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EXACT SOLUTION TO TIME FRACTIONAL FIFTH-ORDER KORTEWEG-DE VRIES EQUATION BY USING (G'/G)-EXPANSION METHOD

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Abstract. In this paper, we have established the (G'/G)-expansion method to find exact solutions for to time fractional fifth-order Korteweg-de Vries equation (FKdV5). This method is an effective method in finding exact traveling wave solutions of nonlinear evolution equations (NLEEs) in mathematical physics.

The effectiveness of this manageable method has been shown by applying it to several particular cases of FKdV5. The present approach has the potential to be applied to other nonlinear fractional differential equations. All of the numerical calculations in the present study have been performed on a PC applying some programs written in *Mathematica*.

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1. Introduction

Nonlinear phenomena have very important roles in applied mathematics and physics. The accurate calculation of numerical solutions, in particular the traveling wave solutions of nonlinear equations in mathematical physics, has a significant role in soliton theory [1, 2].

There have been many powerful methods proposed for finding exact solutions of NLEEs, such as ansatz method and topological solitons [3], the tanh-function method [4], simplest equation method [5], the homogeneous balance method [6], the F-expansion method [7], Hirotas direct method [8], the exp-function method [9], the Adomian decomposition method [10], the extended tanh-function method [11], the auxiliary equation method [12], the Jacobi elliptic function method [13], the Weierstrass elliptic function method [14], the modified exp-function method [15], the modified simple equation method [16], and so on. A method called the

(G'/G)-expansion method was introduced by Wang et al. [17] to obtain traveling wave solutions of the nonlinear partial differential equations. Following that, this method was used by many researchers to construct the traveling wave solutions of the Nonlinear Evolution Equations (NLEEs). For example, Ebadi and Biswas [18] applied the same method to find traveling wave solutions for nonlinear diffusion equations with nonlinear source term. Also, Zayed [19], studied the various aspects of the higher dimensional NLEEs by using the same method in order to obtain solutions.

Numerous other researchers investigated the applicability of the proposed methods in different areas. Naher et al. [20], for instance, investigated the higher order Caudrey-Dodd-Gibbon equation through applying (G'/G)-expansion method to construct traveling wave solutions. Further, Liu et al. [21] implemented the same method to the NLEEs to get exact solutions. The method has also been applied by Feng et al. [22] in seeking traveling wave solutions of the Kolmogorov-Petrovskii-Piskunov equation. In Ref. [23], Zhang applied this method for the complex KdV equation to construct analytical solutions. Moreover, Zayed and Al-Joudi [24] extended the application of the method by using it to obtain traveling wave solutions of nonlinear partial differential equations in mathematical physics. Ayhan and Bekir [25] investigated various aspects of nonlinear lattice equations. Finally, Ozis and Aslan [26] surveyed the Kawahara type equations for the purpose of finding solutions via this method.

We have established (G'/G)-expansion method to obtain exact solutions for fractional partial differential equations (FPDEs) in the sense of conformable derivative as defined by Khalil, R., et al. [27] .

In this work, we have limited our focus of attention to the study of the following form of the time fractional fifth-order Korteweg-de Vries equation (FKdV5)

$$D_t^{\alpha} u + p\, u\, u_{xxx} + q\, u_x u_{xx} + r\, u^2\, u_x + u_{xxxxx} = 0, \quad 0 < \alpha \leq 1 \tag{1}$$

with four constant parameters p, q, r, s (for $\alpha = 1$, see [28]).

The present paper is arranged in five sections. In Section 1, a brief introduction is given. In Section 2, we briefly describe the modified conformable derivative with properties. In Section 3, the main steps of the (G'/G)-expansion method are presented. In Section 4, this method is used to find solutions to time fractional standard Lax equation, time fractional Sawada-Kotera (SK) equation, time fractional Sawada-Kotera-Parker-Dye (SKPD) equation, time fractional Caudrey-Dodd-Gibbon (CDG) equation, time fractional Kaup-Kupershmidt (KK) equation, time fractional Kaup-Kupershmidt-Parker-Dye (KKPD) equation, time fractional Ito equation. Finally, a conclusion is given in section 5.

