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UNIVALENCE CONDITIONS FOR A NEW GENERAL INTEGRAL OPERATOR

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Abstract. In this paper, we obtain univalence conditions for a new general integral operator defined on the space of normalized analytic functions in the open unit disk U. Some corollaries follow as special cases.

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1. Introduction

Let $U = \{z : |z| < 1\}$ be the open unit disk of the complex plane and $\mathcal{A}$ the class of all functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 \dots , \qquad (1)$$

which are analytic in U and satisfy the condition $f(0) = f'(0) - 1 = 0$.
Consider $S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}$.

A function $f \in \mathcal{A}$ is said to be starlike of order δ, $0 \le \delta < 1$, that is $f \in S^*(\delta)$, if and only if

$$Re\left[\frac{zf'(z)}{f(z)}\right] > \delta \quad (z \in U).$$

A function $f \in \mathcal{A}$ is said to be convex of order δ, $0 \le \delta < 1$, that is $f \in K(\delta)$, if and only if

$$Re\left[\frac{zf''(z)}{f'(z)} + 1\right] > \delta \quad (z \in U).$$

It is well known that $S^*(0) \equiv S^*$ and $K(0) \equiv K$ are the classes of starlike and convex functions in U, respectively.

Frasin and Jahangiri [8] defined the family $B(\mu,\lambda), \mu \geq 0, 0 \leq \lambda < 1$ consisting of functions $f \in \mathcal{A}$ which satisfy the condition

$$\left| f^{'}(z)\left[\frac{z}{f(z)}\right]^{\mu} - 1\right| < 1-\lambda, \quad (z \in U). \tag{2}$$

The family $B(\mu,\lambda)$ is a comprehensive class of analytic functions. For instance, we have $B(1,\lambda) = S^{*}(\lambda)$ and $B(2,\lambda) = B(\lambda)$ (see Frasin and Darus [9]).

In the present paper, we define a new integral operator given by

$$F_{n,\zeta}(z) = \left(\zeta \int_0^z t^{\zeta-1} \prod_{i=1}^{n} \left[\frac{tf_i^{'}(t)}{g_i(t)} e^{h_i(t)}\right]^{\alpha_i} dt\right)^{\frac{1}{\zeta}}, \tag{3}$$

where parameters $\zeta \in \mathbb{C} \setminus \{0\}$, $\alpha_i \in \mathbb{C}$ and the functions $f_i, g_i, h_i \in \mathcal{A}$, $i \in \{1,\ldots,n\}$, are so constrained that the integral operator (3) exists.

The operator $F_{n,\zeta}$ extend the following integral operators:

(i) For $\zeta = 1$, $e^{h_i(t)} = 1, g_i(t) = t$ and $\alpha_i > 0$ we have $I_n(f)(z) = \int_0^z \prod_{i=1}^{n}\left[f_i^{'}(t)\right]^{\alpha_i} dt$, that was defined by D. Breaz, S. Owa and N. Breaz in [1], and this operator is a generalization of the integral operator $I_\alpha(f)(z) = \int_0^z \left[f^{'}(t)\right]^{\alpha} dt$, discussed in [10, 16, 18].

(ii) For $\zeta = 1$, $g_i(t) = t$ we get $G_n(z) = \int_0^z \prod_{i=1}^{n}\left[f_i^{'}(t)e^{h_i(t)}\right]^{\alpha_i} dt$ which was studied by A. Oprea and D. Breaz in [14] and this operator is a generalization of the integral operator $I_1(f,h)(z) = \int_0^z \left[f^{'}(t)e^{h(t)}\right]^{\alpha} dt$, defined and studied by N. Ularu and D. Breaz in [12, 13].

(iii) For $\zeta = 1$ and $e^{h_i(t)} = 1$ we obtain $I_n(z) = \int_0^z \prod_{i=1}^{n}\left[\frac{tf_i^{'}(t)}{g_i(t)}\right]^{\alpha_i} dt$, that was introduced by R. Bucur and D. Breaz in [6] and this operator extends the integral operator $I_\alpha(z) = \int_0^z \left[\frac{tf^{'}(t)}{g(t)}\right]^{\alpha} dt$, defined and studied in [2].

(iv) For $\zeta = 1$, $n = 1$ and $f_1(t) = t$ we have $G_\alpha(z) = \int_0^z \left[\frac{te^{h(t)}}{g(t)}\right]^{\alpha} dt$ defined and disscused by R. Bucur, L. Andrei and D. Breaz in [3].

