be an ultrametric field of characteristic zero and* $\overline{\mathbb{K}}$ *its completion, F be a complete ultrametric* $\mathbb{K}$*-vector space,* $\delta \in \mathbb{R}^+$ *and P be a polynomial function of degree n,* $n \ge 1$*, with rational variable and with coefficients in F. Suppose that*

$$\|P(z)\| \le \delta \text{ for all } z \in \mathbb{Q}. \tag{15}$$

Then, there exists a prime number p such that $\mathbb{Q}_p \subset \overline{\mathbb{K}}$ *and*

$$P(z) = P(0) \text{ for all } z \in \mathbb{Q}_p,$$

i.e. all non-constant coefficients of P are zero.

Proof. There exist $a_0, a_1, ..., a_n \in F$ such that

$$P(z) = \sum_{i=0}^{n} a_i z^i,\ z \in \mathbb{Q}.$$

The theorem of Ostrowski shows that there exists a prime number p for which $\mathbb{Q}_p \subset \overline{\mathbb{K}}$. An extension by continuity of the external law of F from $\mathbb{K}$ to $\overline{\mathbb{K}}$ allows us to write,

$$\|P(z)\| \le \delta \text{ for } z \in \mathbb{Q}_p.$$

Let $\varphi : F \to \mathbb{Q}_p$ be a continuous $\mathbb{Q}_p$-linear functional. Taking into account the previous inequality we have for all $z \in \mathbb{Q}_p$:

$$\|\varphi(P(z))\| \le \delta \|\varphi\| \text{ for } z \in \mathbb{Q}_p,$$

wich means that
$$\left\|\sum_{i=0}^{n}\varphi(a_i)z^i\right\| \leq \delta\,\|\varphi\| \ \text{ for } z\in\mathbb{Q}_p.$$
It results, since a polynomial function is bounded if and only if it is constant, that $\varphi(a_i)=0$ for $1\leq i\leq n$ and for any continuous $\mathbb{Q}_p$-linear functional $\varphi: F\to\mathbb{Q}_p$. Thus ultrametric version Hahn Banach Theorem gives $a_i=0,\ 1\leq i\leq n$ i.e. $P(z)=P(0)$ for all $z\in\mathbb{Q}_p$.

In the following theorem, using the operatorial approach we obtain the non-Archimedean stability in the sense of Hyers-Ulam of the generalised Φ-Jensen functional equation.

Theorem 8. *Assume that Φ is a finite subgroup of the group of automorphisms of E, $N=card(\Phi)$, $\{a_\lambda,\lambda\in\Phi\}$ are arbitrary elements of E and $f:E\to F$ satisfying the following inequality:*
$$\left\|\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda)-Nf(x)\right\|\leq\delta, \tag{16}$$
for all $x,y\in E$. Then there exists, up to a constant, a unique GP function $P:E\to F$ solution of the equation (3), of degree at most $N-1$, such that
$$\|f(x)-f(0)-P(x)\|\leq\frac{\delta}{|N|^2}\ \text{ for all } x\in E.$$

Proof. Suppose that f satisfies the inequality (16). Letting $y=0$ and $x=0$ in (16), respectively, we get
$$\left\|\sum_{\lambda\in\Phi} f(x+a_\lambda)-Nf(x)\right\|\leq\delta,\quad x\in E,$$
and
$$\left\|\sum_{\lambda\in\Phi} f(\lambda y+a_\lambda)-Nf(0)\right\|\leq\delta,\quad y\in E.$$
By replacing, in the last inequality, y by μy we obtain
$$\begin{aligned}&\left\|N^2f(0)-N\sum_{\nu\in\Phi}f(\nu y)\right\|\\ &\qquad\leq\max\left\{\left\|N^2f(0)-\sum_{\mu\in\Phi}\sum_{\lambda\in\Phi}f(\mu\lambda y+a_\lambda)\right\|,\left\|\sum_{\nu\in\Phi}\sum_{\lambda\in\Phi}f(\nu y+a_\lambda)-N\sum_{\nu\in\Phi}f(\nu y)\right\|\right\}\\ &\qquad\leq\delta,\end{aligned}\tag{17}$$

