Acta Universitatis Apulensis No. 44/2015
ISSN: 1582-5329 pp. 53-74
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.44.05

# DEMICLOSED PRINCIPLE AND CONVERGENCE THEOREMS FOR TWO GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

G. S. Saluja

Abstract. The purpose of this paper is to establish existence theorem, demiclosed principle, $\Delta$-convergence and some strong convergence theorems of modified $S$-iteration process for two generalized asymptotically nonexpansive mappings in the framework of CAT(0) spaces. The class of generalized asymptotically nonexpansive mappings is wider than the class of nonexpansive, asymptotically nonexpansive and asymptotically nonexpansive mappings in the intermediate sense. Our results extend and generalize the previous work from the current existing literature.

2010 *Mathematics Subject Classification*: 54H25, 54E40.

*Keywords:* Generalized asymptotically nonexpansive mapping, modified $S$-iteration scheme, common fixed point, strong convergence, $\Delta$-convergence, CAT(0) space.

## 1. Introduction

A metric space $X$ is a CAT(0) space if it is geodesically connected and if every geodesic triangle in $X$ is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space (see [14]). Other examples include pre-Hilbert spaces, $\mathbb{R}$-trees (see [4]) and Euclidean buildings (see [5]).

Fixed point theory in CAT(0) space has been first studied by Kirk (see [18, 19]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT($k$) space with $k \leq 0$ since any CAT($k$) space is a CAT($m$) space for every $m \geq k$ (see, Bridson and Haefliger [4], "Metric spaces of non-positive curvature").

The concept of $\Delta$-convergence in a general metric space was introduced by Lim [22]. In 2008, Kirk and Panyanak [20] used the notion of $\Delta$-convergence introduced by Lim [22] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [11] obtained $\Delta$-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the existence problem and the $\Delta$-convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping, total asymptotically nonexpansive mapping, asymptotically nonexpansive mappings in the intermediate sense and asymptotically quasi-nonexpansive type mappings through Picard, Mann [23], Ishikawa[15] and modified $S$-iteration process [2] have been rapidly developed in the framework of CAT(0) space and many papers have appeared in this direction (see, e.g.,[1, 8, 11, 17, 25, 26, 27, 28]).

In 2010, Nanjaras and Panyanak [24] proved the demiclosedness principle for asymptotically nonexpansive mappings and gave the $\Delta$-convergence theorem of the modified Mann iteration process for above mentioned mappings in a CAT(0) space. In 2014, Kumam, Saluja and Nashine [21] studied modified $S$-iteration process involving two mappings and investigate the existence and convergence theorems in the setting of CAT(0) spaces for a class of mappings which is wider than that of asymptotically nonexpansive mappings.

Recently, Saluja and Postolache [29] studied modified $S$-iteration process for two asymptotically nonexpansive mappings in the intermediate sense in the framework of CAT(0) spaces and investigate the existence and convergence theorems for the iteration scheme and mappings.

The purpose of this article is to establish existence theorem, demiclosed principle, $\Delta$-convergence and some strong convergence theorems of modified $S$-iteration process for two generalized asymptotically nonexpansive mappings in the framework of CAT(0) spaces. This class of mappings is wider than the class of nonexpansive, asymptotically nonexpansive and asymptotically nonexpansive mappings in the intermediate sense. Our results extend and generalize several known results from the current existing literature.

## 2. Preliminaries

Let $F(T) = \{x \in K : Tx = x\}$ denotes the set of fixed points of the mapping $T$. We begin with the following definitions.

**Definition 1.** *Let $(X, d)$ be a metric space and $K$ be its nonempty subset. Then $T\colon K \to K$ said to be*

*(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;*

*(2) asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;*

*(3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;*

*(4) semi-compact if for a sequence $\{x_n\}$ in $K$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$ as $k \to \infty$.*

*(5) a sequence $\{x_n\}$ in $K$ is called approximate fixed point sequence for $T$ (AFPS, in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive with the asymptotical sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [12] in 1972, is an important generalization of the class of nonexpansive mapping and they proved that if $C$ is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self mapping of $C$ has a fixed point.

$T$ is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,\, y \in K} (d(T^n x, T^n y) - d(x, y)) \leq 0. \tag{1}$$

Putting $c_n = \max\big\{0, \sup_{x,\, y\in K}(d(T^n x, T^n y) - d(x, y))\big\}$, we see that $c_n \to 0$ as $n \to \infty$. Then, (1) is reduced to the following:

$$d(T^n x, T^n y) \quad \leq \quad d(x, y) + c_n, \quad \forall\, x,\, y\, \in K,\, n \geq 1.$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck et al. [7] as a generalization of the class of asymptotically nonexpansive mappings. It is known that if $K$ is a nonempty closed convex and bounded subset of a real Hilbert space, then every asymptotically nonexpansive self mapping in the intermediate sense has a fixed point (see [33], for more details).

$T$ is said to be generalized asymptotically nonexpansive [3] if it is continuous and there exists a positive sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x,\, y \in K} (d(T^n x, T^n y) - k_n\, d(x, y)) \leq 0. \tag{2}$$

Putting $c_n = \max\big\{0, \sup_{x,\, y\in K}(d(T^n x, T^n y) - k_n\, d(x, y))\big\}$, we see that $c_n \to 0$ as $n \to \infty$. Then, (2) is reduced to the following:

$$d(T^n x, T^n y) \quad \leq \quad k_n\, d(x, y) + c_n, \quad \forall\, x,\, y\, \in K,\, n \geq 1.$$

We remark that if $k_n = 1$, then the class of generalized asymptotically nonexpansive mappings is reduced to the class of asymptotically nonexpansive mappings in the intermediate sense.

We now give the definition and some basic properties of CAT(0) space.

Let $(X, d)$ be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry, and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) *segment* joining $x$ and $y$. We say that $X$ is (i) a *geodesic space* if any two points of $X$ are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining $x$ to $y$.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [4]).