2. Conformable fractional derivative

The expression [27] below is used to define the Modified conformable fractional derivative

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x) = \lim_{\epsilon \longrightarrow 0} \frac{f(x+\epsilon x^{1-\alpha}) - f(x)}{\epsilon}, \quad 0 < \alpha \leq 1, \tag{2}$$

in which, $f : [0, \infty) \longrightarrow \mathbb{R}$ and $x > 0$.
For $0 < \alpha \leq 1$, some properties for the suggested modified conformable fractional derivative given in [27] are as follows

$$\frac{\partial^\alpha}{\partial t^\alpha} x^\gamma = \gamma\, x^{\gamma-\alpha}, \quad \gamma \in \mathbb{R} \tag{3}$$

$$(u(x)v(x))^{(\alpha)} = u^{(\alpha)}(x)\, v(x) + u(x)\, v^{(\alpha)}(x) \tag{4}$$

$$(f[u(x)])^{(\alpha)} = f'_u(u)\, u^{(\alpha)}(x). \tag{5}$$

The above equations have a significant role in fractional calculus as it is shown in the following sections.

3. The methodology

Following the introduction above, we have presented the main steps of the fractional (G^α/G)-expansion method as follows.
Step 1. Suppose that a nonlinear FDEs, say in two independent variables x and t, is given by

$$P\left(D_t^\alpha u,\ u, u_x, u_{xx}, \ldots\right) = 0, \quad 0 < \alpha \leq 1. \tag{6}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in u and their various partial derivatives including fractional time and space derivatives.
Step 2. To obtain the solution of Eq.(6), we introduce the variable transformation

$$u(x,t) = y(\zeta), \quad \zeta = x - \left(\frac{c}{\alpha}\right) t^\alpha, \tag{7}$$

where c is constant to be determined later. Using Eq.(7) changes the Eq. (6) to an ODE

$$Q\left(y,\ \frac{\partial y}{\partial \zeta},\ \frac{\partial^2 y}{\partial \zeta^2},\ \frac{\partial^3 y}{\partial \zeta^3}, \ldots\right) = 0, \quad 0 < \alpha \leq 1, \tag{8}$$

in which $y = y(\zeta)$ is an unknown function, Q is a polynomial in the variable y and its derivatives.

Step 3. Suppose that the solution of (8) can be expressed by a polynomial in (G'/G) as follows:

$$y(\zeta)=\sum_{n=1}^{N} a_i\left(G'/G\right)^i, \tag{9}$$

in which the coefficients $a_i,\ i=1,\ldots,N$, are constants to be determined later with $a_N\neq 0$, and (G'/G) are the functions that satisfy some ordinary differential equations.
In this paper, we use the ordinary differential equations

$$G''(\zeta)+\lambda G'(\zeta)+\mu G(\zeta)=0, \tag{10}$$

where λ and μ are unknown constants.
By using (10) repeatedly, we can express (G'/G) in term of series in (G'/G).
By the General solutions of Eq.(1), we have

$$\frac{G'(\zeta)}{G(\zeta)}=\begin{cases} -\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta+B\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta}{A\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta+B\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta}\right), & \lambda^2-4\mu>0,\\ -\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-A\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta+B\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta}{A\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta+B\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta}\right), & \lambda^2-4\mu<0,\\ -\frac{\lambda}{2}+\frac{B}{A+B\zeta}, & \lambda^2-4\mu=0. \end{cases} \tag{11}$$

Step 4. If we substitute (9) into (8) and using (10), and calculate all terms with the same order of (G'/G) together, the left-hand side of (9) is converted into another polynomial in (G'/G). After solving the equation system and using (11), a variety of exact solutions can be constructed for Eq.(6).
Equating each coefficient of this polynomial to zero results in a set of algebraic equations for $\mu,\ \lambda,\ a_i(i=0,1,2,\ldots,N)$.
Remark. We define the degree of $y(\zeta)$ as $D\left[y(\zeta)\right]=N$, which results in the degrees of other expressions as $D\left[y^{(n)}\right]=N+n$, $D\left[y^m\left(y^{(n)}\right)^s\right]=Nm+(n+N)s$. By balancing the highest order derivative terms and the non linear terms in (8), we can find the parameter N

i) If N is a positive integer, we suppose that the solution to system (8) has the form (9).

ii) If $N=\frac{n}{m}$, we introduce the transformation in the following forms: $y(\zeta)=V^{\frac{m}{n}}(\zeta)$ and then return to step 1 and determine the parameter N.

iii) If N is a negative integer, we make the following transformation: $y(\zeta)=V^N(\zeta)$.

4. Applications

In this section, we apply (G'/G)-expansion method to several particular cases of FKdV5. All of the computing for this equations have been performed on a PC applying some programs written in $Mathematica$.