for all $y \in E$. It follows, by taking $g := f - f(0)$ and the use of (16) and (17) that

$$\begin{aligned}
&\left\| \sum_{\lambda\in\Phi} g(x+\lambda y+a_\lambda) - Ng(x) - \sum_{\lambda\in\Phi} g(\lambda y) \right\| \\
= &\left\| \sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) - Nf(x) + Nf(0) - \sum_{\lambda\in\Phi} f(\lambda y) \right\| \\
\leq &\max\left\{ \left\| \sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) - Nf(x) \right\|, \left\| Nf(0) - \sum_{\lambda\in\Phi} f(\lambda y) \right\| \right\} \\
\leq &\frac{\delta}{|N|},
\end{aligned}$$

for all $x, y \in E$. In virtue of Theorem 3, there exists, in the class of function $g : E \to F$ with $g(0) = 0$, a GP function P of degree at most N solution of the functional equation

$$\sum_{\lambda\in\Phi} g(x+\lambda y+a_\lambda) = Ng(x) + \sum_{\lambda\in\Phi} g(\lambda y) \tag{18}$$

such that

$$\|g(x) - P(x)\| \leq \frac{\delta}{|N|^2} \text{ for all } x \in E. \tag{19}$$

According to Theorem 2, $P(x) = \sum_{i=1}^{N} \mathcal{A}_i^*(x)$ with

$$\sum_{\lambda\in\Phi}\sum_{i=1}^{N} \mathcal{A}_i^*(a_\lambda) = 0 \tag{20}$$

and

$$\sum_{i=max}^{N} \binom{i}{k}\binom{i-k}{j} \sum_{\lambda\in\Phi} \mathcal{A}_i(\underbrace{x, ..., x}_{k}, a_\lambda, ..., a_\lambda, \underbrace{\lambda y, ..., \lambda y}_{j}) = 0 \tag{21}$$

for all $x, y \in E$, $0 \leq k \leq N-1$, $0 \leq j \leq N-k$ and $2 \leq max = max(k+1, j+1, k+j)$. In addition by (17),

$$\begin{aligned}
\left\| \sum_{\lambda\in\Phi} P(\lambda y) \right\| &\leq \max\left\{ \left\| \sum_{\lambda\in\Phi} (P(\lambda y) - g(\lambda y)) \right\|, \left\| \sum_{\lambda\in\Phi} g(\lambda y) \right\| \right\} \\
&\leq \frac{\delta}{|N|^2}
\end{aligned}$$

for all $y \in E$. In view of Lemma 4, Theorem 1 and Lemma 7, we have

$$\mathcal{A}_N = 0 \tag{22}$$

and by Lemma 7,

$$\sum_{\lambda\in\Phi}\mathcal{A}_i^*(\lambda y)=0,\ y\in E,\ 1\le i\le N-1. \tag{23}$$

Taking into account of (20) (21), (22) and (23) we get

$$\sum_{i=max(k+j,k+1)}^{N-1}\binom{i}{k}\binom{i-k}{j}\sum_{\lambda\in\Phi}\mathcal{A}_i(\underbrace{x,...,x}_{k},a_\lambda,...,a_\lambda,\underbrace{\lambda y,...,\lambda y}_{j})=0,\ x,y\in E,$$

$0\le k\le N-2,\ 0\le j\le N-k-1$. This shows, using Theorem 5, that P is a solution of the Eq. (3).
The uniqueness is giving by Lemma 7. In fact, let Q be another GP function of degree at most $N-1$, solution of Eq. (3) and satisfying the inequality (19) then we get

$$\begin{aligned}\|P(x)-Q(x)\| &\le max(\|P(x)-g(x)\|,\|g(x)-Q(x)\|)\\ &\le \frac{\delta}{|N|^2},\ x\in E.\end{aligned}$$

According to Lemma 7 we get $P-Q$ is constant. This completes the proof.