**CAT(0) space.** A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in $X$, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \quad \leq \quad d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{3}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [16]). If $x, y_1, y_2$ are points of a $CAT(0)$ space and $y_0$ is the midpoint of the geodesic segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \quad \leq \quad \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{4}$$

Inequality (4) is the $(CN)$ inequality of Bruhat and Tits [6]. The above inequality was extended in [11] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad & \alpha d^2(z, x) + (1-\alpha) d^2(z, y) \\ & -\alpha(1-\alpha) d^2(x, y), \end{aligned} \tag{5}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the $(CN)$ inequality (see, [4], p. 163). Moreover, if $X$ is a $CAT(0)$ space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d(z, x) + (1-\alpha) d(z, y), \tag{6}$$

for any $z \in X$ and $[x, y] = \Big\{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\Big\}$.

A subset $K$ of a $CAT(0)$ space $X$ is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

In the sequel, we need the following definitions and useful lemmas to prove our main results.

**Lemma 1.** *(See [24]) Let $X$ be a $CAT(0)$ space.*

*(i) For $x,\, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x,\, y]$ such that*

$$d(x,\, z) = t\, d(x,\, y) \quad \textit{and} \quad d(y,\, z) = (1-t)\, d(x,\, y).$$

*We denote such unique $z$ as $(1-t)x \oplus ty$.*

*(ii) For $x,\, y \in X$ and $t \in [0, 1]$, we have*

$$d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z).$$

Let $\{x_n\}$ be a bounded sequence in a closed convex subset $K$ of a CAT(0) space $X$. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \big\{x \in X : r(\{x_n\}) = r(x, \{x_n\})\big\}.$$

It is known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [[9], Proposition 7].

We now recall the definition of $\Delta$-convergence and weak convergence ($\rightharpoonup$) in CAT(0) space.

**Definition 2.** *(See [20]) A sequence $\{x_n\}$ in a CAT(0) space $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta$-$\lim_n x_n = x$ and call $x$ is the $\Delta$-limit of $\{x_n\}$.*

Recall that a bounded sequence $\{x_n\}$ in $X$ is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In the Banach space, it is known that, every bounded sequence has a regular subsequence ([13], Lemma 15.2).

**Lemma 2.** *(See [24], Lemma 2.4) Every bounded sequence in a complete CAT(0) space has a $\Delta$-convergent subsequence.*

**Lemma 3.** *(See [11], Lemma 2.8) If $\{x_n\}$ is a bounded sequence in a CAT(0) space $X$ with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

**Lemma 4.** *(See [10], Proposition 2.1) If $K$ is a closed convex subset of a complete CAT(0) space $X$ and $\{x_n\}$ is a bounded sequence in $K$, then the asymptotic center of $\{x_n\}$ is in $K$.*

**Lemma 5.** *(See [24], Proposition 3.12) Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $X$ and let $K$ be a closed convex subset of $X$ which contains $\{x_n\}$. Then*
*(i) $\Delta\text{-}\lim_{n\to\infty} x_n = x$ implies $x_n \rightharpoonup x$,*
*(ii) the converse is true if $\{x_n\}$ is regular.*

**Lemma 6.** *(See [32]) Suppose that $\{a_n\}$, $\{b_n\}$ and $\{r_n\}$ are sequences of nonnegative numbers such that $a_{n+1} \leq (1 + b_n)a_n + r_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

**Example 1.** *Let $X = K = [0, 1]$ with the usual metric $d$, $\{x_n\} = \{\frac{1}{n}\}$, $\{w_{n_m}\} = \{\frac{1}{(m+1)n}\}$, for all $n$, $m \in \mathbb{N}$ are sequences in $K$. Then $A(\{x_n\}) = \{0\}$ and $A(\{w_{n_m}\}) = \{0\}$. This shows that $\{x_n\}$ $\Delta$-converges to $0$, that is, $\Delta\text{-}\lim_{n\to\infty} x_n = 0$. The sequence $\{x_n\}$ also converges strongly to $0$, that is, $|x_n - 0| \to 0$ as $n \to \infty$. Also it is weakly convergent to $0$, that is, $x_n \rightharpoonup 0$ as $n \to \infty$, by Lemma 5. Thus, we conclude that*

$$\textit{strong convergence} \quad \Rightarrow \quad \Delta\textit{-convergence} \quad \Rightarrow \quad \textit{weak convergence},$$

*but the converse is not true in general.*

The following example shows that if the sequence $\{x_n\}$ is weakly convergent then it is not $\Delta$-convergent.

**Example 2.** *[24] Let $X = \mathbb{R}$, $d$ be the usual metric on $X$, $K = [-1, 1]$, $\{x_n\} = \{1, -1, 1, -1, \dots\}$, $\{u_n\} = \{-1, -1, -1, \dots\}$ and $\{v_n\} = \{1, 1, 1, \dots\}$. Then $A(\{x_n\}) = A_K(\{x_n\}) = \{0\}$, $A(\{u_n\}) = \{-1\}$ and $A(\{v_n\}) = \{1\}$. This shows that $\{x_n\} \rightharpoonup 0$ but it does not have a $\Delta$-limit.*

**Algorithm 1.** The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n)x_n, \quad n \geq 1, \end{cases} \tag{7}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in (0,1), is called a modified Mann iterative sequence (see [23]).

**Algorithm 2.** The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n)x_n, \\ \quad\quad y_n = \beta_n T^n x_n \oplus (1-\beta_n)x_n, \quad n \geq 1, \end{cases} \tag{8}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1], is called modified Ishikawa iterative sequence (see [15]).

**Algorithm 3.** The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n)T^n x_n, \\ \quad\quad y_n = \beta_n T^n x_n \oplus (1-\beta_n)x_n, \quad n \geq 1, \end{cases} \tag{9}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1), is called modified S-iterative sequence (see [2]).

Inspired and motivated by the works of Agarwal et al. [2], Başarir and Şahin [25] and some others, we modify iteration scheme (9) for two mappings in a CAT(0) space as follows.

