Acta Universitatis Apulensis No. 44/2015
ISSN: 1582-5329 pp. 127-151
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.44.09

CERTAIN NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING A NON-ZERO POLYNOMIAL WITH FINITE WEIGHT

A. Banerjee and S. Majumder

Abstract. In this paper we shall stress on the generalization of the specific type of differential polynomials as used in [14] and [15]. Actually we use the notion of weighted sharing to study different relationship of meromorphic functions when the generalized non-linear differential polynomials, used in the paper share a non-zero polynomial. Two examples are provided to show that certain conditions used in the paper are the best possible when the differential polynomial takes the special form.

2010 *Mathematics Subject Classification*: 30D35.

Keywords: Meromorphic function, Weighted Sharing, Non-linear differential polynomials.

1. Introduction, Definitions and Results

Observations: In this paper, by meromorphic functions we shall always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7], [18] and [19]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o\{T(r,h)\}(r\to\infty, r\notin E)$.

For two arbitrary meromorphic functions f_1 and g_1, we denote by $T(r) = \max\{T(r,f_1),T(r,g_1)\}$ and $S(r)=o(T(r)),(r\to\infty,r\notin E)$. Let f and g be two non-constant meromorphic functions. For $a\in\mathbb{C}\cup\{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if $f-a$ and $g-a$ have the same set of zeros with the same multiplicities and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities.

Next we recall the following definition.

Definition 1. [9] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid= 1)$ the counting function of simple a points of f. For a positive integer p we denote by $N(r,a;f \mid\leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r,a;f \mid\geq p)$ and $\overline{N}(r,a;f \mid\geq p)$.

We put

$$\Theta(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

and

$$\Theta_{k)}(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f \mid\leq k)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, Hayman [6] proved the following result.

Theorem A. Let f be a transcendental entire function, and let $n(\geq 1)$ be an integer. Then $f^n f' = 1$ has infinitely many zeros.

Fang and Fang [5] found the uniqueness theorem of more generalized expression corresponding to the above result in the following manner.

Theorem B. Let f and g be two non-constant entire functions, and let $n(\geq 8)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.

In 2002, Fang [4] first investigated the value sharing of certain non-linear differential polynomials which are the k-th derivative of some linear expression. Fang's result is given below.

Theorem C. Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n \geq 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.

Lin and Yi [13] extended Theorem B for meromorphic functions in the following manner.

Theorem D. Let f and g be two non-constant meromorphic functions with $\Theta(\infty,f) > 2/(n+1)$, and let $n(\geq 12)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.

In the same way, in 2008, Zhang [20] extended Theorem C to meromorphic function and obtained the following result.

Theorem E. Suppose that f is a transcendental meromorphic function with finite number of poles, g is a transcendental entire function, and let n, k be two positive integers with $n \geq 2k+6$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f \equiv g$.

We now give the following definition introduced by I. Lahiri [8], known as weighted sharing of values, which is a scaling between CM and IM sharing.

Definition 2. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g *share* (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Recently Xu-Yi-Cao [16] and Li [12] employed weighted sharing of values to obtain some results concerning the value sharing of differential polynomials of the form $[h^n(h-1)]^{(k)}$ $(h = f,\ g)$ and uniqueness of the corresponding meromorphic functions. In 2011, in the same direction as mentioned earlier, the present first author [2] proved the following results first one of which improves Theorem E.

Theorem F. Let f and g be two transcendental meromorphic functions and $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Suppose for two nonzero constants a and b $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 3k+9$ or if $l = 1$ and $n \geq 4k+10$ or if $l = 0$ and $n \geq 9k+18$, then $f = g$ or $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} = 1$. The possibility $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} = 1$ does not occur for $k = 1$.

Theorem G. Let f and g be two transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two nonzero constants a and b $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 2k+6$ or if $l = 1$ and $n \geq 5k/2+7$ or if $l = 0$ and $n \geq 5k+12$, then $f = g$.

Observing the above results it is quiet natural to ask the following questions.

Question 1. Is it possible in any way to remove the second conclusion of Theorem F?

Question 2. What can be said if one replace the sharing value 1 by a nonzero polynomial in Theorems F and G?

In the direction of the first question Xia-Xu [15] obtained the following results.

Theorem H. Let f and g be two transcendental meromorphic functions such that $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$, share $(1,\infty)$, where n, k, m, be three positive integers. If $m > k$ and $n > 3k+m+8$, and $\Theta(\infty,f) > \frac{2m(n+m)}{(n+m)^2-4k^2}$ or $\Theta(\infty,g) > \frac{2m(n+m)}{(n+m)^2-4k^2}$ then either $f = g$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1,w_2) = w_1^n(w_1-1)^m - w_2^n(w_2-1)^m$.

Theorem I. Let f and g be two transcendental meromorphic functions such that $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$, share $(1,\infty)$, where n, k, m, be three positive integers. If $m \leq k$ and $n > 3k+m+8$, and

$$\Theta(\infty,f) + \Theta_{[k/m]}(1;f) > 1 + \frac{2m(n+m)}{(n+m)^2-4k^2}$$

or

$$\Theta(\infty;g) + \Theta_{[k/m]}(1,g) > 1 + \frac{2m(n+m)}{(n+m)^2-4k^2}$$

then the conclusion of Theorem H holds.

Very recently, Sahoo and Seikh [14] further improved above two theorems in the following manner.

Theorem J. Let f and g be two transcendental meromorphic functions, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 1)$ and $l(\geq 0)$ be four integers such that $\Theta(\infty,f)+\Theta(\infty,g) > 4/n$ and $\alpha \not\equiv 0,\infty$ be a small function of f and g. Suppose for two nonzero constants a and b, $[f^n(af+b)^m]^{(k)}$ and $[g^n(ag+b)^m]^{(k)}$ share (α,l). If $m > k$ and i) $l \geq 2$ and $n \geq \max\{2k+3m, 3k+m+8\}$ or if ii) $l = 1$ and $n \geq \max\{2k+3m, 4k+3m/2+9\}$ or if iii) $l = 0$ and $n \geq \max\{2k+3m, 9k+4m+14\}$, then either $f = tg$ for a constant t such that $t^d = 1$, where $d = \gcd\{n+m,\ldots,n+m-i,\ldots,n+1,n\}$, $a_{m-i} \neq 0$ for some $i = 0,1,\ldots,m$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1,w_2) = w_1^n(aw_1+b)^m - w_2^n(aw_2+b)^m$.