Corollary 9. *Assume that a,b are arbitrary elements of E and $f:E\to F$ satisfying the following inequality:*

$$\|f(x+y+a)+f(x+\sigma(y)+b)-2f(x)\|\le\delta, \tag{24}$$

for all $x,y\in E$. Then there exists, up to a constant, a unique GP function $P:E\to F$ solution of the equation (6), of degree at most 1, such that

$$\|f(x)-f(0)-P(x)\|\le\frac{\delta}{|4|}\ \ \textit{for all } x\in E.$$

Proof. The proof follows on putting $\Phi=\{I,\sigma\}$ in Theorem 8.

Corollary 10. *Let p be a prime number, $\mathbb{C}_p=\mathbb{Q}_p+i\mathbb{Q}_p$, $(i^2=-1)$, j be a primitive cube root of unity, a be a nonzero complex number and $f:\mathbb{C}_p\to\mathbb{C}_p$, be a continuous function satisfying the following inequality*

$$\left\|f(x+y+ja)+f(x+jy+j^2a)+f(x+j^2y+a)-3f(x)\right\|\le\delta,\ \ x,y\in\mathbb{C}_p, \tag{25}$$

for all $x,y\in\mathbb{C}_p$. Then there exists, up to a constant, a unique GP function $P:\mathbb{C}_p\to\mathbb{C}_p$ of degree at most 2, solution of the equation

$$f(x+y+ja)+f(x+jy+j^2a)+f(x+j^2y+a)=3f(x),\ x,y\in\mathbb{C}_p, \tag{26}$$

such that

$$\|f(x) - P(x)\| \leq \frac{\delta}{|9|},\ x \in E.$$

Now we investigate the non-Archimedean stability, in the sense of Hyers-Ulam, of the equation (4).

Theorem 11. *Assume that Φ is a finite subgroup of the group of automorphisms of E, $N = card(\Phi)$, $\{a_\lambda, \lambda \in \Phi\}$ are arbitrary elements of E and $f : E \to F$ satisfying the following inequality:*

$$\left\|\sum_{\lambda\in\Phi} f(x + \lambda y + a_\lambda) - Nf(x) - Nf(y)\right\| \leq \delta, \tag{27}$$

for all $x, y \in E$. Then there exists a unique GP function $P : E \to F$ solution of the equation (4), of degree at most N, such that

$$\|f(x) - P(x)\| \leq \frac{\delta}{|N|^2} \quad \textit{for } x \in E.$$

Proof. Suppose that f satisfies the inequality (27). Letting $x = y = 0$, $y = 0$ and $x = 0$, respectively, in (27) we obtain

$$\left\|\sum_{\lambda\in\Phi} f(a_\lambda) - 2Nf(0)\right\| \leq \delta,$$

$$\left\|\sum_{\lambda\in\Phi} f(x + a_\lambda) - Nf(x) - Nf(0)\right\| \leq \delta,$$

$$\left\|\sum_{\lambda\in\Phi} f(\lambda x + a_\lambda) - Nf(x) - Nf(0)\right\| \leq \delta,$$

for all $x, y \in E$. Taking into account the above inequalities and (27) we get that

$$\begin{aligned}
&\left\|N^2 f(x) + N\sum_{\mu\in\Phi} f(\mu y) - N^2 f(0) - N\sum_{\nu\in\Phi} f(x + \nu y)\right\| \\
\leq\ & \max\left\{\left\|N^2 f(x) + N\sum_{\mu\in\Phi} f(\mu y) - \sum_{\lambda\in\Phi}\sum_{\mu\in\Phi} f(x + \lambda\mu y + a_\lambda)\right\|,\right. \\
&\left.\left\|\sum_{\nu\in\Phi}\sum_{\lambda\in\Phi} f(x + \nu y + a_\lambda) - N^2 f(0) - N\sum_{\nu\in\Phi} f(x + \nu y)\right\|\right\} \\
\leq\ & \delta,
\end{aligned}$$

for all $x, y \in E$. With the notation $g := f - f(0)$ we can reformulate the previous inequality to

$$\left\| \sum_{\mu\in\Phi} g(x+\mu y) - Ng(x) - \sum_{\mu\in\Phi} g(\mu y) \right\| \leq \frac{\delta}{|N|},$$

for all $x, y \in E$. Theorem 3 shows that there exists a GP function $Q : E \to F$ of degree at most N solution of the equation