Let $K$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $S, T \colon K \to K$ be two generalized asymptotical nonexpansive mappings with $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, \\ \quad\quad y_n = (1-\beta_n)S^n x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n$, $\beta_n \leq 1$ for all $n \geq 1$.

**Remark 1.** *If we take $S = I$, where $I$ is the identity mapping and $\beta_n = 0$ for all $n \geq 1$, then (10) reduces to the modified Mann iteration process (7).*

In this paper, we study the modified two-step iteration process (10) involving two generalized asymptotical nonexpansive mappings and investigate the strong and $\Delta$-convergence theorems for the mappings and iteration scheme in the framework of CAT(0) spaces. Our results generalize, unify and extend several comparable results in the existing literature.

## 3. Main Results

**Existence theorem and Demiclosed principle**

Denote by $\mathbb{N}$ the set of positive integers. Put

$$c_n = \max\Big\{0, \sup_{x,\, y\in K}\Big(d(T^n x, T^n y) - k_n d(x,y)\Big)\Big\}.$$

**Theorem 7.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. If $T\colon K \to K$ be a generalized asymptotically nonexpansive mapping, then $T$ has a fixed point.*

*Proof.* Fix $x \in K$. We can consider the sequence $\{T^n x\}_{n=1}^{\infty}$ as a bounded sequence in $K$. Let $\Phi\colon K \to [0,\infty)$ be a function defined by

$$\Phi(u) = \limsup_{n\to\infty} d(T^n x, u) \quad \text{for all} \quad u \in K.$$

Then there exists $z \in K$ such that $\Phi(z) = \inf\{\Phi(u) : u \in K\} = \Phi_0$. Since $T$ is a generalized asymptotically nonexpansive mapping, for each $n, m \in \mathbb{N}$, we have

$$d(T^{n+m}x, T^m u) \le k_m\, d(T^n x, u) + c_m.$$

On taking limit as $n \to \infty$, we obtain

$$\Phi(T^m u) \le k_m\, \Phi(u) + c_m \tag{11}$$

for any $m \in \mathbb{N}$.

Then by (11), for any $n \in \mathbb{N}$, we have

$$\Phi(T^n z) \le k_n\, \Phi(z) + c_n = k_n\, \Phi_0 + c_n. \tag{12}$$

In view of inequality (4), we obtain

$$\begin{aligned} d\Big(T^n x, \frac{T^m z \oplus T^h z}{2}\Big)^2 &\le \frac{1}{2} d(T^n x, T^m z)^2 + \frac{1}{2} d(T^n x, T^h z)^2 \\ &\quad - \frac{1}{4} d(T^m z, T^h z)^2 \end{aligned}$$

which on taking limit as $n \to \infty$ gives

$$\begin{aligned} \Phi_0^2 &\le \Phi\Big(\frac{T^m z \oplus T^h z}{2}\Big)^2 \\ &\le \frac{1}{2}\Phi(T^m z)^2 + \frac{1}{2}\Phi(T^h z)^2 - \frac{1}{4} d(T^m z, T^h z)^2. \end{aligned} \tag{13}$$

Using (12) in (13), we have

$$d(T^m z, T^h z)^2 \leq 2(k_m \Phi_0 + c_m)^2 + 2(k_h \Phi_0 + c_h)^2 - 4\Phi_0^2.$$

As $T$ is a generalized asymptotically noonexpansive mapping, so we have $\limsup_{m,h\to\infty} d(T^m z, T^h z) \leq 0$. Therefore, $\{T^n z\}_{n=1}^{\infty}$ is a Cauchy sequence in $K$ and hence converges to some point $v^* \in K$. Since $T$ is continuous,

$$T(v^*) = T\Big(\lim_{n\to\infty} T^n(z)\Big) = \lim_{n\to\infty} T^{n+1}(z) = v^*.$$

This shows that $T$ has a fixed point in $K$. This completes the proof.

**Theorem 8.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. If $T\colon K \to K$ be a generalized asymptotically nonexpansive mapping, then $F(T)$ is closed and convex.*

*Proof.* As $T$ is continuous, so $F(T)$ is closed. In order to prove $F(T)$ is convex, it is sufficient to show that $\frac{x\oplus y}{2} \in F(T)$ whenever $x, y \in F(T)$. Set $f = \frac{x\oplus y}{2}$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(T^n f, f)^2 &= d(T^n f, \frac{x \oplus y}{2})^2 \\ &\leq \frac{1}{2} d(x, T^n f)^2 + \frac{1}{2} d(y, T^n f)^2 - \frac{1}{4} d(x,y)^2. \end{aligned} \tag{14}$$

In view of Lemma 1(ii), we obtain

$$\begin{aligned} d(x, T^n f)^2 &= d(T^n x, T^n f)^2 \\ &\leq [k_n(f,x) + c_n]^2 \\ &= \Big\{k_n\, d\Big(\frac{x\oplus y}{2}, x\Big) + c_n\Big\}^2 \\ &\leq \Big\{\frac{k_n}{2} d(x,y) + c_n\Big\}^2. \end{aligned} \tag{15}$$

Similarly,

$$d(y, T^n f)^2 \leq \Big\{\frac{k_n}{2} d(x,y) + c_n\Big\}^2. \tag{16}$$

From (14), (15) and (16), we get

$$d(T^n f, f)^2 \leq \frac{(k_n^2 - 1)}{4} d(x,y)^2 + c_n(c_n + k_n d(x,y))$$

for any $n \in \mathbb{N}$. Since $c_n \to 0$ and $k_n \to 1$ as $n \to \infty$, hence $\lim_{n\to\infty} T^n f = f$ and $Tf = f \in F(T)$. This completes the proof.