(v) For $n = 1$ and $e^{h_1(t)} = 1$ we have $I_\alpha^{\zeta}(z) = \left\{\zeta \int_0^z t^{\alpha+\zeta-1}\left[\frac{f^{'}(t)}{g(t)}\right]^{\alpha} dt\right\}^{\frac{1}{\zeta}}$, which was studied by R. Bucur, L. Andrei and D. Breaz in [4].

(vi) For $\zeta = 1$ we obtain the integral operator $F(z) = \int_0^z \prod_{i=1}^{n}\left[\frac{tf_i^{'}(t)}{g_i(t)}e^{h_i(t)}\right]^{\alpha_i} dt$, introduced and studied by R. Bucur and D. Breaz in [5].

Recently, many authors studied the sufficient conditions for the univalence and convexity of certain families of integral operators in the open unit disk and some of them motivated our work(see [7, 19]).

In order to derive our main results, we have to recall here the following:

Lemma 1. *(Pascu [15]) Let γ be a complex number,* $\mathrm{Re}\gamma > 0$ *and let the function $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma} \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in U$, then for any complex number ζ, $\mathrm{Re}\zeta \geq \mathrm{Re}\gamma$*, the function*

$$H_\zeta(z) = \left[\zeta \int_0^z t^{\zeta-1} f'(t)dt\right]^{\frac{1}{\zeta}} \tag{4}$$

is regular and univalent in U.

Lemma 2. *(Pescar [17]) Let ζ, c be complex numbers with* $\mathrm{Re}\zeta > 0$ *and* $|c| \leq 1$, $c \neq -1$. *If $f \in \mathcal{A}$ satisfies*

$$\left|c|z|^{2\zeta} + (1-|z|^{2\zeta})\frac{zf''(z)}{\zeta f'(z)}\right| \leq 1, \tag{5}$$

for all $z \in U$, then the function H_ζ given by (4) is univalent in U.

Lemma 3. *(The General Schwarz Lemma [11]) Let f be regular function in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, for M fixed. If f has in $z = 0$ one zero with multiply bigger than m, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad (z \in U_R).$$

The equality case hold only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

2. Main results

In the folowing theorem we give sufficient conditions of univalence of the operator $F_{n,\zeta}$ defined in (3), by using Pascu univalence criterion.

Theorem 4. *Let* $\gamma, \alpha_i \in \mathbb{C}$, $\mathrm{Re}\gamma > 0$ *and* N_i, M_i, $P_i \geq 1$, $i \in \{1, \ldots, n\}$, *such that*

$$(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}} \sum_{\mathrm{i}=1}^{\mathrm{n}} |\alpha_i|[1+(2-\lambda_i)N_i^{\mu_i-1}] + 2\mathrm{Re}\gamma \sum_{\mathrm{i}=1}^{\mathrm{n}} |\alpha_\mathrm{i}|[\mathrm{M_i} + (2-\eta_\mathrm{i})\mathrm{P_i^{\nu_i}}]$$
$$\leq \mathrm{Re}\gamma(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}. \tag{6}$$

If $f_i \in \mathcal{A}$, $g_i \in B(\mu_i, \lambda_i)$, $h_i \in B(\nu_i, \eta_i)$ *satisfies*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M_i, \quad |g_i(z)| < N_i, \quad |h_i(z)| < P_i, \tag{7}$$

for all $z \in U$, $i \in \{1, \dots, n\}$, *then for every complex number* ζ, $\mathrm{Re}\zeta \ge \mathrm{Re}\gamma$, *the function* $F_{n,\zeta}$ *given by (3) is in the class* S.

Proof. We begin by considering the function J be defined by

$$J(z) = \int_0^z \prod_{i=1}^{n} \left[\frac{t f_i'(t)}{g_i(t)} e^{h_i(t)}\right]^{\alpha_i} dt, \quad z \in U. \tag{8}$$

After we calculate the first-order and the second-order derivatives, we obtain

$$\frac{zJ''(z)}{J'(z)} = \sum_{i=1}^{n} \alpha_i \left[1 + \frac{z f_i''(z)}{f_i'(z)} - \frac{z g_i'(z)}{g_i(z)} + z h_i'(z)\right]. \tag{9}$$