$$\sum_{\mu\in\Phi} g(x+\mu y) = Ng(x) + \sum_{\mu\in\Phi} g(\mu y),\ x, y \in E$$

such that

$$\|g(x) - Q(x)\| \leq \frac{\delta}{|N|^2},\ \mathrm{x} \in E. \tag{28}$$

Then there exist k-additive and symmetric functions $\mathcal{A}_k : E^k \to F$, $k \in \{1, 2, ..., N\}$ such that $Q(x) = \sum\limits_{i=1}^{N} \mathcal{A}_i^*(x),\ x \in E$ and we have

$$\sum_{\mu\in\Phi} Q(x+\mu y) = NQ(x) + \sum_{\mu\in\Phi} Q(\mu y),\ x, y \in E.$$

Let P be the GP function defined by

$$P(x) = Q(x) + \frac{1}{N} \sum_{\lambda\in\Phi} \sum_{i=1}^{N} \mathcal{A}_i^*(a_\lambda),\ x \in E,$$

so we have the following inequality

$$\begin{aligned} \|f(x) - P(x)\| &= \left\| g(x) - Q(x) - \frac{1}{N}\Big(\sum_{\lambda\in\Phi} f(a_\lambda) + 2Nf(0)\Big) \right\| \\ &\leq \max\Big(\frac{\delta}{|N|^2}, \frac{\delta}{|N|}\Big) \\ &\leq \frac{\delta}{|N|^2}, \end{aligned}$$

for all $x \in E$. To prove that P is a solution of the equation (4) we define the functions $I_P, J_P : E \times E \to F$ by the formulas

$$I_P(x, y) = \sum_{\nu\in\Phi} P(x+\nu y + a_\nu) - NP(x) - NP(y),\ x, y \in E$$

and

$$J_P(x,y) = I_P(x,y) - I_P(0,y),\ x,y \in E.$$

We have therefore

$$\begin{aligned} I_P(0,0) &= \sum_{\nu\in\Phi} P(a_\nu) - 2NP(0) \\ &= \left\{\sum_{\nu\in\Phi} Q(a_\nu) + \sum_{\nu\in\Phi}\sum_{i=1}^{N} A_i^*(a_\nu)\right\} - 2\left\{\sum_{\nu\in\Phi}\sum_{i=1}^{N} A_i^*(a_\nu)\right\} \\ &= 0. \end{aligned}$$

Furthermore we have,

$$\begin{aligned} \|I_P(x,y)\| &\le \max\left\{\left\|\sum_{\lambda\in\Phi} P(x+\lambda y+a_\lambda) - f(x+\lambda y+a_\lambda)\right\|,\right. \\ &\qquad \left. \|NP(x)-Nf(x)\|, \|NP(y)-Nf(y)\|, \delta\right\} \\ &\le \max(\frac{\delta}{|N|^2}, \delta) \\ &\le \frac{\delta}{|N|^2}, \end{aligned}$$

for all $x,y \in E$. Replacing P by its expression (as a GP function) in $I_P(0,y),\ I_P(x,y)$ we get, that for all $x,y \in E$

$$\begin{aligned} I_P(0,y) &= \sum_{\lambda\in\Phi} P(\lambda y + a_\lambda) - NP(0) - NP(y) \\ &= \sum_{\lambda\in\Phi}\sum_{i=1}^{N} \mathcal{A}_i^*(\lambda y + a_\lambda) - N\sum_{i=1}^{N} \mathcal{A}_i^*(y) - NP(0) \\ &= \sum_{i=1}^{N}\sum_{j=0}^{i} \binom{i}{j} \sum_{\lambda\in\Phi} \mathcal{A}_i(\underbrace{\lambda y, ..., \lambda y}_{j}, a_\lambda, ..., a_\lambda) - N\sum_{i=1}^{N} \mathcal{A}_i^*(y) - NP(0) \\ &= \sum_{j=1}^{N}\left(\sum_{i=j}^{N} \binom{i}{j} \sum_{\lambda\in\Phi} \mathcal{A}_i(\underbrace{\lambda y, ..., \lambda y}_{j}, a_\lambda, ..., a_\lambda) - N\mathcal{A}_j^*(y)\right) \end{aligned}$$

and

$$J_P(x,y) = \sum_{\lambda\in\Phi}\sum_{j=0}^{N-k}\sum_{k=1}^{N-1}\sum_{i=max(k+j,k+1)\le N} \left(\begin{array}{c} i \\ j \end{array}\right)\left(\begin{array}{c} i-j \\ k \end{array}\right) \mathcal{A}_i(\underbrace{x, ...x}_{k}, a_\lambda, ..., a_\lambda, \underbrace{\lambda y, ..., \lambda y}_{j}).$$