**Theorem 9.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. If $T\colon K \to K$ be a generalized asymptotically nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in $K$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{x_n\} \rightharpoonup w$, then $Tw = w$.*

*Proof.* Define $\Phi(x) = \limsup_{n\to\infty} d(T^m x_n, x)$ for all $x \in K$ and $m \in \mathbb{N}$. Then as observed in (11), we have

$$\Phi(T^m w) \quad \leq \quad k_m\,\Phi(w) + c_m \quad \text{for all } x \in K \text{ and } m \in \mathbb{N}.$$

Hence

$$\limsup_{m\to\infty} \Phi(T^m w) \quad \leq \quad \Phi(w). \tag{17}$$

In view of inequality (4), we have

$$\begin{aligned} d\Big(T^n x_n, \frac{w \oplus T^m w}{2}\Big)^2 \quad &\leq \quad \frac{1}{2} d(T^m x_n, w)^2 + \frac{1}{2} d(T^m x_n, T^m w)^2 \\ &\quad -\frac{1}{4} d(w, T^m w)^2 \end{aligned}$$

for all $n, m \in \mathbb{N}$. On taking limit as $n \to \infty$, we get

$$\Phi\Big(\frac{w \oplus T^m w}{2}\Big)^2 \quad \leq \quad \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(T^m w)^2 - \frac{1}{4} d(w, T^m w)^2$$

for any $m \in \mathbb{N}$. Since $\Delta - \lim_{n\to\infty} x_n = w$, so $A(\{x_n\}) = w$, letting $n \to \infty$, we get that

$$\Phi(w)^2 \quad \leq \quad \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(T^m w)^2 - \frac{1}{4} d(w, T^m w)^2.$$

That is,

$$\begin{aligned} 4\Phi(w)^2 \quad &\leq \quad \Phi\Big(\frac{w \oplus T^m w}{2}\Big)^2 \\ &\leq \quad 2\Phi(w)^2 + 2\Phi(T^m w)^2 - d(w, T^m w)^2 \end{aligned}$$

for any $m \in \mathbb{N}$, which implies that

$$d(w, T^m w)^2 \quad \leq \quad 2\Phi(T^m w)^2 - 2\Phi(w)^2. \tag{18}$$

By (17) and (18), we have

$$\lim_{m\to\infty} d(w, T^m w) = 0$$

and $Tw = w$. This completes the proof.

In the light of Lemma 5, we get the following result from Theorem 9.

**Corollary 10.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and $T\colon K \to K$ be a generalized asymptotically nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in $K$ $\Delta$-converges to $x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in K$ and $Tx = x$.*

Now, we prove the following lemma using modified two-step iteration scheme for two generalized asymptotically nonexpansive mappings (10) needed in the sequel.

**Lemma 11.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and let $S, T\colon K \to K$ be two generalized asymptotical nonexpansive mappings with sequences $\{k'_n\}, \{k''_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (10). Put*

$$A_n = \max\Big\{0, \sup_{x,\, y\in K}\Big(d(S^n x, S^n y) - k'_n d(x,y)\Big)\Big\} \tag{19}$$

*and*

$$B_n = \max\Big\{0, \sup_{x,\, y\in K}\Big(d(T^n x, T^n y) - k''_n d(x,y)\Big)\Big\} \tag{20}$$

*such that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$. Then*

*(i)*$\lim_{n\to\infty} d(x_n, p)$ *exists for each $p \in F(S,T)$.*
*(ii)* $\lim_{n\to\infty} d(x_n, F(S,T))$ *exists.*

*Proof.* Let $p \in F(S,T)$ and let $k_n = \max\{k'_n, k''_n\}$ with $k_n \to 1$ as $n \to \infty$. From (10), (19), (20) and Lemma 1(ii), we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)S^n x_n \oplus \beta_n T^n x_n, p) \\ &\leq (1-\beta_n) d(S^n x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1-\beta_n)[k'_n\, d(x_n, p) + A_n] + \beta_n [k''_n\, d(x_n, p) + B_n] \\ &\leq (1-\beta_n)[k_n\, d(x_n, p) + A_n] + \beta_n [k_n\, d(x_n, p) + B_n] \\ &\leq k_n\, d(x_n, p) + A_n + B_n. \end{aligned} \tag{21}$$

Again using (10), (19)-(21) and Lemma 1(ii), we have

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, p) \\ &\leq (1-\alpha_n)d(T^n x_n,p)+\alpha_n d(T^n y_n,p) \\ &\leq (1-\alpha_n)[k_n'' \, d(x_n,p)+B_n]+\alpha_n[k_n' \, d(y_n,p)+A_n] \\ &\leq (1-\alpha_n)[k_n \, d(x_n,p)+B_n]+\alpha_n[k_n \, d(y_n,p)+A_n] \\ &= (1-\alpha_n)k_n d(x_n,p)+\alpha_n k_n d(y_n,p)+(1-\alpha_n)B_n+\alpha_n A_n \\ &\leq (1-\alpha_n)k_n d(x_n,p)+\alpha_n k_n[k_n d(x_n,p)+A_n+B_n] \\ &\quad +(1-\alpha_n)B_n+\alpha_n A_n \\ &\leq (1-\alpha_n)k_n^2 d(x_n,p)+\alpha_n k_n[k_n d(x_n,p)+A_n+B_n] \\ &\quad +(1-\alpha_n)k_n B_n+\alpha_n k_n A_n \\ &\leq k_n^2 d(x_n,p)+2\alpha_n k_n A_n+k_n B_n \\ &= [1+(k_n^2-1)]d(x_n,p)+2\alpha_n k_n A_n+k_n B_n \\ &= [1+W_n]d(x_n,p)+V_n \end{aligned} \tag{22}$$

where $W_n = (k_n^2-1) = (k_n+1)(k_n-1)$ and $V_n = 2\alpha_n k_n A_n + k_n B_n$. Since $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty} A_n<\infty$ and $\sum_{n=1}^{\infty} B_n<\infty$, it follows that $\sum_{n=1}^{\infty} W_n<\infty$ and $\sum_{n=1}^{\infty} V_n<\infty$.