Theorem K. Let f and g be two transcendental meromorphic functions, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 1)$ and $l(\geq 0)$ be four integers such that $\Theta(\infty,f)+\Theta(\infty,g) > 4/n$ and

$\alpha \not\equiv 0, \infty$ be a small function of f and g. Suppose for two nonzero constants a and b, $[f^n(af+b)^m]^{(k)}$ and $[g^n(ag+b)^m]^{(k)}$ share (α, l). If $m \le k$,

$$\Theta(\infty; f) + \Theta_{[k/m])}(0, af+b) > 1 + \frac{2m(n+m)}{(n+m)^2 - 4k^2}$$

or

$$\Theta(\infty, g) + \Theta_{[k/m])}(0, ag+b) > 1 + \frac{2m(n+m)}{(n+m)^2 - 4k^2}$$

and one of i) $l \ge 2$, $n \ge 3k+m+8$ or ii) $l = 1$, $n \ge 4k + 3m/2 + 9$ or iii) $l = 0$ and $n \ge 9k + 4m + 14$, is satisfied, then the conclusion of Theorem J is satisfied.

In the proof of Theorem 1, *Case 3* [15] and Lemma 2.8 of [14] the authors have assumed that f and g will have poles and so both are true for non-entire meromorphic functions, but it is not clear whether the same is true for entire functions. In this paper we shall not only consider this case but also extend, improve and generalize the results of [14] and [15].

Throughout this paper, we always use $Q(\omega)$ to denote an arbitrary polynomial of degree n as follows,

$$Q(\omega) = a_n\omega^n + a_{n-1}\omega^{n-1} + \ldots + a_0 = a_n(\omega - c_{d_1})^{d_1}(\omega - c_{d_2})^{d_2} \ldots (\omega - c_{d_s})^{d_s}, \quad (1.1)$$

where $a_i (i = 0, 1, \ldots, n-1)$, $a_n \ne 0$ and $c_{d_j} (j = 1, 2, \ldots, s)$ are distinct finite complex numbers; $d_1, d_2, \ldots, d_s$, $s \ge 2$, n and k are all positives integers with

$$\sum_{i=1}^{s} d_i = n.$$

Let

$$d = \max\{d_1, d_2, \ldots, d_s\},$$

such that

$$d > \max_{\substack{d_i \ne d \\ i=1,\ldots,r}} \{d_1, d_2, \ldots, d_r\}, \text{ where } r = s-1.$$

We set an arbitrary non-zero polynomial $P(\omega_1)$ by

$$P(\omega_1) = a_n \prod_{\substack{i=1 \\ d_i \ne d}}^{s} (\omega_1 + c_d - c_{d_i})^{d_i} = b_m\omega_1^m + b_{m-1}\omega_1^{m-1} + \ldots + b_0, \quad (1.2)$$

where $a_n = b_m$, $\omega_1 = \omega - c_d$ and $m = n - d$.
Obviously

$$Q(\omega) = \omega_1^d P(\omega_1). \quad (1.3)$$

Let $P(\omega_1) = b_m \prod_{i=1}^{r} (\omega_1 - \alpha_i)^{d_i}$, where $\alpha_i = c_{d_i} - c_d$, $i = 1, 2, \ldots, r$ be the distinct zeros of $P(\omega_1)$.

Following theorems are the main results of the paper.

Theorem 1. Let f and g be two transcendental meromorphic functions such that $f_1 = f - c_d$, $g_1 = g - c_d$ and $[Q(f)]^{(k)} - P_1$ and $[Q(g)]^{(k)} - P_1$ share $(0, l)$, where P_1 is a non-zero polynomial. If $d_i \leq k$, for $i = 1, 2, \ldots, r$,

$$\left[\Theta(\infty, f) + \sum_{i=1}^{r} \Theta_{[k/d_i])}(c_{d_i}, f) > 1 + \frac{2mn}{n^2 - 4k^2}\right] \tag{1.4}$$

or

$$\left[\Theta(\infty, g) + \sum_{i=1}^{r} \Theta_{[k/d_i])}(c_{d_i}, g) > 1 + \frac{2mn}{n^2 - 4k^2}\right] \tag{1.5}$$

and one of the following conditions is satisfied

(a) $l \geq 2$ and $d > 3k + m + 8$;

(b) $l = 1$ and $d > 4k + \frac{3m}{2} + 9$;

(c) $l = 0$ and $d > 9k + 4m + 14$,

then one of the following two conclusions hold

(I1) $f_1(z) \equiv t g_1(z)$ for a constant t such that $t^l = 1$, where $l = \mathrm{g}cd(d + m, \ldots, d + m - i, \ldots, d)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m$;

(I2) f_1 and g_1 satisfy the algebraic equation $R(f_1, g_1) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^d(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^d(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$, except for $P(w) = a_1 w + a_2$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{d}$.

Theorem 2. Let f and g be two transcendental meromorphic functions such that $f_1 = f - c_d$, $g_1 = g - c_d$, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 1)$ and $l(\geq 0)$ be four integers. Let $P(z)$ be defined as in Theorem 1. Suppose that $[Q(f)]^{(k)} - P_1$ and $[Q(g)]^{(k)} - P_1$ share $(0, l)$, where P_1 is a non-zero polynomial. If $d_i > k$, for $i = 1, 2, \ldots, r$, $\min\{\Theta(\infty; f), \Theta(\infty; g)\} > 1 + \frac{2mn}{n^2 - 4k^2} - r$ and the conditions (a), (b) and (c) of Theorem 1 is satisfied then conclusion of Theorem 1 holds.

Remark 1. The following examples show that the condition $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{d}$ in Theorem 1 is sharp when $m = 1$, $c_d = 0$, $P(z) = z - 1$ and $d > 3k + 9 (k \leq 2)$.