Making the substitution y by $Zy,\ Z \in \mathbb{Q}$ in $I_P(0, y)$ we obtain a polynomial function $R(Z)$ with rational variable and with coefficients in F,

$$R(Z) = \sum_{j=1}^{N} Z^j \left(\sum_{i=j}^{N} \binom{i}{j} \sum_{\lambda \in \Phi} \mathcal{A}_i(\underbrace{\lambda y, ..., \lambda y}_{j}, a_\lambda, ..., a_\lambda) - N \mathcal{A}_j^*(y) \right),\ y \in E,\ Z \in \mathbb{Q}. \tag{29}$$

It satisfies

$$\|R(Z)\| \leq \frac{\delta}{|N|^2},\ Z \in \mathbb{Q}.$$

In view of Lemma 7, $R(Z) = 0,\ Z \in \mathbb{Q}_p$. Consequently $J_P(x, y) = I_P(x, y),\ x, y \in E$. In addition, a similar reasoning, making the substitution x by $Zx,\ Z \in \mathbb{Q}$ in $J_P(x, y)$, we can show that $I_P(x, y) = 0,\ x, y \in E$ which means that (p, q) is a solution of the equation (4).

It is left to prove the uniqueness statement. Let T be another GP function of degree at most N, solution of the Eq. (4) such that

$$\|g(x) - T(x)\| \leq \frac{\delta}{|N|^2},\ x \in E. \tag{30}$$

From (28) and (30) we infer that we have

$$\begin{aligned} \|P(x) - T(x)\| &= \|P(x) - g(x) + g(x) - T(x)\| \\ &\leq \max\{\|P(x) - g(x)\|, \|g(x) - T(x)\|\} \\ &\leq \frac{\delta}{|N|^2}, \end{aligned}$$

for all $x \in E$. So, by Lemma 7 we conclude that $T - P$ is a constant, and by the fact that T and P are solution of the Eq. (4) we get $T = P$. This completes the proof of Theorem 11.

Corollary 12. *Assume that a, b are arbitrary elements of E and $f : E \to F$ satisfying the following inequality:*

$$\|f(x + y + a) + f(x + \sigma(y) + b) - 2f(x) - 2f(y)\| \leq \delta, \tag{31}$$

for all $x, y \in E$. Then there exists a unique GP function $P : E \to F$ solution of the equation (7), of degree at most 2, such that

$$\|f(x) - P(x)\| \leq \frac{\delta}{|4|} \quad \text{for all } x \in E.$$

Proof. The proof follows on putting $\Phi = \{I, \sigma\}$ in Theorem 11.

Corollary 13. *Let w be a primitive N^{th} root of unity, $N \geq 2$, let a be a complex constant, p be a prime number, $\mathbb{C}_p = \mathbb{Q}_p + i\mathbb{Q}_p$, $i^2 = -1$ and $f : \mathbb{C}_p \to \mathbb{C}_p$ be a continuous function satisfying the inequality*

$$\left\| \sum_{n=0}^{N-1} f(x + w^n y + \overline{w}^{n+1} a) - Nf(x) - Nf(y) \right\| \leq \delta, \quad x, y \in \mathbb{C}_p.$$

Then there exist a unique GP function $P : \mathbb{C}_p \to \mathbb{C}_p$, of degree at most N, solution of the equation,

$$\sum_{n=0}^{N-1} f(x + w^n y + \overline{w}^{n+1} a) = Nf(x) + Nf(y), \quad x, y \in \mathbb{C}_p,$$

such that

$$\|f(z) - P(z)\| \leq \frac{\delta}{|N|^2}, \quad z \in \mathbb{C}_p.$$

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