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PRESERVING PROPERTIES AND ESTIMATION OF THE COEFFICIENTS FOR FUNCTIONS THAT BELONG TO THE SUBCLASS OF ANALYTIC FUNCTIONS

P. Dicu, R. Diaconu

Abstract. In this paper we give preserving properties and estimation of the coefficients for functions that belong to the subclass of analytic functions $TS_\gamma(f, g; \alpha, \beta)$.

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1. Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

that are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$.

Let T denote the subclass of S consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ (a_k \geq 0). \tag{1.2}$$

Definition 1.1. *[5] Let I_A be a Alexander integral operator defined as:*

$$I_A : A \to A, \ I_A(F) = f, \ \textit{where}$$

$$f(z) = \int_0^z \frac{F(t)}{t} dt. \tag{1.3}$$

Definition 1.2. *[1] Let I_a be a Bernardi integral operator defined as:*

$$I_a : A \to A,\ I_a(F) = f,\ a = 1, 2, 3, \ldots,\ \textit{where}$$

$$f(z) = \frac{a+1}{z^a} \int\limits_0^z F(t) \cdot t^{a-1} dt. \tag{1.4}$$

Definition 1.3. *[1] Let L_a be a generalization of the previously integral operator defined as:*

$$L_a : A \to A,\ L_a(F) = f,\ a \in \mathbb{C}, \mathrm{Re\,a} \geq 0,\ \textit{where}$$

$$f(z) = \frac{a+1}{z^a} \int\limits_0^z F(t) \cdot t^{a-1} dt. \tag{1.5}$$

Definition 1.4. *[5] Let $I_{c+\delta}$ be the integral operator defined as: $I_{c+\delta} : A \to A$, $0 < u \leq 1$, $1 \leq \delta < \infty$, $0 < c < \infty$,*

$$f(z) = I_{c+\delta}(F)(z) = (c+\delta) \int_0^1 u^{c+\delta-2} F(uz) du. \tag{1.6}$$

Remark 1.1. *[5] For $\delta = 1$ and c=1,2,..., from the integral operator $I_{c+\delta}$ we obtain the Bernardi integral operator defined by (1.4).*

Definition 1.5. *[5] Let $F \in A$, $F(z) = z + b_2 z^2 + \cdots + b_n z^n + \ldots$, and $a \in \mathbb{R}^*$. We define the integral operator $L : A \to A$ by*

$$f(z) = L(F)(z) = \frac{1+a}{z^a} \int_0^z F(t) \left(t^{a-1} + t^{a+1}\right) dt. \tag{1.7}$$

2. Preliminary results

Lemma 2.1. *[3] For $-1 \leq \alpha < 1$, $0 \leq \gamma \leq 1$ and $\beta \geq 0$, we let $S_\gamma(f, g; \alpha, \beta)$ be the subclass of S consisting of functions $f(z)$ of the form (1.1) and functions g(z) given by*

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \ \ (b_k \geq 0), \tag{2.1}$$

and satisfying the analytic criterion:

$$\text{Re}\left\{\frac{z\left(f*g\right)'(z)+\gamma z^2\left(f*g\right)''(z)}{(1-\gamma)\left(f*g\right)(z)+\gamma z\left(f*g\right)'(z)}-\alpha\right\}> \tag{2.2}$$

$$>\beta\left|\frac{z\left(f*g\right)'(z)+\gamma z^2\left(f*g\right)''(z)}{(1-\gamma)\left(f*g\right)(z)+\gamma z\left(f*g\right)'(z)}-1\right|.$$

Further, we define the class $TS_\gamma(f,g;\alpha,\beta)$ by

$$TS_\gamma(f,g;\alpha,\beta)=S_\gamma(f,g;\alpha,\beta)\cap T.$$

Lemma 2.2. *[3] A function $f(z)$ of the form (1.1) is in the class $TS_\gamma(f,g;\alpha,\beta)$ if*

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]|a_k|b_k\leq 1-\alpha, \tag{2.3}$$

where $-1\leq\alpha<1,\ \beta\geq 0$ and $0\leq\gamma\leq 1$.