4.1. Time fractional standard Lax equation

When $p = 30$, $q = 20$, $r = 10$, (1) becomes time fractional standard Lax equation:

$$D_t^\alpha u + 30\, u\, u_{xxx} + 20\, u_x u_{xx} + 10\, u^2\, u_x + u_{xxxxx} = 0, \quad 0 < \alpha \leq 1. \tag{12}$$

With the help of $Mathematica$, we find

1) $a_0 = \mp\frac{1}{30}\left(\sqrt{5}\sqrt{6c - (\lambda^2 - 4\mu)^2} + 5\lambda^2 + 40\mu\right)$, $a_1 = -2\lambda$, $a_2 = -2$.

2) $a_0 = \pm\sqrt{\frac{2c}{7}} - \frac{3\lambda^2}{2}, a_1 = -6\lambda$, $a_2 = -6$, $\mu = \frac{1}{28}\left(7\lambda^2 \pm \sqrt{14c}\right)$.

Variable c is arbitrary constant.
Then the exact solution to nonlinear time fractional standard Lax equation can be written as
For case 1. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{1}{30}\left(60\mu \mp \left(\sqrt{5}\sqrt{6c - (\lambda^2 - 4\mu)^2} + 5\lambda^2 + 40\mu\right) + \frac{15(A-B)(A+B)\left(\lambda^2 - 4\mu\right)}{\left(A\cosh\left(\frac{\sqrt{\lambda^2-4\mu}(\alpha x - ct^\alpha)}{2\alpha}\right) + B\sinh\left(\frac{\sqrt{\lambda^2-4\mu}(\alpha x - ct^\alpha)}{2\alpha}\right)\right)^2}\right). \tag{13}$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{1}{30}\left(60\mu \mp \left(\sqrt{5}\sqrt{6c - (\lambda^2 - 4\mu)^2} + 5\lambda^2 + 40\mu\right) + \frac{15\left(A^2 + B^2\right)\left(\lambda^2 - 4\mu\right)}{\left(A\cos\left(\frac{\sqrt{4\mu-\lambda^2}(\alpha x - ct^\alpha)}{2\alpha}\right) + B\sin\left(\frac{\sqrt{4\mu-\lambda^2}(\alpha x - ct^\alpha)}{2\alpha}\right)\right)^2}\right). \tag{14}$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = \frac{1}{30}\left(-\frac{60B^2}{\left(A + B\left(x - \frac{ct^\alpha}{\alpha}\right)\right)^2} \mp \left(\sqrt{30c} + 40\mu\right) - 5\lambda^2(\pm 1 - 3)\right). \tag{15}$$

For case 2. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{\sqrt{c}\left(\left(A^2-B^2\right)(\pm 3 \mp 2) - (\pm 3 \pm 2)\left(\left(A^2+B^2\right)\cosh\left(\sqrt[4]{\frac{2}{7}}L\right) + 2AB\sinh\left(\sqrt[4]{\frac{2}{7}}L\right)\right)\right)}{2\sqrt{14}\left(A\cosh\left(\frac{L}{2^{3/4}\sqrt[4]{7}}\right) + B\sinh\left(\frac{L}{2^{3/4}\sqrt[4]{7}}\right)\right)^2} \tag{16}$$

where

$$L = \sqrt{\pm\sqrt{c}}\left(x - \frac{ct^\alpha}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{\sqrt{c}\left(\left(A^2+B^2\right)(\pm 3 \mp 2) \pm 3 \pm 2(K)\right)}{2\sqrt{14}\left(A\cosh\left(\frac{\sqrt[4]{c}\sqrt{\pm 1}\left(x-\frac{ct^\alpha}{\alpha}\right)}{2^{3/4}\sqrt[4]{7}}\right) + B\sin\left(\frac{\sqrt{\mp 1\sqrt{c}}\left(x-\frac{ct^\alpha}{\alpha}\right)}{2^{3/4}\sqrt[4]{7}}\right)\right)^2} \tag{17}$$

where

$$K = \left(B^2 - A^2\right)\cosh\left(\sqrt[4]{\frac{2}{7}}\sqrt{\pm\sqrt{c}}\left(x - \frac{ct^\alpha}{\alpha}\right)\right) - 2AB\sin\left(\sqrt[4]{\frac{2}{7}}\sqrt{\mp\sqrt{c}}\left(x - \frac{ct^\alpha}{\alpha}\right)\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = \pm\sqrt{\frac{2c}{7}} - \frac{6B^2}{\left(A + B\left(x - \frac{ct^\alpha}{\alpha}\right)\right)^2}. \tag{18}$$