Example 1. Let $f = \frac{1-h^d}{1-h^{d+1}}$ and $g = h\frac{1-h^d}{1-h^{d+1}}$, where $h = \frac{\alpha^2(e^z-1)}{e^z-\alpha}$, $\alpha = exp(\frac{2\pi i}{d+1})$ and $d(> 3k+9)$ is an integer.
Clearly

$$f^d(f-1) \equiv g^d(g-1).$$

Note that $T(r,f) = dT(r,h) + O(1)$, $T(r,g) = dT(r,h) + O(1)$ and $T(r,h) = T(r,e^z)+O(1)$. Next we see that $h \neq \alpha, \alpha^2$ and so for any complex number $\gamma \neq \alpha, \alpha^2$, we have $\overline{N}(r,\gamma;h) \sim T(r,h)$. Also we note that $h = 1$ is not a pole and zero of f and g. Hence

$$\Theta(\infty,f) = \Theta(\infty,g) = \frac{2}{d}.$$

On the other hand we have

$$f-1 = h^d\frac{h-1}{1-h^{d+1}}, \quad g-1 = \frac{h-1}{1-h^{d+1}}.$$

Note that

$$\Theta_{k)}(1,f) = 1 - \limsup_{r\longrightarrow\infty}\frac{\overline{N}_{k)}(r,0;h^d)}{dT(r,h)+O(1)} = 1$$

and

$$\Theta_{k)}(1,g) = 1 - \limsup_{r\longrightarrow\infty}\frac{\overline{N}_{k)}(r,\infty;h^d)}{dT(r,h)+O(1)} = 1.$$

Clearly both (1.4), (1.5) hold and $\Theta(\infty,f)+\Theta(\infty,g) = \frac{4}{d}$, but $f(z) \not\equiv g(z)$.

Example 2. Let f and g be as in Example 1, where $h = \frac{\alpha(\alpha e^z-1)}{e^z-1}$, $\alpha = exp(\frac{2\pi i}{d+1})$ and $d > 3k+9(k \leq 2)$ is an integer.

However the following question is still open :

Question 3. Keeping all other conditions intact, are Theorems 1 and *2* true for rational functions also ?

Though the standard definitions and notations of the value distribution theory are available in [7], we explain some definitions and notations which are used in the paper.

Definition 3. [8] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \ldots + \overline{N}(r,a;f \mid\geq k).$$

Definition 4. Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ ($\overline{N}(r, a; f \mid \geq p \mid g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

Definition 5. {*cf.[1], 2*} *Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.*

Definition 6. {*cf.[1], 2*} *Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}\ (r, 1; g)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\overline{N}_{g>k}\ (r, 1; f)$ is defined analogously.*

Definition 7. [8] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 8. Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 1. [17] Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \ldots + a_1 f(z) + a_0,$$

where $a_n(\neq 0)$, a_{n-1}, ..., a_1, a_0 are complex numbers. Then

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

Lemma 2. [21] Let f be a non-constant meromorphic function, and p, k be positive integers. Then

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

Lemma 3. [11] If $N(r, 0; f^{(k)} \mid f \neq 0)$ denote the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid< k) + k\overline{N}(r, 0; f \mid\geq k) + S(r, f).$$

Lemma 4. [7, 18] Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r, a_i(z)) = S(r, f)$, i=1,2. Then

$$T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 5. Let f and g be two non-constant transcendental meromorphic functions; n, k, m be three positive integers. If $d_i > k$ and $\min\{\Theta(\infty; f), \Theta(\infty; g)\}$ $> 1 + \frac{2m(n+m)}{(n+m)^2 - 4k^2} - r$ or if $d_i \leq k$ and (1.4) or (1.5) holds, then

$$(Q(f))^{(k)}(Q(g))^{(k)} \neq p^2,$$

where p, is a non-zero polynomial.

Proof. Set $f_1 = f - c_d$, $g_1 = g - c_d$. If possible, let

$$(Q(f))^{(k)}(Q(g))^{(k)} = p^2, \tag{2.1}$$

i.e.,

$$(f_1^d P(f_1))(g_1^d P(g_1)) = p^2.$$

First suppose that f and g both are transcendental entire functions and so are f_1 and g_1. From above, it is clear that the zeros of f_1 (g_1) will be neutralized by the zeros of p and so they will be finite in numbers, it follows that $f_1(z) = p_* e^{\alpha}$, where

α is a non-constant entire function and p_* is a polynomial. Then by induction we get

$$(a_i f_1^{d+i})^{(k)} = t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, p_*, p_*^{'}, \ldots, p_*^{(k)}) e^{(d+i)\alpha}, \tag{2.2}$$

where $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, p_*, p_*^{'}, \ldots, p_*^{(k)})$ $(i = 0, 1, \ldots, m)$ are differential polynomials in α', α'', $\ldots$, $\alpha^{(k)}$ and $p_*, p_*^{'}, \ldots, p_*^{(k)}$. Obviously

$$t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, p_*, p_*^{'}, \ldots, p_*^{(k)}) \not\equiv 0$$

for $i = 0, 1, 2, \ldots, m$, since otherwise by (2.2), f_1 and so f would reduce to a polynomial, which is not the case. From (2.1) and (2.2) we obtain

$$\overline{N}(r, 0; t_m e^{m\alpha(z)} + \ldots + t_0) \leq N(r, 0; p^2) = S(r, f_1). \tag{2.3}$$

Since α is an entire function, we know $T(r, \alpha^{(j)}) = S(r, f_1)$ for $j = 1, 2, \ldots, k$. Hence $T(r, t_i) = S(r, f_1)$ for $i = 0, 1, 2, \ldots, m$.

So from (2.3), Lemmas 1 and 3 we obtain

$$\begin{aligned}
& mT(r, f_1) \\
= \; & T(r, t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + S(r, f_1) \\
\leq \; & \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha} + t_0) + S(r, f_1) \\
\leq \; & \overline{N}(r, 0; t_m e^{(m-1)\alpha} + \ldots + t_1) + S(r, f_1) \\
\leq \; & (m-1)T(r, f) + S(r, f_1),
\end{aligned}$$

which is a contradiction.

Next suppose both f and g and so f_1 and g_1 are non entire transcendental meromorphic functions.