Lemma 2.3. *[3] A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $TS_\gamma(f,g;\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]a_kb_k\leq 1-\alpha. \tag{2.4}$$

Corollary 2.1. *[3] Let the function $f(z)$ be defined by (1.2) be in the class $TS_\gamma(f,g;\alpha,\beta)$. Then*

$$a_k\leq\frac{1-\alpha}{\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]b_k},\quad (k\geq 2). \tag{2.5}$$

3. Main results

Theorem 3.1. *The Alexander integral operator defined by (1.3) preserves the class $TS_\gamma(f,g;\alpha,\beta)$, that is: If $F\in TS_\gamma(f,g;\alpha,\beta)$, then $f(z)=I_AF(z)\in TS_\gamma(f,g;\alpha,\beta)$, for $F(z)=z-\sum_{k=2}^{\infty}a_kz^k$, $a_k\geq 0$.*

Proof. Let $F\in T$, $F(z)=z-\sum_{k=2}^{\infty}a_kz^k,\ a_k\geq 0$. Then

$$f(z)=I_AF(z)=\int_0^z\frac{F(t)}{t}dt=$$

$$= \int_0^z \frac{1}{t}\left(t - \sum_{k=2}^{\infty} a_k\, t^k\right) dt =$$

$$= z - \sum_{k=2}^{\infty} \frac{a_k}{k} z^k$$

$$= z - \sum_{k=2}^{\infty} c_k z^k, \text{ with}$$

$c_k = \dfrac{a_k}{k} \geq 0,\ k \geq 2$. It follows that $f \in T$. We have now to prove that $f \in TS_\gamma(f, g; \alpha, \beta)$. Using Lemma 2.3 we need to prove that:

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)]\,[1+\gamma(k-1)]\, c_k b_k \leq 1-\alpha. \tag{3.1}$$

for $k \geq 2$, $-1 \leq \alpha < 1,\ \beta \geq 0$ and $0 \leq \gamma \leq 1$. This means:

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)]\,[1+\gamma(k-1)]\, \frac{a_k}{k} b_k \leq 1-\alpha. \tag{3.2}$$

But we have $\dfrac{a_k}{k} \leq a_k,$ for k $\geq 2,$ and by using (2.4) and (3.2), we observe that inequality (3.1) is fulfilled.This means that $f \in TS_\gamma(f, g; \alpha, \beta)$.

Theorem 3.2. *The integral operator $I_{c+\delta}$ defined by (1.6) preserves the class $TS_\gamma(f, g; \alpha, \beta)$, that is: If $F \in TS_\gamma(f, g; \alpha, \beta)$, then $f(z) = I_{c+\delta}(F)(z) \in TS_\gamma(f, g; \alpha, \beta)$, for $F(z) = z - \sum_{k=2}^{\infty} a_k\, z^k,\ a_k \geq 0$.*

Proof. Let $F \in TS_\gamma(f, g; \alpha, \beta),\ F(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0.$

We have, from Lemma 2.3:

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)]\,[1+\gamma(k-1)]\, a_k b_k \leq 1-\alpha. \tag{3.3}$$

From (1.6) we obtain $f(z) = I_{c+\delta}(F)(z) = z - \sum_{k=2}^{\infty} \dfrac{c+\delta}{c+k+\delta-1}\, a_k\, z^k,$ *where* $0 < c < \infty,\ 1 \leq \delta < \infty.$

We also remark that for $0 < c < \infty,\ k \geq 2$ and $1 \leq \delta < \infty$, we have

$$0 < \frac{c+\delta}{c+k+\delta-1} < 1 \tag{3.4}$$

Thus $f \in T$ and by using Lemma 2.3 we have only to prove that.

$$\sum_{k=2}^{\infty} [k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,\frac{c+\delta}{c+k+\delta-1} a_k b_k \leq 1-\alpha. \tag{3.5}$$

where $-1 \leq \alpha < 1,\ \beta \geq 0$ and $0 \leq \gamma \leq 1,\ \ 0 < c < \infty$ and $1 \leq \delta < \infty$.
By using the relation (3.4) we have

$$\frac{c+\delta}{c+k+\delta-1} \cdot a_k < a_k,$$

for $0 < c < \infty,\ k \geq 2,\ 1 \leq \delta < \infty$, and thus from (3.3) we conclude that the condition (3.5) take place and thus the proof it is complete.

The following theorem is proved similarly (see Remark 1.1):

Corollary 3.1. *The Bernardi integral operator defined by (1.4) preserves the class* $TS_\gamma(f,g;\alpha,\beta)$*, that is: If* $F \in TS_\gamma(f,g;\alpha,\beta)$*, then* $f(z) = I_a F(z) \in TS_\gamma(f,g;\alpha,\beta)$*, for* $F(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0.$

Theorem 3.3. *Let* $F \in TS_\gamma(f,g;\alpha,\beta)$ *with* $-1 \leq \alpha < 1,\ \beta \geq 0$ *and* $0 \leq \gamma \leq 1$, $F(z) = z - \sum_{k=2}^{\infty} b_k z^k,\ \ b_k \geq 0.$ *For* $f(z) = L_a(F)(z),\ \ f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0,\ z \in U$*, where the integral operator* L_a *it is defined by (1.5), we have:*

$$a_k \leq \left| \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} \cdot \frac{a+1}{a+k} \right|,\ k \geq 2.$$