4.2. Time fractional Sawada-Kotera equation

When $p = 5,\ q = 5,\ r = 5$, (1) becomes time fractional Sawada-Kotera (SK) equation:

$$D_t^\alpha u + 5\,u\,u_{xxx} + 5\,u_x u_{xx} + 5\,u^2\,u_x + u_{xxxxx} = 0, \quad 0 < \alpha \le 1. \tag{19}$$

With the help of *Mathematica*, we find

1) $a_1 = -6\lambda,\ a_2 = -6,\ c = 5a_0\lambda^2 + 40a_0\mu + 5a_0^2 + \lambda^4 + 22\lambda^2\mu + 76\mu^2.$

2) $a_0 = -\lambda^2 - 8\mu,\ a_1 = -12\lambda,\ a_2 = -12,\ c = \left(\lambda^2 - 4\mu\right)^2.$

Then the exact solution to nonlinear time fractional Sawada-Kotera equation can be written as
For case 1. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = a_0 + \frac{3\left(A^2 - B^2\right)\left(\lambda^2 - 4\mu\right)}{2(A\cosh(L) - B\sinh(L))^2} + 6\mu, \tag{20}$$

where

$$L = \frac{\sqrt{\lambda^2 - 4\mu}}{2\alpha}\left(t^\alpha\left(5a_0\left(a_0 + \lambda^2 + 8\mu\right) + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right) - \alpha x\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = a_0 + \frac{3\left(A^2 + B^2\right)\left(\lambda^2 - 4\mu\right)}{2(A\cos(K) - B\sin(K))^2} + 6\mu, \tag{21}$$

where

$$K = \frac{\sqrt{4\mu - \lambda^2}}{2\alpha}\left(t^\alpha\left(5a_0\left(a_0 + \lambda^2 + 8\mu\right) + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right) - \alpha x\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = a_0 - \frac{6\alpha^2 B^2}{\left(\alpha(A + Bx) - Bt^\alpha\left(5a_0\left(a_0 + \lambda^2 + 8\mu\right) + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right)\right)^2} + \frac{3\lambda^2}{2}. \tag{22}$$

In case 1, a_0 is arbitrary constant.
For case 2. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{3(A - B)(A + B)\left(\lambda^2 - 4\mu\right)}{(A\cosh(L) + B\sinh(L))^2} - \lambda^2 + 4\mu, \tag{23}$$

where

$$L = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(x - \frac{\left(\lambda^2 - 4\mu\right)^2 t^\alpha}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{3\left(A^2 + B^2\right)\left(\lambda^2 - 4\mu\right)}{(A\cos(K) + B\sin(K))^2} - \lambda^2 + 4\mu, \tag{24}$$

where

$$K = \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(x - \frac{\left(\lambda^2 - 4\mu\right)^2 t^\alpha}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = -\frac{12B^2}{(A+Bx)^2}. \tag{25}$$

4.3. Time fractional Sawada-Kotera-Parker-Dye equation

When $p = 45$, $q = -15$, $r = -15$, (1) becomes time fractional Sawada-Kotera-Parker-Dye (SKPD) equation:

$$D_t^\alpha u + 45\,u\,u_{xxx} - 15\,u_x u_{xx} - 15\,u^2\,u_x + u_{xxxxx} = 0, \quad 0 < \alpha \leq 1. \tag{26}$$

With the help of $Mathematica$, we find

1) $a_1 = 2\lambda,\ a_2 = 2,\ c = -15a_0\lambda^2 - 120a_0\mu + 45a_0^2 + \lambda^4 + 22\lambda^2\mu + 76\mu^2.$

2) $a_0 = \frac{1}{3}\left(\lambda^2 + 8\mu\right),\ a_1 = 4\lambda,\ a_2 = 4, c = \left(\lambda^2 - 4\mu\right)^2.$