Let $d_i > k$, for $i = 1, 2, \ldots, r$. Let $z_1 \notin \{z : p(z) = 0\}$ be a zero of f_1 with multiplicity $t_1(\geq 1)$. Then it follows from (2.1) that z_1 is a pole of g_1 of order $q_1(\geq 1)$(say). So we have

$$dt_1 - k = (d + m)q_1 + k. \tag{2.4}$$

From (2.4) we get $mq_1 + 2k = d(t_1 - q_1) \geq d$, i.e., $q_1 \geq \frac{d-2k}{m}$. Thus (2.4) yields $dt_1 = (d + m)q_1 + 2k$, and so

$$p_1 \geq \frac{d + m - 2k}{m}.$$

Let $z_2 \notin \{z : p(z) = 0\}$ be a zero of order s_i of the factor $f_1 - \alpha_i$ of $P(f_1)$ and it is also a zero of $P(f_1)$ of order t_2. So $t_2 = s_i d_i$. Since $d_i > k$, from (2.3) it follows that if z_2 is a pole of g_1 of order $q_2(\geq 1)$, then we have $t_2 - k = s_i d_i - k = (d+m)q_2 + k$, i.e.,

$$s_i \geq \frac{d+m+2k}{d_i},$$

for $i = 1, 2, \ldots, r$.

Using the second fundamental theorem of Nevanlinna and lemma 1 we get

$$\begin{aligned} rT(r,f) &= rT(r,f_1) + O(1) \\ &\leq \overline{N}(r,\infty;f_1) + \overline{N}(r,0;f_1) + \sum_{i=1}^{r} \overline{N}(r,\alpha_i;f_1) + S(r,f_1) \\ &\leq \overline{N}(r,\infty;f) + \left[\frac{m}{d+m-2k} + \sum_{i=1}^{r} \frac{d_i}{d+m+2k}\right] T(r,f_1) + S(r,f_1) \\ &\leq \overline{N}(r,\infty;f) + \left[\frac{m}{d+m-2k} + \frac{m}{d+m+2k}\right] T(r,f) + S(r,f). \quad (2.5) \end{aligned}$$

In a similar way we can obtain

$$rT(r,g) \leq \overline{N}(r,\infty;g) + \left[\frac{m}{d+m-2k} + \frac{m}{d+m+2k}\right] T(r,g) + S(r,g). \quad (2.6)$$

Adding (2.5) and (2.6) we get

$$\begin{aligned} & r\{T(r,f) + T(r,g)\} \\ \leq\ & (1 - \Theta(\infty;f) + \frac{2mn}{n^2-4k^2} + \varepsilon)T(r,f) \\ & +(1 - \Theta(\infty;g) + \frac{2mn}{n^2-4k^2} + \varepsilon)T(r,g) + S(r,f) + S(r,g). \quad (2.7) \end{aligned}$$

Since $\varepsilon > 0$ and $\min\{\Theta(\infty;f), \Theta(\infty;g)\} > 1 + \frac{2mn}{n^2-4k^2} - r$ we have from (2.7),

$$\begin{aligned} & \left[r + \Theta(\infty;f) - 1 - \frac{2mn}{n^2-4k^2} + \varepsilon\right] T(r,f) \\ & + \left[r + \Theta(\infty;g) - 1 - \frac{2mn}{n^2-4k^2} + \varepsilon\right] T(r,g) \leq S(r,f) + S(r,g), \quad (2.8) \end{aligned}$$

which gives a contradiction.

Next suppose $d_i \leq k$ for $i = 1, 2, \ldots, r$. Let $z_3 \notin \{z : p(z) = 0\}$ be a zero of order $s_i \geq [\frac{k}{d_i}] + 1$, $i = 1, 2, \ldots, r$ of the factor $f_1 - \alpha_i$ of $P(f_1)$. Then it is also a zero of

$(f_1^d P(f_1))^{(k)}$ of multiplicity $s_i d_i - k (\geq 1)$. It follows from (2.1) that z_3 is a pole of g_1 of order $q_3(\geq 1)$, such that

$$s_i = \frac{(d+m)q_3 + 2k}{d_i} \geq \frac{d+m+2k}{d_i},$$

for $i = 1, 2, \ldots, r$.

Using the second fundamental theorem of Nevanlinna and lemma 1 we get

$$\begin{aligned} & rT(r,f) = rT(r,f_1) + O(1) \\ \leq\ & \overline{N}(r,\infty;f_1) + \overline{N}(r,0;f_1) + \sum_{i=1}^{r} \overline{N}(r,\alpha_i;f_1 \mid\leq ([k/d_i]) \\ & + \sum_{i=1}^{r} \overline{N}(r,\alpha_i;f_1 \mid\geq [k/d_i]+1) + S(r,f_1) \\ \leq\ & \overline{N}(r,\infty;f) + \left[\frac{m}{d+m-2k}\right] N(r,c_d;f) + \sum_{i=1}^{r} \overline{N}(r,c_{d_i};f \mid\leq [k/d_i]) \\ & + \sum_{i=1}^{r} \frac{d_i}{d+m+2k} N(r,c_{d_i};f \mid\geq [k/d_i]+1) + S(r,f). \end{aligned} \tag{2.9}$$

(2.9) yields,

$$\left[\Theta(\infty,f) + \sum_{i=1}^{r} \Theta_{[k/d_i]})(c_{d_i},f) - 1 - \frac{2mn}{n^2-4k^2}\right] T(r,f) \leq S(r,f),$$

which is a contradiction to (1.4). In the same way we can deduce a contradiction to (1.5) .

Lemma 6. Let f and g be two non-constant meromorphic functions such that

$$\Theta(\infty,f) + \Theta(\infty,g) > \frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b) = g^n(ag+b)$$

implies $f = g$, where a, b are two nonzero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [10].

Lemma 7. Let f and g be two non-constant meromorphic functions. Let $Q(z)$ be defined as in Theorem 1 and let $f_1 = f - c_d$, $g_1 = g - c_d$ where $d > 3k + m$. If $[Q(f)]^{(k)} \equiv [Q(g)]^{(k)}$, then $f_1^d P(f_1) \equiv g_1^d P(g_1)$.