Proof. For $f = L_a(F)(z)$ with $F(z) = z - \sum_{k=2}^{\infty} b_k z^k$ and $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ we have

$$a_k = b_k \cdot \frac{a+1}{a+k},$$

where $a \in \mathbb{C}$, $\mathrm{Re}\, a \geq 0$, $k \geq 2$.
The coefficient bounds for the functions belonging to the class $TS_\gamma(f,g;\alpha,\beta)$ are

$$b_k \leq \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k},\quad (k \geq 2).$$

For $k \geq 2$ we obtain

$$a_k = |b_k| \cdot \left| \frac{a+1}{a+k} \right| \leq$$

$$\leq \left| \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} \right| \cdot \left| \frac{a+1}{a+k} \right| =$$

$$= \left| \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} \cdot \frac{a+1}{a+k} \right|.$$

Hence the theorem is proved.

Theorem 3.4. *Let $F \in TS_\gamma(f,g;\alpha,\beta)$ with $-1 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \gamma \leq 1$, $F(z) = z - \sum_{k=2}^{\infty} b_k z^k$, $b_k \geq 0$. For $f(z) = L(F)(z)$, $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0$, $z \in U$, where the integral operator L it is defined by (1.7), we have:*

$$a_2 \leq \frac{1-\alpha}{(2-\alpha+\beta)\,(1+\gamma)\,b_2} \cdot \frac{a+1}{a+2}.,$$

$$a_3 \leq \left[\frac{1-\alpha}{(3-\alpha+2\beta)\,(1+2\gamma)\,b_3} + 1 \right] \cdot \frac{a+1}{a+3}.$$

$$a_k \leq \frac{(1-\alpha)(a+1)}{a+k} \cdot (r_k + r_{k-2}),$$

where

$$r_k = \frac{1}{[(k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}$$

$$r_{k-2} = \frac{1}{[(k-2)(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-3)]\,b_{k-2}}.$$

Proof. For $f = L(F)(z)$ with $F(z) = z - \sum_{k=2}^{\infty} b_k z^k$ and $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ we have:

$$a_2 = b_2 \cdot \frac{a+1}{a+2},$$

$$a_3 = (b_3+1) \cdot \frac{a+1}{a+3},$$

$$a_k = (b_k + b_{k-2}) \cdot \frac{a+1}{a+k},$$

where $a \in \mathbb{R}^*,\ k \geq 4$.
The coefficient bounds for the functions belonging to the class $TS_\gamma(f, g; \alpha, \beta)$ are :

$$b_k \leq \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}, \quad (k \geq 2).$$

For $k \geq 4$ we obtain:

$$a_k = (b_k + b_{k-2}) \cdot \frac{a+1}{a+k} \leq$$

$$\leq \frac{1-\alpha}{[(k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} \cdot \frac{a+1}{a+k} +$$

$$+ \frac{1-\alpha}{[(k-2)(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-3)]\,b_{k-2}} \cdot \frac{a+1}{a+k}.$$

$$a_k \leq \frac{(1-\alpha)(a+1)}{a+k} \cdot [\frac{1}{[(k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k} +$$

$$+ \frac{1}{[(k-2)(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-3)]\,b_{k-2}}] = \frac{(1-\alpha)(a+1)}{a+k} \cdot (r_k + r_{k-2}),$$

where

$$r_k = \frac{1}{[(k(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-1)]\,b_k}$$

$$r_{k-2} = \frac{1}{[(k-2)(1+\beta)-(\alpha+\beta)]\,[1+\gamma(k-3)]\,b_{k-2}}$$

For $k = 2$ we have:

$$a_2 = b_2 \cdot \frac{a+1}{a+2} \leq$$

$$\leq \frac{1-\alpha}{(2-\alpha+\beta)\,(1+\gamma)\,b_2} \cdot \frac{a+1}{a+2}.$$

Similarly for $k = 3$ we have:

$$a_3 \leq \left[\frac{1-\alpha}{(3-\alpha+2\beta)\,(1+2\gamma)\,b_3} + 1\right] \cdot \frac{a+1}{a+3}.$$

Hence the theorem is proved.