Taking infimum over all $p \in F(S,T)$, we have

$$d(x_{n+1},F(S,T)) \leq [1+W_n]d(x_n,F(S,T))+V_n. \tag{23}$$

Since $\sum_{n=1}^{\infty} W_n<\infty$ and $\sum_{n=1}^{\infty} V_n<\infty$, it follows from Lemma 6, (22) and (23) that $\lim_{n\to\infty} d(x_n,p)$ and $\lim_{n\to\infty} d(x_n,F(S,T))$ both exist.

**Lemma 12.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and let $S$, $T\colon K\to K$ be two generalized asymptotical nonexpansive mappings with sequences $\{k_n'\},\{k_n''\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $F(S,T)=F(S)\cap F(T)\neq\emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (10) and $A_n$ and $B_n$ be taken as in Lemma 11. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[a,b]$ for some $a,b\in(0,1)$. If $d(x,Sx)\leq d(Tx,Sx)$ for all $x\in K$, then $\lim_{n\to\infty} d(x_n,Sx_n)=0$ and $\lim_{n\to\infty} d(x_n,Tx_n)=0$.*

*Proof.* Using (10), (19), (20) and (5), we have

$$\begin{aligned} d^2(y_n,p) &= d^2((1-\beta_n)S^n x_n \oplus \beta_n T^n x_n, p) \\ &\le \beta_n d^2(T^n x_n,p) + (1-\beta_n) d^2(S^n x_n,p) \\ &\quad -\beta_n(1-\beta_n) d^2(T^n x_n, S^n x_n) \\ &\le \beta_n [k_n'' d(x_n,p)+B_n]^2 + (1-\beta_n)[k_n' d(x_n,p)+A_n]^2 \\ &\quad -\beta_n(1-\beta_n) d^2(T^n x_n, S^n x_n) \\ &\le \beta_n [k_n d(x_n,p)+B_n]^2 + (1-\beta_n)[k_n d(x_n,p)+A_n]^2 \\ &\quad -\beta_n(1-\beta_n) d^2(T^n x_n, S^n x_n) \\ &\le k_n^2 d^2(x_n,p) + P_n + Q_n - \beta_n(1-\beta_n) d^2(T^n x_n, S^n x_n) \qquad (24) \end{aligned}$$

where $P_n = B_n^2 + 2k_n B_n d(x_n,p)$ and $Q_n = A_n^2 + 2k_n A_n d(x_n,p)$, since by hypothesis $\sum_{n=1}^{\infty} An < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, it follows that $\sum_{n=1}^{\infty} P_n < \infty$ and $\sum_{n=1}^{\infty} Q_n < \infty$. Again using (10), (19), (20), (24) and (5), we have

$$\begin{aligned} d^2(x_{n+1},p) &= d^2((1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, p) \\ &\le \alpha_n d^2(S^n y_n,p) + (1-\alpha_n) d^2(T^n x_n,p) \\ &\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \\ &\le \alpha_n [k_n' d(y_n,p)+A_n]^2 + (1-\alpha_n)[k_n'' d(x_n,p)+B_n]^2 \\ &\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \\ &\le \alpha_n [k_n d(y_n,p)+A_n]^2 + (1-\alpha_n)[k_n d(x_n,p)+B_n]^2 \\ &\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \\ &\le \alpha_n [k_n^2 d^2(y_n,p)+R_n] + (1-\alpha_n)[k_n^2 d^2(x_n,p)+S_n] \\ &\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \\ &\le \alpha_n k_n^2 d^2(y_n,p) + (1-\alpha_n) k_n^2 d^2(x_n,p) + \alpha_n R_n + (1-\alpha_n) S_n \\ &\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \\ &\le \alpha_n k_n^2 [k_n^2 d^2(x_n,p) + P_n + Q_n - \beta_n(1-\beta_n) d^2(T^n x_n, S^n x_n)] \\ &\quad +(1-\alpha_n) k_n^4 d^2(x_n,p) + \alpha_n R_n + (1-\alpha_n) S_n \\ &\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \\ &\le k_n^4 d^2(x_n,p) - \alpha_n\beta_n(1-\beta_n) k_n^2 d^2(T^n x_n, S^n x_n) \\ &\quad +(P_n+Q_n+R_n+S_n) k_n^2 - \alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \\ &= k_n^4 d^2(x_n,p) - \alpha_n\beta_n(1-\beta_n) d^2(T^n x_n, S^n x_n) \\ &\quad +T_n - \alpha_n(1-\alpha_n) d^2(S^n y_n, T^n x_n) \qquad (25) \end{aligned}$$

where $R_n = A_n^2 + 2k_n A_n d(y_n,p)$, $S_n = B_n^2 + 2k_n B_n d(x_n,p)$ and $T_n = (P_n + Q_n + R_n + S_n)k_n^2$, since by hypothesis $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} B_n < \infty$, $\sum_{n=1}^{\infty} P_n < \infty$,

$\sum_{n=1}^{\infty} Q_n < \infty$, it follows that $\sum_{n=1}^{\infty} R_n < \infty$, $\sum_{n=1}^{\infty} S_n < \infty$ and $\sum_{n=1}^{\infty} T_n < \infty$. This implies that

$$\begin{aligned} d^2(T^n x_n, S^n x_n) &\leq \frac{1}{\alpha_n \beta_n (1-\beta_n)}[k_n^4 d^2(x_n, p) - d^2(x_{n+1}, p)] \\ &\quad + \frac{T_n}{\alpha_n \beta_n (1-\beta_n)} \\ &\leq \frac{1}{a^2(1-b)}[k_n^4 d^2(x_n, p) - d^2(x_{n+1}, p)] \\ &\quad + \frac{T_n}{a^2(1-b)} \end{aligned} \tag{26}$$

and

$$\begin{aligned} d^2(S^n y_n, T^n x_n) &\leq \frac{1}{\alpha_n (1-\alpha_n)}[k_n^4 d^2(x_n, p) - d^2(x_{n+1}, p)] \\ &\quad + \frac{T_n}{\alpha_n (1-\alpha_n)} \\ &\leq \frac{1}{a(1-b)}[k_n^4 d^2(x_n, p) - d^2(x_{n+1}, p)] \\ &\quad + \frac{T_n}{a(1-b)}. \end{aligned} \tag{27}$$