Then the exact solution to nonlinear time fractional Sawada-Kotera-Parker-Dye can be written as
For case 1. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = a_0 - \frac{(A-B)(A+B)\left(\lambda^2 - 4\mu\right)}{2(A\cosh(L) + B\sinh(L))^2} - 2\mu, \tag{27}$$

where

$$L = \frac{\sqrt{\lambda^2 - 4\mu}}{2\alpha}\left(\alpha x - t^\alpha\left(-15a_0\left(\lambda^2 + 8\mu\right) + 45a_0^2 + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right)\right),$$

and a_0 is arbitrary constant.
As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = a_0 - \frac{\left(A^2 + B^2\right)\left(\lambda^2 - 4\mu\right)}{2(A\cos(K) + B\sin(K))^2} - 2\mu, \tag{28}$$

where

$$K = \frac{\sqrt{4\mu - \lambda^2}}{2\alpha}\left(\alpha x - t^\alpha\left(-15a_0\left(\lambda^2 + 8\mu\right) + 45a_0^2 + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right)\right),$$

and a_0 is arbitrary constant.
As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = a_0 + \frac{2\alpha^2 B^2}{\left(\alpha(A+Bx) - Bt^\alpha\left(-15a_0\left(\lambda^2+8\mu\right)+45a_0^2+\lambda^4+22\lambda^2\mu+76\mu^2\right)\right)^2} - \frac{\lambda^2}{2}, \tag{29}$$

where a_0 is arbitrary constant.
For case 2. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{1}{3}\left(\lambda^2-4\mu\right)\left(1 - \frac{3(A-B)(A+B)}{(A\cosh(L)+B\sinh(L))^2}\right), \tag{30}$$

where

$$L = \frac{\sqrt{\lambda^2-4\mu}}{2}\left(x - \frac{\left(\lambda^2-4\mu\right)^2 t^\alpha}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{1}{3}\left(\lambda^2-4\mu\right)\left(1 - \frac{3\left(A^2+B^2\right)}{(A\cos(K)+B\sin(K))^2}\right), \tag{31}$$

where

$$K = \frac{1}{2}\sqrt{4\mu-\lambda^2}\left(x - \frac{\left(\lambda^2-4\mu\right)^2 t^\alpha}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = \frac{4B^2}{(A+Bx)^2}. \tag{32}$$

4.4. Time fractional Caudrey-Dodd-Gibbon equation

When $p = 180$, $q = 30$, $r = 30$, (1) becomes time fractional Caudrey-Dodd-Gibbon (CDG) equation:

$$D_t^\alpha u + 180\,u\,u_{xxx} + 30\,u_x u_{xx} + 30\,u^2\,u_x + u_{xxxxx} = 0, \quad 0 < \alpha \leq 1. \tag{33}$$

With the help of *Mathematica*, we find

1) $a_1 = -\lambda,\ a_2 = -1, c = 30a_0\lambda^2 + 240a_0\mu + 180a_0^2 + \lambda^4 + 22\lambda^2\mu + 76\mu^2.$

2) $a_0 = \frac{1}{6}\left(-\lambda^2 - 8\mu\right),\ a_1 = -2\lambda,\ a_2 = -2, c = \left(\lambda^2 - 4\mu\right)^2.$

Then the exact solution to nonlinear time fractional Caudrey-Dodd-Gibbon equation can be written as
For case 1. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = a_0 + \frac{(A-B)(A+B)\left(\lambda^2 - 4\mu\right)}{4(A\cosh(L) - B\sinh(L))^2} + \mu, \tag{34}$$

where

$$L = \frac{\sqrt{\lambda^2 - 4\mu}\left(t^\alpha\left(30a_0\left(6a_0 + \lambda^2 + 8\mu\right) + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right) - \alpha x\right)}{2\alpha}.$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = a_0 + \frac{\left(A^2 + B^2\right)\left(\lambda^2 - 4\mu\right)}{4(A\cos(K) - B\sin(K))^2} + \mu, \tag{35}$$

where

$$K = \frac{\sqrt{4\mu - \lambda^2}}{2\alpha}\left(t^\alpha\left(30a_0\left(6a_0 + \lambda^2 + 8\mu\right) + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right) - \alpha x\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = -\frac{\alpha^2 B^2}{\left(\alpha(A + Bx) - Bt^\alpha\left(30a_0\left(6a_0 + \lambda^2 + 8\mu\right) + \lambda^4 + 22\lambda^2\mu + 76\mu^2\right)\right)^2} + a_0 + \frac{\lambda^2}{4}. \tag{36}$$

In case 1, a_0 is arbitrary constant.