Therefore

$$\left|\frac{zJ''(z)}{J'(z)}\right| \le \sum_{i=1}^{n} |\alpha_i| \left\{1 + |z| \cdot \left|\frac{f_i''(z)}{f_i'(z)}\right| + \left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right| \cdot \left|\frac{g_i(z)}{z}\right|^{\mu_i - 1} + \left|h_i'(z)\left(\frac{z}{h_i(z)}\right)^{\nu_i}\right| \cdot \frac{|h_i(z)|^{\nu_i}}{|z|^{\nu_i - 1}}\right\}. \tag{10}$$

By applying the General Schwarz Lemma to the functions $g_1, \dots, g_n$, $h_1, \dots, h_n$, we obtain

$$|g_i(z)| \le N_i |z| \text{ and } |h_i(z)| \le P_i |z| \quad (z \in U, \, i \in \{1, \dots, n\}). \tag{11}$$

Replacing (11) in inequality (10), we find that

$$\left|\frac{zJ''(z)}{J'(z)}\right| \le \sum_{i=1}^{n} |\alpha_i| \cdot \left\{1 + |z| \cdot \left|\frac{f_i''(z)}{f_i'(z)}\right| + \left(\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i} - 1\right| + 1\right) N_i^{\mu_i - 1} + |z| \cdot \left(\left|h_i'(z)\left(\frac{z}{g_i(z)}\right)^{\nu_i} - 1\right| + 1\right) P_i^{\nu_i}\right\}. \tag{12}$$

Next, using the hypothesis, we obtain

$$\frac{1 - |z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma} \cdot \left|\frac{zJ''(z)}{J'(z)}\right| \le \frac{1 - |z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma} \cdot \sum_{i=1}^{n} |\alpha_i| \cdot [1 + (2 - \lambda_i) N_i^{\mu_i - 1}] + \frac{1 - |z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma} \cdot |z| \cdot \sum_{i=1}^{n} |\alpha_i| \cdot [M_i + (2 - \eta_i) P_i^{\nu_i}]. \tag{13}$$

Since

$$\max_{|z|\leq 1}\frac{1-|z|^{2Re\gamma}}{Re\gamma}\cdot|z|=\frac{2}{(2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}}},\tag{14}$$

we have

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\cdot\left|\frac{zJ''(z)}{J'(z)}\right|\leq\frac{1}{\mathrm{Re}\gamma}\sum_{i=1}^{n}|\alpha_i|\cdot[1+(2-\lambda_i)N_i^{\mu_i-1}]$$
$$+\frac{2}{(2Re\gamma+1)^{\frac{2Re\gamma+1}{2Re\gamma}}}\sum_{i=1}^{n}|\alpha_i|\cdot[M_i+(2-\eta_i)P_i^{\nu_i}].\tag{15}$$

If we make use of (6), the last inequality yields

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\cdot\left|\frac{zJ''(z)}{J'(z)}\right|\leq 1,\quad z\in U.\tag{16}$$

Finally by applying Theorem 1, it results that function $F_{n,\zeta}$ is in the class S.

Letting $\mu_i=\nu_i=M_i=N_i=P_i=1$ and $\eta_i=\lambda_i$ for all $i\in\{1,\ldots,n\}$ in Theorem 4, we have

Corollary 5. *Let* γ, $\alpha_1,\ldots,\alpha_n\in\mathbb{C}$, $\mathrm{Re}\gamma>0$, $0\leq\lambda_{\mathrm{i}}<1$ *such that*

$$\left[(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}+2\mathrm{Re}\gamma\right]\sum_{i=1}^{n}|\alpha_i|(3-\lambda_i)\leq\mathrm{Re}\gamma(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}},\ \mathrm{i}\in\{1,\ldots,\mathrm{n}\}.\tag{17}$$

If $f_i\in\mathcal{A}$, g_i, $h_i\in S^*(\lambda_i)$ *and*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right|\leq 1,\quad |g_i(z)|<1,\quad |h_i(z)|<1\quad (z\in U;\, i\in\{1,\ldots,n\}),\tag{18}$$

then for every complex number ζ, $\mathrm{Re}\zeta\geq\mathrm{Re}\gamma$, *the function* $F_{n,\zeta}$ *given by (3) is in the class* S.