Since $T_n \to 0$ and $k_n \to 1$ as $n \to \infty$ and $d(x_n, p)$ is convergent, therefore on taking limit as $n \to \infty$ in (26) and (27), we get

$$\lim_{n\to\infty} d(T^n x_n, S^n x_n) = 0 \tag{28}$$

and

$$\lim_{n\to\infty} d(S^n y_n, T^n x_n) = 0. \tag{29}$$

Now

$$\begin{aligned} d(T^n x_n, x_n) &\leq d(T^n x_n, S^n x_n) + d(S^n x_n, x_n) \\ &\leq d(T^n x_n, S^n x_n) + d(S^n x_n, T^n x_n) \\ &= 2d(T^n x_n, S^n x_n) \to 0 \text{ as } n \to \infty, \end{aligned} \tag{30}$$

and

$$d(S^n x_n, x_n) \leq d(S^n x_n, T^n x_n) + d(T^n x_n, x_n)$$

by (22) and (24), we obtain

$$\lim_{n\to\infty} d(S^n x_n, x_n) = 0. \tag{31}$$

Again note that

$$\begin{aligned} d(x_{n+1}, T^n x_n) &= d((1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, T^n x_n) \\ &\le (1-\alpha_n)d(T^n x_n, T^n x_n) + \alpha_n d(S^n y_n, T^n x_n) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{32}$$

By (30) and (32), we have

$$\begin{aligned} d(x_{n+1}, x_n) &\le d(x_{n+1}, T^n x_n) + d(T^n x_n, x_n) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{33}$$

Let $\rho_n = d(T^n x_n, x_n)$, by (30), we have $\rho_n \to 0$ as $n \to \infty$. Now, we have

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + k''_{n+1} d(x_{n+1}, x_n) + B_{n+1} \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + k_{n+1} d(x_{n+1}, x_n) + B_{n+1} \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\le \rho_{n+1} + (1 + k_{n+1}) d(x_n, x_{n+1}) + B_{n+1} + d(T^{n+1}x_n, Tx_n) \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{34}$$

by (30), (33), $B_{n+1} \to 0$ and uniform continuity of $T$. Similarly, we can prove that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{35}$$

This completes the proof.

Now we prove the $\Delta$-convergence and strong convergence results.

**Theorem 13.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and let $S, T \colon K \to K$ be two generalized asymptotical nonexpansive mappings with sequences $\{k'_n\}, \{k''_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (10) and $A_n$ and $B_n$ be taken as in Lemma 11. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ $\Delta$-converges to a point of $F(S,T)$.*

*Proof.* It follows from Lemma 12 that $\lim_{n\to\infty} d(Sx_n, x_n) = 0$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Now let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n) \subseteq F$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2 and 3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$ and $\lim_{n\to\infty} d(v_n, Sv_n) = 0$, then by Corollary 10, $v \in F(T)$ and $v \in F(S)$ and so $v \in F(S,T)$. By Lemma 11, $\lim_{n\to\infty} d(x_n, F(S,T))$ exists so by Lemma 3, $v = u$, i.e., $\omega_w(x_n) \subseteq F(S,T)$.

To show that $\{x_n\}$ $\Delta$-converges to a point in $F(S,T)$, it is sufficient to show that $\omega_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in \omega_w(x_n) \subseteq F(S,T)$ and by Lemma 11, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 3, we have $x = w \in F(S,T)$. Thus $\omega_w(x_n) = \{x\}$. This shows that $\{x_n\}$ $\Delta$-converges to a point in $F(S,T)$. This completes the proof.

**Theorem 14.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and let $S$, $T\colon K \to K$ be two generalized asymptotical nonexpansive mappings with sequences $\{k'_n\}, \{k''_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (10) and $A_n$ and $B_n$ be taken as in Lemma 11. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[a,b]$ for some $a, b \in (0,1)$. If $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(S,T)) = 0$, where $d(x, F(S,T)) = \inf_{p\in F(S,T)} d(x,p)$, then the sequence $\{x_n\}$ converges strongly to a point in $F(S,T)$.*

*Proof.* From (23) of Lemma 11, we have

$$d(x_{n+1}, p) \quad \leq \quad [1 + W_n] d(x_n, F(S,T)) + V_n$$

where $p \in F(S,T)$. Since $\sum_{n=1}^{\infty} W_n < \infty$ and $\sum_{n=1}^{\infty} V_n < \infty$, by Lemma 6 and $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(S,T)) = 0$ gives that

$$\lim_{n\to\infty} d(x_n, F(S,T)) = 0. \tag{36}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in $K$. With the help of inequality

$1+x \leq e^x$, $x \geq 0$. For any integer $m \geq 1$, therefore from (22), we have

$$\begin{aligned} d(x_{n+m},p) &\leq (1+W_{n+m-1})d(x_{n+m-1},p)+V_{n+m-1} \\ &\leq e^{W_{n+m-1}}d(x_{n+m-1},p)+V_{n+m-1} \\ &\leq e^{W_{n+m-1}}[e^{W_{n+m-2}}d(x_{n+m-2},p)+V_{n+m-2}]+V_{n+m-1} \\ &\leq e^{(W_{n+m-1}+W_{n+m-2})}d(x_{n+m-2},p)+e^{W_{n+m-1}}[V_{n+m-2}+V_{n+m-1}] \\ &\leq \dots \\ &\leq \big(e^{\sum_{k=n}^{n+m-1}W_k}\big)d(x_n,p)+\big(e^{\sum_{k=n}^{n+m-1}W_k}\big)\sum_{k=n}^{n+m-1}V_k \\ &\leq \big(e^{\sum_{n=1}^{\infty}W_n}\big)d(x_n,p)+\big(e^{\sum_{n=1}^{\infty}W_n}\big)\sum_{k=n}^{n+m-1}V_k \\ &= M\,d(x_n,p)+M\sum_{k=n}^{n+m-1}V_k, \end{aligned} \tag{37}$$