For case 2. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{1}{6}\left(\frac{3(A-B)(A+B)\left(\lambda^2 - 4\mu\right)}{(A\cosh(L) + B\sinh(L))^2} - \lambda^2 + 4\mu\right), \tag{37}$$

where

$$L = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(x - \frac{\left(\lambda^2 - 4\mu\right)^2 t^\alpha}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{1}{6}\left(\frac{3\left(A^2 + B^2\right)\left(\lambda^2 - 4\mu\right)}{(A\cos(K) + B\sin(K))^2} - \lambda^2 + 4\mu\right), \tag{38}$$

where

$$K = \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(x - \frac{\left(\lambda^2 - 4\mu\right)^2 t^{\alpha}}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = \frac{A^2\lambda^2 + B\left(\lambda^2 x(2A + Bx) - 6B\right) - 4\mu(A + Bx)^2}{3(A + Bx)^2}. \tag{39}$$

4.5. Time fractional Kaup-Kupershmidt equation

When $p = 20$, $q = 25$, $r = 10$, (1) becomes time fractional Kaup-Kupershmidt (KK) equation:

$$D_t^{\alpha} u + 20\,u\,u_{xxx} + 25\,u_x u_{xx} + 10\,u^2\,u_x + u_{xxxxx} = 0, \quad 0 < \alpha \leq 1. \tag{40}$$

With the help of *Mathematica*, we find

1) $a_0 = -\lambda^2 - 8\mu,\ a_1 = -12\lambda,\ a_2 = -12, c = 11\left(\lambda^2 - 4\mu\right)^2.$

2) $a_0 = \frac{1}{8}\left(-\lambda^2 - 8\mu\right),\ a_1 = -\frac{1}{2}(3\lambda),\ a_2 = -\frac{3}{2}, c = \frac{1}{16}\left(\lambda^2 - 4\mu\right)^2.$

Then the exact solution to nonlinear time fractional Kaup-Kupershmidt can be written as
For case 1. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{3(A - B)(A + B)\left(\lambda^2 - 4\mu\right)}{(A\cosh(L) + B\sinh(L))^2} - \lambda^2 + 4\mu, \tag{41}$$

where

$$L = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(x - \frac{11\left(\lambda^2 - 4\mu\right)^2 t^{\alpha}}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{3\left(A^2 + B^2\right)\left(\lambda^2 - 4\mu\right)}{(A\cos(K) + B\sin(K))^2} - \lambda^2 + 4\mu, \tag{42}$$

where

$$K = \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(x - \frac{11\left(\lambda^2 - 4\mu\right)^2 t^{\alpha}}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = -\frac{12B^2}{(A+Bx)^2}. \tag{43}$$

For case 2. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{1}{8}\left(\frac{3(A-B)(A+B)\left(\lambda^2-4\mu\right)}{(A\cosh(L)+B\sinh(L))^2} - \lambda^2 + 4\mu\right), \tag{44}$$

where

$$L = \frac{1}{2}\sqrt{\lambda^2-4\mu}\left(x - \frac{\left(\lambda^2-4\mu\right)^2 t^\alpha}{16\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{1}{8}\left(\frac{3\left(A^2+B^2\right)\left(\lambda^2-4\mu\right)}{(A\cos(K)+B\sin(K))^2} - \lambda^2 + 4\mu\right), \tag{45}$$

where

$$K = \frac{1}{2}\sqrt{4\mu-\lambda^2}\left(x - \frac{\left(\lambda^2-4\mu\right)^2 t^\alpha}{16\alpha}\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = \frac{3B^2}{2(A+Bx)^2}. \tag{46}$$

4.6. Time fractional Kaup-Kupershmidt-Parker-Dye equation

When $p = 45$, $q = -\frac{75}{2}$, $r = -15$, (1) becomes time fractional Kaup-Kupershmidt-Parker-Dye (KKPD) equation:

$$D_t^\alpha u + 45\,u\,u_{xxx} - \frac{75}{2}\,u_x u_{xx} - 15\,u^2\,u_x + u_{xxxxx} = 0, \quad 0 < \alpha \leq 1. \tag{47}$$

With the help of *Mathematica*, we find

1) $a_0 = \frac{1}{12}\left(\lambda^2+8\mu\right),\ a_1 = \lambda,\ a_2 = 1, c = \frac{1}{16}\left(\lambda^2-4\mu\right)^2.$

2) $a_0 = \frac{2}{3}\left(\lambda^2+8\mu\right),\ a_1 = 8\lambda, a_2 = 8,\ c = 11\left(\lambda^2-4\mu\right)^2.$