Letting $n=1$ in Theorem 4, we have

Corollary 6. *Let* γ, $\alpha\in\mathbb{C}$ *with* $\mathrm{Re}\gamma>0$ *and* N, M, $P\geq 1$. *Suppose that* $f\in\mathcal{A}$, $g\in B(\mu,\lambda)$, $h\in B(\nu,\eta)$ *such that*

$$\left|\frac{f''(z)}{f'(z)}\right|\leq M,\quad |g(z)|<N,\quad |h(z)|<P,\tag{19}$$

for all $z \in U$. *If*

$$(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}|\alpha|[1+(2-\lambda)\mathrm{N}^{\mu-1}]+2\mathrm{Re}\gamma|\alpha|[\mathrm{M}+(2-\eta)\mathrm{P}^{\nu}] \\ \leq \mathrm{Re}\gamma(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}, \tag{20}$$

then for every complex number ζ, $\mathrm{Re}\zeta \geq \mathrm{Re}\gamma$, *the function*

$$F_\zeta(z)=\left(\zeta\int_0^z t^{\alpha+\zeta-1}\left[\frac{f'(t)}{g(t)}e^{h(t)}\right]^{\alpha}dt\right)^{\frac{1}{\zeta}} \tag{21}$$

is in the class S.

Letting $g_i(z)=z$, $i\in\{1,\dots,n\}$, Theorem 4 reduces to the following result.

Example 1. *Let* γ, $\alpha_i\in\mathbb{C}$, $\mathrm{Re}\gamma>0$ *and* M_i, $P_i\geq 1$, $i\in\{1,\dots,n\}$, *such that*

$$3(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}\sum_{\mathrm{i}=1}^{\mathrm{n}}|\alpha_{\mathrm{i}}|+2\mathrm{Re}\gamma\sum_{\mathrm{i}=1}^{\mathrm{n}}|\alpha_{\mathrm{i}}|[\mathrm{M_i}+(2-\eta_{\mathrm{i}})\mathrm{P_i^{\nu_i}}] \\ \leq \mathrm{Re}\gamma(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}. \tag{22}$$

If $f_i\in\mathcal{A}$, $h_i\in B(\nu_i,\eta_i)$ *satisfies*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right|\leq M_i,\quad |h_i(z)|<P_i, \tag{23}$$

for all $z\in U$, $i\in\{1,\dots,n\}$, *then for every complex number* ζ, $\mathrm{Re}\zeta\geq\mathrm{Re}\gamma$, *the function*

$$J_1(z)=\left(\zeta\int_0^z t^{\zeta-1}\prod_{i=1}^{n}\left[f_i'(t)e^{h_i(t)}\right]^{\alpha_i}dt\right)^{\frac{1}{\zeta}}, \tag{24}$$

is in the class S.

Theorem 7. *Let* $c,\alpha_1,\dots,\alpha_n,\zeta\in\mathbb{C}$ *with* $\mathrm{Re}\zeta>0$ *and* M_i, N_i, $P_i\geq 1$, $i\in\{1,\dots,n\}$. *Suppose that* $f_i\in\mathcal{A}$, $g_i\in B(\mu_i,\lambda_i)$, $h_i\in B(\nu_i,\eta_i)$ *satisfies*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right|<M_i,\quad |g_i(z)|<N_i,\quad |zh_i'(z)|<P_i,\quad z\in U,\ i\in\{1,\dots,n\}. \tag{25}$$

If

$$\mathrm{Re}\zeta \geq \sum_{\mathrm{i}=1}^{\mathrm{n}} |\alpha_{\mathrm{i}}|[1 + \mathrm{M_i} + \mathrm{P_i} + (2-\lambda_{\mathrm{i}})\mathrm{N_i}^{\mu_{\mathrm{i}}-1}]$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\zeta}\sum_{i=1}^{n} |\alpha_i|[1 + M_i + P_i + (2-\lambda_i)N_i^{\mu_i-1}]$$

for all $i \in \{1,\ldots,n\}$, then the function $F_{n,\zeta}$ given by (3) is in the class S.

Proof. Let the function J be defined in (8). So, for a given constant $c \in \mathbb{C}$, we obtain

$$\begin{aligned} &\left| c|z|^{2\zeta} + (1-|z|^{2\zeta})\frac{zJ''(z)}{\zeta J'(z)} \right| \\ &= \left| c|z|^{2\zeta} + (1-|z|^{2\zeta})\frac{1}{\zeta}\sum_{i=1}^{n} \alpha_i \cdot \left[1 + \frac{zf_i''(z)}{f_i'(z)} - \frac{zg_i'(z)}{g_i(z)} + zh_i'(z) \right] \right| \\ &\leq |c| + \frac{1}{|\zeta|}\sum_{i=1}^{n} |\alpha_i| \cdot \left[1 + \left|\frac{zf_i''(z)}{f_i'(z)}\right| + \left|\frac{zg_i'(z)}{g_i(z)}\right| + \left|zh_i'(z)\right| \right]. \end{aligned} \tag{26}$$