where $M = e^{\sum_{n=1}^{\infty}W_n} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$ and $\sum_{n=1}^{\infty} V_n < \infty$, therefore for any $\varepsilon > 0$, there exists a natural number $n_0$ such that $d(x_n, F(S,T)) < \varepsilon/8M$ and $\sum_{k=n}^{n+m-1} V_k < \varepsilon/2M$ for all $m,n \geq n_0$. So, we can find $p^* \in F(S,T)$ such that $d(x_{n_0}, p^*) < \varepsilon/4M$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},p^*)+d(x_n,p^*) \\ &\leq M\,d(x_{n_0},p^*)+M\sum_{k=n}^{n+m-1}V_k \\ &\quad +M\,d(x_{n_0},p^*) \\ &= 2M\,d(x_{n_0},p^*)+M\sum_{k=n}^{n+m-1}V_k \\ &< 2M.\frac{\varepsilon}{4M}+M.\frac{\varepsilon}{2M}=\varepsilon. \end{aligned} \tag{38}$$

This proves that $\{x_n\}$ is a Cauchy sequence in $K$. Thus, the completeness of $X$ implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = q$. Since $K$ is closed, therefore $q \in K$. Next, we show that $q \in F(S,T)$. Since $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$ we get $d(q, F(S,T)) = 0$, closedness of $F(S,T)$ gives that $q \in F(S,T)$. Thus $\{x_n\}$ converges strongly to a point in $F(S,T)$. This completes the proof.

**Theorem 15.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and let $S, T\colon K \to K$ be two generalized asymptotical nonexpansive mappings with sequences $\{k'_n\}, \{k''_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (10) and $A_n$ and $B_n$ be taken as in Lemma 11. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[a, b]$ for some $a, b \in (0, 1)$. If either $S$ or $T$ is semi-compact, then the the sequence $\{x_n\}$ converges strongly to a point of $F(S,T)$.*

*Proof.* Suppose that $T$ is semi-compact. By Lemma 12, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. Now Lemma 12 guarantees that $\lim_{n_j\to\infty} d(x_{n_j}, Tx_{n_j}) = 0$ and so $d(p, Tp) = 0$. Similarly, we can show that $d(p, Sp) = 0$. Thus $p \in F(S,T)$. By (22), we have

$$d(x_{n+1}, p) \leq [1 + W_n]d(x_n, p) + V_n.$$

Since $\sum_{n=1}^{\infty} W_n < \infty$ and $\sum_{n=1}^{\infty} V_n < \infty$, by Lemma 6, $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_j} \to p \in F(S,T)$ gives that $x_n \to p \in F(S,T)$. This shows that $\{x_n\}$ converges strongly to a point of $F(S,T)$. This completes the proof.

We recall the following definition.

A mapping $T\colon K \to K$, where $K$ is a subset of a metric space $(X, d)$, is said to satisfy condition $(A)$ [31] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$ where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T) \neq \emptyset\}$.

We modify this definition for two mappings.

Two mappings $S, T\colon K \to K$, where $K$ is a subset of a metric space $(X, d)$, is said to satisfy condition $(B)$ if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $a_1\, d(x, Sx) + a_2\, d(x, Tx) \geq f(d(x, F(S,T)))$ for all $x \in K$ where $d(x, F(S,T)) = \inf\{d(x, p) : p \in F(S,T) \neq \emptyset\}$ and $a_1$ and $a_2$ are two nonnegative real numbers such that $a_1 + a_2 = 1$. It is to be noted that Condition (B) is weaker than compactness of the domain $K$.

**Remark 2.** *Condition $(B)$ reduces to condition $(A)$ when $S = T$.*

As an application of Theorem 14, we establish some strong convergence results as follows.

**Theorem 16.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and let $S, T\colon K \to K$ be two generalized asymptotical nonexpansive mappings with sequences $\{k'_n\}, \{k''_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (10) and $A_n$ and $B_n$ be taken as in Lemma 11. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence* *in $[a,b]$ for some $a,b \in (0,1)$. If $S$ and $T$ satisfy condition $(B)$, then the sequence $\{x_n\}$ converges strongly to a point of $F(S,T)$.*

*Proof.* By Lemma 12, we know that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{39}$$

From condition $(B)$ and (39), we get

$$\lim_{n\to\infty} f(d(x_n, F(S,T))) \leq a_1. \lim_{n\to\infty} d(x_n, Sx_n) + a_2. \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

i.e.,

$$\lim_{n\to\infty} f(d(x_n, F(S,T))) = 0$$

.

Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore we obtain

$$\lim_{n\to\infty} d(x_n, F(S,T)) = 0.$$

The conclusion now follows from Theorem 14. This completes the proof.

**Theorem 17.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$ and let $S,T\colon K \to K$ be two uniformly continuous asymptotical nonexpansive mappings with sequences $\{k'_n\}, \{k''_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process (10). Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(S,T)) = 0$, where $d(x, F(S,T)) = \inf_{p\in F(S,T)} d(x,p)$, then the sequence $\{x_n\}$ converges strongly to a point of $F(S,T)$.*

*Proof.* Let $p \in F(S,T)$ and let $k_n = \max\{k'_n, k''_n\}$ with $k_n \to 1$ as $n \to \infty$. Since $S$ is uniformly continuous and asymptotically nonexpansive mapping, we know that there exist a sequence $\{k'_n\} \subset [1,\infty)$ with $k'_n \to 1$ as $n \to \infty$ such that

$$\begin{aligned} d(S^n x, S^n y) &\leq k'_n\, d(x,y) \\ &\leq k_n\, d(x,y), \ \forall\, x,\, y \in K,\, n \geq 1. \end{aligned}$$

This implies that

$$d(S^n x, S^n y) - k_n\, d(x,y) \quad \leq \quad 0, \ \forall\, x,\, y \in K,\, n \geq 1.$$

Therefore we have

$$\limsup_{n\to\infty}\Big\{\sup_{x,\,y\in K}\Big(d(S^n x, S^n y) - k_n\, d(x,y)\Big)\Big\} \leq 0.$$

This implies that $S$ is a generalized asymptotically nonexpansive mapping. By similar fashion, we can show that $T$ is also a generalized asymptotically nonexpansive mapping. Thus the conclusion of Theorem 17 follows from Theorem 14 immediately. This completes the proof.