Proof. We have $[Q(f)]^{(k)} \equiv [Q(g)]^{(k)}$ implies $[f_1^d P(f_1)]^{(k)} \equiv [g_1^d P(g_1)]^{(k)}$.
When $k \geq 2$, integrating we get

$$[f_1^d P(f_1)]^{(k-1)} \equiv [g_1^d P(g_1)]^{(k-1)} + \beta_{k-1}.$$

If possible suppose $\beta_{k-1} \neq 0$.
Now in view of Lemma 2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned}
& (d+m)T(r, f_1) \\
\leq\ & T(r, [f_1^d P(f_1)]^{(k-1)}) - \overline{N}(r, 0; [f_1^d P(f_1)]^{(k-1)}) + N_k(r, 0; f_1^d P(f_1)) + S(r, f_1) \\
\leq\ & \overline{N}(r, 0; [f_1^d P(f_1)]^{(k-1)}) + \overline{N}(r, \infty; f_1) + \overline{N}(r, \beta_{k-1}; [f_1^d P(f_1)]^{(k-1)}) \\
& -\overline{N}(r, 0; [f_1^d P(f_1)]^{(k-1)}) + N_k(r, 0; f_1^d P(f_1)) + S(r, f_1) \\
\leq\ & \overline{N}(r, \infty; f_1) + \overline{N}(r, 0; [g_1^d P(g_1)]^{(k-1)}) + k\overline{N}(r, 0; f_1) + N(r, 0; P(f_1)) + S(r, f_1) \\
\leq\ & (k+m+1)\ T(r, f_1) + (k-1)\overline{N}(r, \infty; g_1) + N_k(r, 0; g_1^d P(g_1)) + S(r, f_1) \\
\leq\ & (k+m+1)T(r, f_1) + (k-1)\overline{N}(r, \infty; g_1) + k\overline{N}(r, 0; g_1) + N(r, 0; P(g_1)) \\
& +S(r, f_1) \\
\leq\ & (k+m+1)T(r, f_1) + (2k+m-1)T(r, g_1) + S(r, f_1) + S(r, g_1) \\
\leq\ & (3k+2m)T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(d+m)\ T(r, g_1) \leq (3k+2m)T(r) + S(r).$$

Combining these we get

$$(d-m-3k)\ T(r) \leq S(r),$$

which is a contradiction since $d > 3k + m$.
Therefore $\beta_{k-1} = 0$ and so $[f_1^d P(f_1)]^{(k-1)} \equiv [g_1^d P(g_1)]^{(k-1)}$. Repeating $k-1$ times, we obtain

$$f_1^d P(f_1) \equiv g_1^d P(g_1) + \beta_0.$$

If $k = 1$, clearly integrating once we obtain the above. If possible suppose $\beta_0 \neq 0$.

Now using the second fundamental theorem we get

$$\begin{aligned} & (d+m)T(r,f_1) \\ \leq\ & \overline{N}(r,0;f_1^dP(f_1)) + \overline{N}(r,\infty;f_1^dP(f_1)) + \overline{N}(r,\beta_0;f_1^dP(f_1)) \\ \leq\ & \overline{N}(r,0;f_1) + mT(r,f_1) + \overline{N}(r,\infty;f_1) + \overline{N}(r,0;g_1^dP(g_1)) \\ \leq\ & (m+2)\,T(r,f_1) + \overline{N}(r,0;g_1) + m\,T(r,g_1) + S(r,f_1) \\ \leq\ & (m+2)\,T(r,f_1) + (m+1)\,T(r,g_1) + S(r,f_1) + S(r,g_1) \\ \leq\ & (3+2m)\,T(r) + S(r). \end{aligned}$$

Similarly we get

$$(d+m)\,T(r,g_1) \leq (2m+3)T(r) + S(r).$$

Combining these we get

$$(d-m-3)\,T(r) \leq S(r),$$

which is a contradiction since $d > 3+m$.
Therefore $\beta_0 = 0$ and so

$$f_1^dP(f_1) \equiv g_1^dP(g_1).$$

This completes the proof of the Lemma.

Lemma 8. Let f, g be two non-constant meromorphic functions such that $f_1 = f - c_d$ and $g_1 = g - c_d$. Let $F = \frac{[f_1^dP(f_1)]^{(k)}}{P_1}$, $G = \frac{[g_1^dP(g_1)]^{(k)}}{P_1}$, where $P_1(z)$ be a non-zero polynomial, $d(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ be positive integers such that $d > 3k+m+3$ and $P(z)$ be defined as in Theorem 1. Suppose $d_i > k$ and $\min\{\Theta(\infty;f),\Theta(\infty;g)\} > 1 + \frac{2m(n+m)}{(n+m)^2-4k^2} - r$ or $d_i \leq k$ and (1.4) or (1.5) holds. If $H \equiv 0$ then one of the following two cases hold

(I) $f_1(z) \equiv tg_1(z)$ for a constant t such that $t^l = 1$, where $l = gcd(d+m,\ldots,d+m-i,\ldots,d)$, $a_{m-i} \neq 0$ for some $i = 1,2,\ldots,m$;

(I) f_1 and g_1 satisfy the algebraic equation $R(f_1,g_1) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^d(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^d(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$, except for $P_1(z) = a_1z + a_2$ and $\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{n}$;

Proof. Since $H \equiv 0$, on integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{2.10}$$

where a, b are constants and $a \neq 0$. We now consider the following cases.
Case 1. Let $b \neq 0$ and $a \neq b$.
If $b = -1$, then from (2.10) we have

$$F \equiv \frac{-a}{G - a - 1}.$$

Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
& (d+m)\ T(r, g_1) \\
\leq\ & T(r, G) + N_{k+1}(r, 0; g_1^d P(g_1)) - \overline{N}(r, 0; G) \\
\leq\ & \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, a+1; G) + N_{k+1}(r, 0; g_1^n P(g_1)) \\
& -\overline{N}(r, 0; G) + S(r, g_1) \\
\leq\ & \overline{N}(r, \infty; g_1) + N_{k+1}(r, 0; g_1^d P(g_1)) + \overline{N}(r, \infty; f_1) + S(r, g_1) \\
\leq\ & \overline{N}(r, \infty; f_1) + \overline{N}(r, \infty; g_1) + N_{k+1}(r, 0; g_1^d) + N_{k+1}(r, 0; P(g_1)) + S(r, g_1) \\
\leq\ & \overline{N}(r, \infty; f_1) + \overline{N}(r, \infty; g_1) + (k+1)\overline{N}(r, 0; g_1) + T(r, P(g_1)) + S(r, g_1) \\
\leq\ & T(r, f_1) + (k+m+2)T(r, g_1) + S(r, f_1) + S(r, g_1).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f_1) \leq T(r, g_1)$ for $r \in I$.
So for $r \in I$ we have

$$(d - k - 3)T(r, g_1) \leq S(r, g_1),$$

which is a contradiction since $d > k + 3$.
If $b \neq -1$, from (2.10) we obtain that

$$F - (1 + \frac{1}{b}) \equiv \frac{-a}{b^2[G + \frac{a-b}{b}]}.$$

So

$$\overline{N}(r, \frac{(b-a)}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f_1) + S(r, f_1).$$