Then the exact solution to nonlinear time fractional Kaup-Kupershmidt-Parker-Dye (KKPD) equation can be written as
For case 1. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{1}{12}\left(\lambda^2 - 4\mu\right)\left(1 - \frac{3(A-B)(A+B)}{(A\cosh(L) + B\sinh(L))^2}\right), \tag{48}$$

where

$$L = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(x - \frac{\left(\lambda^2 - 4\mu\right)^2 t^\alpha}{16\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{1}{12}\left(\lambda^2 - 4\mu\right)\left(1 - \frac{3\left(A^2 + B^2\right)}{(A\cos(K) + B\sin(K))^2}\right), \tag{49}$$

where

$$K = \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(x - \frac{\left(\lambda^2 - 4\mu\right)^2 t^\alpha}{16\alpha}\right).$$

As if $\lambda^2 - 4\,\mu = 0$

$$u(x,t) = \frac{B^2}{(A+Bx)^2} - \frac{\lambda^2}{6} + \frac{2\mu}{3}. \tag{50}$$

For case 2. If $\lambda^2 - 4\,\mu > 0$

$$u(x,t) = \frac{2}{3}\left(\lambda^2 - 4\mu\right)\left(1 - \frac{3(A-B)(A+B)}{(A\cosh(L) + B\sinh(L))^2}\right), \tag{51}$$

where

$$L = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(x - \frac{11\left(\lambda^2 - 4\mu\right)^2 t^\alpha}{\alpha}\right).$$

As if $\lambda^2 - 4\,\mu < 0$

$$u(x,t) = \frac{2}{3}\left(\lambda^2 - 4\mu\right)\left(1 - \frac{3\left(A^2 + B^2\right)}{(A\cos(K) + B\sin(K))^2}\right), \tag{52}$$

where

$$K=\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(x-\frac{11\left(\lambda^2-4\mu\right)^2t^{\alpha}}{\alpha}\right).$$

As if $\lambda^2-4\,\mu=0$

$$u(x,t)=\frac{8B^2}{(A+Bx)^2}. \tag{53}$$

4.7. Time fractional Ito equation

When $p=2$, $q=6$, $r=3$, (1) becomes time fractional fractional Ito equation:

$$D_t^{\alpha}u+2\,u\,u_{xxx}+6\,u_xu_{xx}+3\,u^2\,u_x+u_{xxxxx}=0,\quad 0<\alpha\leq 1. \tag{54}$$

With the help of $Mathematica$, we find

$$a_0=\frac{1}{2}(-5)\left(\lambda^2+8\mu\right),\ a_1=-30\lambda,\ a_2=-30,\ c=6\left(\lambda^2-4\mu\right)^2$$

. Then the exact solution to nonlinear time fractional Ito equation can be written as
If $\lambda^2-4\,\mu>0$

$$u(x,t)=\frac{5}{2}\left(\lambda^2-4\mu\right)\left(\frac{3(A-B)(A+B)}{(A\cosh(L)+B\sinh(L))^2}-1\right), \tag{55}$$

where

$$L=\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(x-\frac{6\left(\lambda^2-4\mu\right)^2t^{\alpha}}{\alpha}\right).$$

As if $\lambda^2-4\,\mu<0$

$$u(x,t)=\frac{5}{2}\left(\lambda^2-4\mu\right)\left(\frac{3\left(A^2+B^2\right)}{(A\cos(K)+B\sin(K))^2}-1\right), \tag{56}$$

where

$$K=\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(x-\frac{6\left(\lambda^2-4\mu\right)^2t^{\alpha}}{\alpha}\right).$$

As if $\lambda^2-4\,\mu=0$

$$u(x,t)=-\frac{30B^2}{(A+Bx)^2}. \tag{57}$$

5. Conclusion

In this study, we have proposed the (G'/G)-expansion method and applied it to find exact solutions for nonlinear time fractional fifth-order Korteweg-de Vries equation. The (G'/G)-expansion method is an efficient method in searching solutions for nonlinear time fractional partial differential equations. The method proposed in this paper can also be extended to solve nonlinear time fractional partial differential equations in mathematical physics.

References

[1] AM.Wazwaz, *Analytic study on Burgers, Fisher, Huxley equations and combined forms of these equations*, Appl Math Comput 2008; 195:75461.

[2] M. Wadati, *Introduction to solitons*, Pramana: J. Phys. 57 (5/6) (2001) 841-847.

[3] A. Biswas, *Topological 1-soliton solution of the nonlinear Schrodingers equation with Kerr law nonlinearity in (1+ 2) dimensions*, Commun. Nonlinear Sci. Numer. Simulat. 2009; 14: 2845-7.

[4] H.A. Nassar, M.A.Abdel-Razek, A.K. Seddeek, *Expanding the tanh-function method for solving nonlinear equations*, Appl. Math. (2011); 2: 1096-1104.