**Example 3.** *Let $E = \mathbb{R}$, $K = [-1, 1]$ and $T\colon K \to K$ be a mapping defined by*

$$T(x) = \frac{x}{2}, \quad \text{if } x \in [-1, 1].$$

*Thus $T$ is a nonexpansive mapping and hence it is asymptotically nonexpansive mapping with constant sequence $\{1\}$. Also $T$ is uniformly continuous on $[-1, 1]$. Thus $T$ is asymptotically nonexpansive mapping in the intermediate sense and hence it is generalized asymptotically nonexpansive mapping.*

## 4. Conclusion

In this paper, we establish a $\Delta$ convergence and some strong convergence theorems using iteration scheme (10) which contains modified Mann iteration scheme for a wider class of nonexpansive, asymptotically nonexpansive and asymptotically nonexpansive mappings in the intermediate sense in the setting of CAT(0) spaces. The results presented in this paper extend and generalize several known results from the previous work given in the current existing literature.

## References

[1] M. Abbas, Z. Kadelburg, D.R. Sahu, *Fixed point theorems for Lipschitzian type mappings in CAT(0) spaces*, Math. Comput. Model. 55 (2012), 1418-1427.

[2] R. P. Agarwal, Donal O'Regan, D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, Nonlinear Convex Anal. 8, 1 (2007), 61-79.

[3] R.P. Agarwal, X. Qin, S.M. Kang, *An implicit iterative algorithm with errors for two families of generalized asymptotically nonexpansive mappings*, Fixed Point Theory Appl. 2011:58 (2011).

[4] M.R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Vol. 319 of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany, 1999.

[5] K.S. Brown, *Buildings*, Springer, New York, NY, USA, 1989.

[6] F. Bruhat, J. Tits, *"Groups reductifs sur un corps local"*, Institut des Hautes Etudes Scientifiques. Publications Mathematiques, 41 (1972), 5-251.

[7] R.E. Bruck, T. Kuczumow, S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform opial property*, Colloq Math. 65 (1993), 169-179.

[8] S.S. Chang, L. Wang, H.W. Joesph Lee, C.K. Chan, L. Yang, *Demiclosed principle and* $\Delta$*-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces*, Appl. Math. Comput. 219, 5 (2012), 2611-2617.

[9] S. Dhompongsa, W.A. Kirk, B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal. 65, 4 (2006), 762-772.

[10] S. Dhompongsa, W.A. Kirk, B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. 8, 1 (2007), 35-45.

[11] S. Dhompongsa, B. Panyanak, *On* $\triangle$*-convergence theorem in CAT(0) spaces*, Comput. Math. Appl. 56, 10 (2008), 2572-2579.

[12] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35 (1972), 171-174.

[13] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[14] K. Goebel, S. Reich, *Uniform convexity, Hyperbolic Geometry and Nonexpansive mappings*, Dekker, New York, 1984.

[15] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[16] M.A. Khamsi, W.A. Kirk, *An introduction to metric spaces and fixed point theory*, Pure Appl. Math, Wiley-Interscience, New York, NY, USA, 2001.

[17] S.H. Khan, M. Abbas, *Strong and* $\triangle$*-convergence of some iterative schemes in CAT(0) spaces*, Comput. Math. Appl. 61, 1 (2011), 109-116.

[18] W.A. Kirk, *Geodesic geometry and fixed point theory*, in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Vol. 64 of Coleccion Abierta, 195-225, University of Seville Secretary of Publications, Seville, Spain, 2003.

[19] W.A. Kirk, *Geodesic geometry and fixed point theory II*, in International Conference on Fixed point Theory and Applications, 113-142, Yokohama Publishers, Yokohama, Japan, 2004.

[20] W.A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. 68 (2008), 3689-3696.

[21] P. Kumam. G.S. Saluja, H.K. Nashine, *Convergence of modified S-iteration process for two asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces*, J. Inequality and Applications 2014:368, 2014.

[22] T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. 60 (1976), 179-182.

[23] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4 (1953), 506-510.

[24] B. Nanjaras, B. Panyanak, *Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces*, Fixed Point Theory Appl. 2010 (2010), Art. ID 268780.

[25] A. Şahin, M. Başarir, *On the strong convergence of a modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space*, Fixed Point Theory Appl. 2013, 2013:12.

[26] G.S. Saluja, *On the convergence of modified S-iteration process for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces*, Funct. Anal. Approx. Comput. 6, 2 (2014), 29-38.

[27] G.S. Saluja, J.K. Kim,, *On the convergence of a modified S-iteration process for asymptotically quasi-nonexpansive type mappings in a CAT(0) space*, Nonlinear Funct. Anal. Appl. 19, 3 (2014), 329-339.

[28] G.S. Saluja, *Modified S-iteration process for two asymptotically quasi-nonexpansive mappings in a CAT(0) space*, J. Adv. Math. Stud. 7, 2 (2014), 151-159.

[29] G.S. Saluja, M. Postolache, *Strong and $\Delta$-convergence theorems for two asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces*, Fixed Point Theory Appl. 2015:12(2015), DOI 10.1186/s 13663-015-0259-8.

[30] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. 43, 1 (1991), 153-159.

[31] H.F. Senter, W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. 44 (1974), 375-380.

[32] K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (1993), 301-308.

[33] H.K. Xu, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Anal. 16 (1991), 1139-1146.

G. S. Saluja
Department of Mathematics,
Govt. N. P. G. College of Science,
Raipur-492010 (C.G.), India.
email: *saluja1963@gmail.com*