Using Lemma 2 and the same argument as used in the case when $b = -1$ we can get a contradiction.
Case 2. Let $b \neq 0$ and $a = b$.
If $b = -1$, then from (2.10) we have

$$FG \equiv P_1^2,$$

that is

$$[f_1^d P(f_1)]^{(k)}[g_1^d P(g_1)]^{(k)} \equiv P_1^2,$$

which is impossible in view of Lemma 5.
If $b \neq -1$, from (2.10) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F).$$

So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
& (d+m)\ T(r, g_1) \\
\leq\ & T(r, G) + N_{k+1}(r, 0; g_1^d P(g_1)) - \overline{N}(r, 0; G) + S(r, g_1) \\
\leq\ & \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, \frac{1}{1+b}; G) + N_{k+1}(r, 0; g_1^d P(g_1)) \\
& -\overline{N}(r, 0; G) + S(r, g_1) \\
\leq\ & \overline{N}(r, \infty; g_1) + (k+1)\overline{N}(r, 0; g_1) + T(r, P(g_1)) + \overline{N}(r, 0; F) + S(r, g_1) \\
\leq\ & \overline{N}(r, \infty; g_1) + (k+1)\overline{N}(r, 0; g_1) + T(r, P(g_1)) + (k+1)\overline{N}(r, 0; f_1) + T(r, P(f_1)) \\
& +k\overline{N}(r, \infty; f_1) + S(r, f_1) + S(r, g_1) \\
\leq\ & (k+m+2)T(r, g_1) + (2k+m+1)T(r, f_1) + S(r, f_1) + S(r, g_1).
\end{aligned}$$

So for $r \in I$ we have

$$(d-3k-m-3)T(r, g_1) \leq S(r, g_1),$$

which is a contradiction since $d > 3k+m+3$.
Case 3. Let $b = 0$. From (2.10) we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{2.11}$$

If $a \neq 1$ then from (2.11) we obtain

$$\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in *Case 2*. Therefore $a = 1$ and from (2.11) we obtain

$$F \equiv G,$$

i.e

$$[f_1^d P(f_1)]^{(k)} \equiv [g_1^d P(g_1)]^{(k)}.$$

Noting that $d > 3k + m + 3 > 3k + m$ by Lemma 6 we have

$$f_1^d P(f_1) \equiv g_1^d P(g_1). \tag{2.12}$$

Let $h = \frac{f_1}{g_1}$. If h is a constant, putting $f_1 = g_1 h$ in (2.12) we get

$$a_m g_1^{d+m}(h^{d+m} - 1) + a_{m-1} g_1^{d+m-1}(h^{d+m-1} - 1) + ... + a_0 g_1^d (h^d - 1) = 0,$$

which implies $h^l = 1$, where $l = gcd(d + m, ..., d + m - i, ..., d + 1, d)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$. Thus $f_1 = t g_1$ for a constant t such that $t^l = 1$, $l = gcd(d + m, ..., d + m - i, ..., d + 1, d)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$.

If h is not a constant, then from (2.12) we can say that f_1 and g_1 satisfy the algebraic equation $R(f_1, g_1) = 0$, where $R(\omega_1, \omega_2) = \omega_1^d(a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^d(a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$. In particular when $P_1(z) = a_1 z + a_2$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$ then by Lemma 6 we get from (2.12) that $f_1 \equiv g_1$, i.e., $f \equiv g$.

Lemma 9. [1] If f, g be two non-constant meromorphic functions such that they share $(1, 1)$. Then

$$\begin{aligned} & 2\overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2}(r, 1; f) - \overline{N}_{f>2}(r, 1; g) \\ \leq \; & N(r, 1; g) - \overline{N}(r, 1; g). \end{aligned}$$

Lemma 10. [3] Let f, g share $(1, 1)$. Then

$$\overline{N}_{f>2}(r, 1; g) \leq \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) - \frac{1}{2}N_0(r, 0; f') + S(r, f),$$

where $N_0(r, 0; f')$ is the counting function of those zeros of f' which are not the zeros of $f(f - 1)$.

Lemma 11. [3] Let f and g be two non-constant meromorphic functions sharing $(1, 0)$. Then

$$\begin{aligned} & \overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2}(r, 1; f) - \overline{N}_{f>1}(r, 1; g) - \overline{N}_{g>1}(r, 1; f) \\ \leq \; & N(r, 1; g) - \overline{N}(r, 1; g). \end{aligned}$$

Lemma 12. [3] Let f, g share $(1, 0)$. Then

$$\overline{N}_L(r, 1; f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f)$$

Lemma 13. [3] Let f, g share $(1, 0)$. Then

$$\begin{aligned} (i) \quad & \overline{N}_{f>1}(r, 1; g) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_0(r, 0; f') + S(r, f) \\ (ii) \quad & \overline{N}_{g>1}(r, 1; f) \leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) - N_0(r, 0; g') + S(r, g). \end{aligned}$$

3. Proofs of the Theorems

Proof of Theorem 1. Let $F = [f_1^d P(f_1)]^{(k)}/P_1$ and $G = [g_1^d P(g_1)]^{(k)}/P_1$. It follows that F and G share $(1,l)$ except the zeros of P_1.
Case 1 Let $H \not\equiv 0$.
Subcase 1.1 $l \geq 1$.
From the definition of H it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.
Since H has only simple poles we get

$$\begin{aligned} & N(r,\infty;H) \qquad (3.1)\\ \leq\ & \overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)\\ & +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.
Let z_0 be a simple zero of $F-1$ but $P_1(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1) \leq N(r,0;H) \leq N(r,\infty;H)+S(r,f_1)+S(r,g_1). \qquad (3.2)$$