[5] NK.Vitanov, ZI. Dimitrova, *Application of the method of simplest equation for obtaining exact traveling-wave solutions for two classes of model PDEs from ecology and population dynamics*, Commun. Nonlinear Sci. Numer. Simul. 2010; 15(10):2836-45.

[6] EME. Zayed, HA. Zedan, KA. Gepreel, *On the solitary wave solutions for nonlinear Hirota-Sasuma coupled KDV equations*, Chaos, Solitons Fractals 2004; 22:285-303.

[7] YB. Zhou, ML. Wang, YM. Wang, *Periodic wave solutions to coupled KdV equations with variable coefficients*, Phys Lett A 2003; 308:31-6.

[8] R. Hirota, *The Direct Method in Soliton Theory* , Cambridge University Press; 2004.

[9] MA. Akbar, NHM. Ali, *Exp-function method for Duffing equation and new solutions of (2+1)dimensional dispersive long wave equations*, Prog Appl Math 2011; 1(2):30-42.

[10] G. Adomian, *Solving frontier problems of physics: the decomposition method* , Boston, M A: Kluwer Academic; 1994 .

[11] MA. Abdou, *The extended tanh-method and its applications for solving nonlinear physical models*, Appl. Math. Comput. 2007; 190:988-96.

[12] Sirendaoreji, *New exact travelling wave solutions for the Kawahara and modified Kawahara equations*, Chaos Solitons Fract 2004; 19:147-50.

[13] AT. Ali, *New generalized Jacobi elliptic function rational expansion method*, J Comput Appl Math 2011; 235:4117-27.

[14] MS. Liang and et al, *A method to construct Weierstrass elliptic function solution for nonlinear equations*, Int J Modern Phys B 2011; 25(4):1931-9.

[15] Y. He, S. Li , Y. Long, *Exact solutions of the Klein-Gordon equation by modified Exp-function method*, Int Math Forum 2012; 7(4):175-82.

[16] EME. Zayed, SAH. Ibrahim, *Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method*, Chinese Phys Lett 2012; 29(6):060201.

[17] M. Wang, Li. X, J. Zhang, *The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett. A, 2008; 372: 417-423.32.

[18] G. Ebadi, A. Biswas, *Application of the (G'/G)-expansion method for nonlinear diffusion equations with nonlinear source*, J. Franklin Ins 2010;347: 1391-1398.

[19] EME. Zayed, *Traveling wave solutions for higher dimensional nonlinear evolution equations using the (G'/G)-expansion method* , J. Appl. Math. Informatics 2010; 28 (1-2): 383-395.

[20] H.Naher, F.A. Abdullah, M.A. Akbar, *The (G'/G)-expansion method for abundant travelling wave solutions of Caudrey-Dodd-Gibbon equation* , Math. Prob. Eng 2011; Article ID: 218216, pp: 11, doi:10.1155/2011/218216.

[21] X. Liu, L. Tian, Y. Wu, *Application of (G'/G)-expansion method to two nonlinear evolution equations* , Appl. Math. Computation 2010; 217: 1376-1384.

[22] J. Feng, W. Li, Q. Wan, *Using (G'/G)-expansion method to seek the travelling wave solution of Kolmogorov-Petrovskii-Piskunov equation*, Appl. Math. Computation 2011; 217: 5860-5865.

[23] H. Zhang, *Application of the (G'/G)-expansion method for the complex KdV equation*, Commun Nonlinear Sci Numer Simulat 201015; 1700-1704.

[24] E.M.E. Zayed, S. Al-Joudi, *Applications of an Extended (G'/G)-Expansion Method to find Exact Solutions of Nonlinear PDEs in Mathematical Physics* , Math. Prob. Eng 2010; Article ID 768573, 19 pages, doi:10.1155/2010/768573.

[25] B. Ayhan, A. Bekir, *The (G'/G)-expansion method for the nonlinear lattice equations*, Commun. Nonlinear Sci. Numer. Simul 2012; 17: 3490-3498

[26] T. Ozis, I. Aslan, *Application of the (G'/G)-expansion method to Kawahara type equations using symbolic computation* , Appl. Math. Computation 2010; 216: 2360-2365.

[27] Khalil, R., et al. *A new definition of fractional derivative.* Journal of Computational and Applied Mathematics 264 (2014): 65-70.

[28] C.T. Lee, *Some remarks on the fifth-order KdV equations*, Mathematical Analysis and Applications 425 (1) (2015) 281-294.

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