We remark that for suitable values of the functions f and g and the parameters α and β we obtain similarly results for the following subclasses:

i) $TS_0\left(f, \frac{z}{(1-z)}; \alpha, 1\right) = S_pT(\alpha)$ and $TS_0\left(f, \frac{z}{(1-z)^2}; \alpha, 1\right) =$

$= TS_1\left(f, \dfrac{z}{(1-z)}; \alpha, 1\right) = UCT(\alpha)$ $(-1 \leq \alpha < 1)$ (see Bharati et al. [4]);

ii) $TS_1\left(f, \dfrac{z}{(1-z)}; 0, \beta\right) = UCT(\beta)$ $(\beta \geq 0)$ (see Subramanian et al. [12]);

iii) $TS_0\left(f, z + \sum_{k=2}^{\infty} \dfrac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta\right) = TS(\alpha, \beta)$ $(-1 \leq \alpha < 1, \beta \geq 0,\ c \neq 0, -1, -2, \ldots)$ (see Murugusundaramoorthy and Magesh [6] and [7]);

iv) $TS_0\left(f, z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta\right) = TS(n, \alpha, \beta)$ $(-1 \leq \alpha < 1, \beta \geq 0,\ n \in N_0 = N \cup \{0\}, N = \{1, 2, \ldots\})$ (see Rosy and Murugusundaramoorthy [10]);

v) $TS_0\left(f, z + \sum_{k=2}^{\infty} \binom{k+\lambda-1}{\lambda} z^k; \alpha, \beta\right) = D(\alpha, \beta, \lambda)$ $(-1 \leq \alpha < 1, \beta \geq 0,\ \lambda > -1)$ (see Shams et al. [11]);

vi) $TS_0\left(f, z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n z^k; \alpha, \beta\right) = TS_\lambda(n, \alpha, \beta)$ $(-1 \leq \alpha < 1, \beta \geq 0,\ \lambda \geq 0)$, $n \in N_0$ (see Aouf and Mostafa [2]);

vii) $TS_\gamma\left(f, z + \sum_{k=2}^{\infty} \dfrac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta\right) = TS(\gamma, \alpha, \beta)$ $(-1 \leq \alpha < 1, \beta \geq 0,\ 0 \leq \gamma \leq 1,\ c \neq 0, -1, -2, \ldots)$ (see Murugusundaramoorthy et al. [8]);

viii) $TS_0(f, g; \alpha, \beta) = H_T(g, \alpha, \beta)$ $(-1 \leq \alpha < 1, \beta \geq 0)$ (see Raina and Bansal [9]);

References

[1] M.Acu, *Liberalization Pascu integrated operator and its properties on the functions of uniform star-shaped, convex, convex and alpha almost uniformly convex*, ULBS, Faculty of Science Collection, Mathematics Series, Sibiu 2005.

[2] M. K. Aouf, R. and A.O.Mostafa, *Some properties of a subclass of uniformly convex functions with negative coefficients,* Demonstratio Math., 2(2008), 353-370.

[3] M. K. Aouf, R. M. El-Ashwah and S. M. El-Deeb, *Integral means for certain subclasses of uniformly starlike and convex functions defined by convolution,* Acta Universitatis Apulensis, No 20/2009, pp. 31-42.

[4] R.Bharati, R. Parvatham and A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions,* Tamakang J. Math. 28(1997), 17-32.

[5] R. Diaconu, *Special classes of analytic functions and their transformations by means of integral operators*, Thesis, Piteşti 2014.

[6] G. Murugusundaramoorthy and N. Magesh, *A new subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient,* J. Inequal. Pure Appl.Math. 5(2004), no. 4, Art. 85, 1-10 .

[7] G. Murugusundaramoorthy and N. Magesh, *Linear operators associated with a subclass of uniformly convex functions,* Internat. J. Pure Appl. Math. Sci. 3(2006), no. 2, 113-125 .

[8] G. Murugusundaramoorthy, T. Rosy and K. Muthunagai, *Carlson- Shaffer operator and their applications to certain subclass of uniformly convex function,* General Math. 15(2007), no. 4, 131-143.

[9] R. K. Raina and D. Bansal, *Some properties of a new class of analytic functions defined in terms of a Hadamard product,* J. Inequal. Pure Appl.Math. 9(2008), no. 1, Art. 22, 1-9.

[10] T. Rosy and G. Murugusundaramoorthy, *Fractional calculus and their applications to certain subclass of uniformly convex functions,* Far East J. Math. Sci. (FJMS), 115(2004), no. 2, 231-242.

[11] S. Shams, S. R. Kulkarni and J. M. Jahangiri , *Classes of Ruscheweyh-type analytic univalent functions,* Southwest J. Pure Appl. Math., 2(2003), 105-110.

[12] K. G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahma and H. Silverman, *Subclasses of uniformly convex and uniformly starlike functions,* Math. Japon. 42(1995), no. 3, 517-522.

Petrică Dicu
Department of Mathematics and Informatics
Lucian Blaga University of Sibiu,
Sibiu, 550012, România
email: *petrica.dicu@ulbsibiu.ro*

Radu Diaconu
Department of Mathematics, Faculty of Science,
"Alma Mater" University of Sibiu,
Sibiu, 550003, România
email: *radudyaconu@yahoo.com*