While $l \geq 2$, using (3.1) and (3.2) we get

$$\begin{aligned} & \overline{N}(r,1;F) \qquad (3.3)\\ \leq\ & N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2)\\ \leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\ & +\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f_1)+S(r,g_1). \end{aligned}$$

Now in view of Lemma 3 we get

$$\begin{aligned} & \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\ |\geq 2)+\overline{N}_*(r,1;F,G) \qquad (3.4)\\ \leq\ & \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F|\geq 2)+\overline{N}(r,1;F|\geq 3)\\ =\ & \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;G|\geq 2)+\overline{N}(r,1;G|\geq 3)\\ \leq\ & \overline{N}_0(r,0;G^{'})+N(r,1;G)-\overline{N}(r,1;G)\\ \leq\ & N(r,0;G^{'}\mid G\neq 0) \leq \overline{N}(r,0;G)+\overline{N}(r,\infty;g)+S(r,g_1). \end{aligned}$$

Hence using (3.3), (3.4), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
& (d+m)T(r,f_1) \qquad (3.5)\\
\leq\ & T(r,F) + N_{k+2}(r,0;f_1^dP(f_1)) - N_2(r,0;F) + S(r,f_1)\\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;f_1^dP(f_1)) - N_2(r,0;F)\\
& -N_0(r,0;F^{'})\\
\leq\ & 2\,\overline{N}(r,\infty,f_1) + \overline{N}(r,\infty;g_1) + \overline{N}(r,0;F) + N_{k+2}(r,0;f_1^dP(f_1)) + \overline{N}(r,0;F|\geq 2)\\
& +\overline{N}(r,0;G|\geq 2) + \overline{N}(r,1;F|\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;G^{'}) - N_2(r,0;F)\\
& +S(r,f_1) + S(r,g_1)\\
\leq\ & 2\,\{\overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;g_1)\} + N_{k+2}(r,0;f_1^dP(f_1)) + N_2(r,0;G)\\
& +S(r,f_1) + S(r,g_1)\\
\leq\ & 2\,\{\overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;g_1)\} + N_{k+2}(r,0;f_1^dP(f_1)) + k\,\overline{N}(r,\infty;g_1)\\
& +N_{k+2}(r,0;g_1^dP(g_1)) + S(r,f_1) + S(r,g_1)\\
\leq\ & 2\,\{\overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;g_1)\} + (k+2)\,\overline{N}(r,0;f_1) + T(r,P(f_1))\\
& +(k+2)\,\overline{N}(r,0;g_1) + T(r,P(g_1)) + k\,\overline{N}(r,\infty;g_1) + S(r,f_1) + S(r,g_1)\\
\leq\ & (k+m+4)T(r,f_1) + (2k+m+4)T(r,g_1) + S(r,f_1) + S(r,g_1)\\
\leq\ & (3k+2m+8)T(r) + S(r).
\end{aligned}$$

In a similar way we can obtain

$$(d+m)\,T(r,g_1) \leq (3k+2m+8)\,T(r) + S(r). \tag{3.6}$$

Combining (3.5) and (3.6) we see that

$$(d+m)\,T(r) \leq (3k+2m+8)T(r) + S(r),$$

i.e

$$(d-3k-m-8)T(r) \leq S(r). \tag{3.7}$$

Since $d > 3k+m+8$, (3.7) leads to a contradiction.

While $l = 1$, using Lemmas 3, *9*, *10*, (3.1) and (3.2) we get

$$\begin{aligned}
& \overline{N}(r,1;F) \qquad\qquad (3.8)\\
\leq\; & N(r,1;F|=1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F)\\
\leq\; & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}_*(r,1;F,G)\\
& + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G')\\
& + S(r,f) + S(r,g)\\
\leq\; & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + 2\overline{N}_L(r,1;F)\\
& + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\\
\leq\; & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}_{F>2}(r,1;G)\\
& + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\\
\leq\; & \frac{3}{2}\,\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \frac{1}{2}\,\overline{N}(r,0;F) + \overline{N}(r,0;G|\geq 2)\\
& + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;G') + \overline{N}_0(r,0;F') + S(r,f) + S(r,g)\\
\leq\; & \frac{3}{2}\,\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \frac{1}{2}\,\overline{N}(r,0;F) + \overline{N}(r,0;G|\geq 2)\\
& + N(r,0;G'|G\neq 0) + \overline{N}_0(r,0;F') + S(r,f) + S(r,g)\\
\leq\; & \frac{3}{2}\,\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + \overline{N}(r,0;F|\geq 2) + \frac{1}{2}\,\overline{N}(r,0;F) + N_2(r,0;G)\\
& + \overline{N}_0(r,0;F') + S(r,f) + S(r,g).
\end{aligned}$$

Hence using (3.8), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
& (d+m)T(r,f_1) \qquad (3.9)\\
\leq\ & T(r,F) + N_{k+2}(r,0;f_1^dP(f_1)) - N_2(r,0;F) + S(r,f_1)\\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;f_1^dP(f_1)) - N_2(r,0;F)\\
& -N_0(r,0;F^{'})\\
\leq\ & \frac{5}{2}\ \overline{N}(r,\infty,f_1) + 2\overline{N}(r,\infty;g_1) + N_2(r,0;F) + \frac{1}{2}\overline{N}(r,0;F) + N_{k+2}(r,0;f_1^dP(f_1))\\
& +N_2(r,0;G) - N_2(r,0;F) + S(r,f_1) + S(r,g_1)\\
\leq\ & \frac{5}{2}\overline{N}(r,\infty;f_1) + 2\overline{N}(r,\infty;g_1) + N_{k+2}(r,0;f_1^dP(f_1))) + \frac{1}{2}\overline{N}(r,0;F) + N_2(r,0;G)\\
& +S(r,f_1) + S(r,g_1)\\
\leq\ & \frac{5}{2}\overline{N}(r,\infty;f_1) + 2\overline{N}(r,\infty;g_1) + N_{k+2}(r,0;f_1^dP(f_1)) + k\ \overline{N}(r,\infty;g_1)\\
& +N_{k+2}(r,0;g_1^dP(g_1)) + \frac{1}{2}\{k\overline{N}(r,\infty;f_1) + \overline{N}_{k+1}(r,0;f_1^dP(f_1))\}\\
& +S(r,f_1) + S(r,g_1)\\
\leq\ & \frac{5+k}{2}\ \overline{N}(r,\infty;f_1) + (k+2)\overline{N}(r,\infty;g_1) + \frac{3k+5}{2}\ \overline{N}(r,0;f_1) + \frac{3}{2}\ T(r,P(f_1))\\
& +(k+2)\ \overline{N}(r,0;g_1) + T(r,P(g_1)) + S(r,f_1) + S(r,g_1)\\
\leq\ & (2k+5+\frac{3m}{2})\ T(r,f_1) + (2k+m+4)T(r,g_1) + S(r,f_1) + S(r,g_1)\\
\leq\ & (4k+\frac{5m}{2}+9)\ T(r) + S(r).
\end{aligned}$$

In a similar way we can obtain

$$(d+m)\ T(r,g_1) \leq (4k+\frac{5m}{2}+9)T(r) + S(r). \qquad (3.10)$$

Combining (3.9) and (3.10) we see that

$$(d+m)\ T(r) \leq (4k+\frac{5m}{2}+9)\ T(r) + S(r),$$

i.e

$$(d-4k-\frac{3m}{2}-9)\ T(r) \leq S(r). \qquad (3.11)$$

Since $d > 4k + \frac{3m}{2} + 9$, (3.11) leads to a contradiction.
Subcase 1.2 $l = 0$. Here (3.2) changes to

$$N_E^{1)}(r,1;F \mid= 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G). \qquad (3.12)$$

Using Lemmas 3, *11*, *12*, *13*, (3.1) and (3.12) we get

$$\begin{aligned}
& \overline{N}(r,1;F) \qquad (3.13)\\
\leq\; & N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
\leq\; & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\
& +\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\
& +S(r,f)+S(r,g)\\
\leq\; & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_L(r,1;F)\\
& +2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
\leq\; & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{F>1}(r,1;G)\\
& +\overline{N}_{G>1}(r,1;F)+\overline{N}_L(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})\\
& +\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
\leq\; & 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
& +N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
\leq\; & 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
& +N(r,0;G^{'}|G\neq 0)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
\leq\; & 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
& +\overline{N}(r,0;G)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Hence using (3.13), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
& (d+m)T(r,f_1) \qquad (3.14)\\
\leq\; & T(r,F)+N_{k+2}(r,0;f_1^dP(f_1))-N_2(r,0;F)+S(r,f_1)\\
\leq\; & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f_1^dP(f_1))-N_2(r,0;F)\\
& -N_0(r,0;F^{'})\\
\leq\; & 4\overline{N}(r,\infty,f_1)+3\overline{N}(r,\infty;g_1)+N_2(r,0;F)+2\overline{N}(r,0;F)+N_{k+2}(r,0;f_1^dP(f_1))\\
& +N_2(r,0;G)+\overline{N}(r,0;G)-N_2(r,0;F)+S(r,f_1)+S(r,g_1)\\
\leq\; & 4\overline{N}(r,\infty;f_1)+3\overline{N}(r,\infty;g_1)+N_{k+2}(r,0;f_1^dP(f_1))+2\overline{N}(r,0;F)+N_2(r,0;G)\\
& +\overline{N}(r,0;G)+S(r,f_1)+S(r,g_1)\\
\leq\; & 4\overline{N}(r,\infty;f_1)+3\overline{N}(r,\infty;g_1)+N_{k+2}(r,0;f_1^dP(f_1))+2k\overline{N}(r,\infty;f_1)\\
& +2N_{k+1}(r,0;f_1^dP(f_1))\\
& +k\,\overline{N}(r,\infty;g_1)+N_{k+2}(r,0;g_1^dP(g_1))+k\overline{N}(r,\infty;g_1)+\overline{N}_{k+1}(r,0;g_1^dP(g_1))\\
& +S(r,f_1)+S(r,g_1)\\
\leq\; & (2k+4)\,\overline{N}(r,\infty;f_1)+(2k+3)\overline{N}(r,\infty;g_1)+(3k+4)\overline{N}(r,0;f_1)+3T(r,P(f_1))\\
& +(2k+3)\,\overline{N}(r,0;g_1)+2T(r,P(g_1))+S(r,f_1)+S(r,g_1)\\
\leq\; & (5k+3m+8)\,T(r,f_1)+(4k+2m+6)T(r,g_1)+S(r,f_1)+S(r,g_1)\\
\leq\; & (9k+5m+14)T(r)+S(r).
\end{aligned}$$

In a similar way we can obtain

$$(d+m)\,T(r,g_1)\leq(9k+5m+14)T(r)+S(r). \tag{3.15}$$

Combining (3.14) and (3.15) we see that

$$(d+m)\,T(r)\leq(9k+5m+14)\,T(r)+S(r),$$

i.e.,

$$(d-9k-4m-14)T(r)\leq S(r). \tag{3.16}$$

Since $d>9k+4m+14$, (3.16) leads to a contradiction.
Case 2. Let $H\equiv 0$. Then noting that here $d_i\leq k$ and either (1.4) or (1.4) is satisfied, the theorem follows from Lemma 8.

Proof of Theorem 2. We omit the proof since the same can be carried out in the line of proof of Theorem 1.

Acknowledgements. This research work is supported by the Council Of Scientific and Industrial Research, Extramural Research Division, CSIR Complex, Pusa, New Delhi-110012, India, under the sanction project no. 25(0229)/14/EMR-II.

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Abhijit Banerjee
Department of Mathematics
University of Kalyani, Nadia
West Bengal 741235, India
email: *abanerjee_kal@yahoo.co.in, abanerjeekal@gmail.com.*

Sujoy Majumder
Department of Mathematics, Katwa College
Burdwan, 713130, India
email: *sujoy.katwa@gmail.com, sr.koshigram@gmail.com, smajumder05@yahoo.in*