Now, applying the General Schwarz Lemma to the functions $g_1,\ldots,g_n$, we find that

$$|g_i(z)| \leq N_i|z|. \tag{27}$$

Using the hypothesis and (27) in inequality (26), we have

$$\begin{aligned} &\left| c|z|^{2\zeta} + (1-|z|^{2\zeta})\frac{zJ''(z)}{\zeta J'(z)} \right| \\ &\leq |c| + \frac{1}{|\zeta|}\sum_{i=1}^{n} |\alpha_i| \cdot \left[1 + M_i + P_i + \left(\left| g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i} - 1 \right| + 1 \right) N_i^{\mu_i-1} \right] \\ &\leq |c| + \frac{1}{\mathrm{Re}\zeta}\sum_{i=1}^{n} |\alpha_i| \cdot \left[1 + M_i + P_i + (2-\lambda_i)N_i^{\mu_i-1} \right] \leq 1. \end{aligned} \tag{28}$$

Finally, by applying Lemma 2 to the function J, we deduce that function $F_{n,\zeta}$ is in the class S.

Letting $\mu_i = \nu_i = M_i = N_i = P_i = 1$ and $\eta_i = \lambda_i$ for all $i \in \{1, \ldots, n\}$ in Theorem 7, we have

Corollary 8. *Let* $c, \alpha_1, \ldots, \alpha_n, \zeta \in \mathbb{C}$ *with* $\mathrm{Re}\zeta > 0$. *Suppose that* $f_i \in \mathcal{A}$, g_i, $h_i \in S^*(\lambda_i)$ *satisfies*

$$\left|\frac{zf_i^{''}(z)}{f_i^{'}(z)}\right| < 1, \quad |g_i(z)| < 1, \quad |zh_i^{'}(z)| < 1, \tag{29}$$

for all $z \in U$ *and* $i \in \{1, \ldots, n\}$. *If*

$$\mathrm{Re}\zeta \geq \sum_{\mathrm{i}=1}^{\mathrm{n}} |\alpha_{\mathrm{i}}|(5 - \lambda_{\mathrm{i}}) \quad \textit{and} \quad |\mathrm{c}| \leq 1 - \frac{1}{\mathrm{Re}\zeta}\sum_{\mathrm{i}=1}^{\mathrm{n}} |\alpha_{\mathrm{i}}|(5 - \lambda_{\mathrm{i}}),$$

for all $i \in \{1, \ldots, n\}$, *then the function* $F_{n,\zeta}$ *given by (3) is in the class* S.

Letting $n = 1$ in Theorem 7, we have

Corollary 9. *Let* $c, \alpha, \zeta \in \mathbb{C}$ *with* $\mathrm{Re}\zeta > 0$ *and* $M, N, P \geq 1$. *Suppose that* $f \in \mathcal{A}$, $g \in B(\mu, \lambda)$, $h \in B(\nu, \eta)$ *satisfies*

$$\left|\frac{zf^{''}(z)}{f^{'}(z)}\right| < M, \quad |g(z)| < N, \quad |zh^{'}(z)| < P, \tag{30}$$

for all $z \in U$. *If*

$$\mathrm{Re}\zeta \geq |\alpha|[1 + \mathrm{M} + \mathrm{P} + (2 - \lambda)\mathrm{N}^{\mu-1}]$$

and

$$|c| \leq 1 - \frac{|\alpha|}{\mathrm{Re}\zeta}[1 + M + P + (2 - \lambda)N^{\mu-1}],$$

then the function F_ζ *given by (21) is in the class* S.

Letting $g_i(z) = z$, $i \in \{1, \ldots, n\}$, Theorem 7 reduces to the following result.

Example 2. *Let* $c, \alpha_1, \ldots, \alpha_n, \zeta \in \mathbb{C}$ *with* $\mathrm{Re}\zeta > 0$ *and* M_i, $P_i \geq 1$, $i \in \{1, \ldots, n\}$. *Suppose that* $f_i \in \mathcal{A}$, $h_i \in B(\nu_i, \eta_i)$ *satisfies*

$$\left|\frac{zf_i^{''}(z)}{f_i^{'}(z)}\right| < M_i, \quad |zh_i^{'}(z)| < P_i, \quad z \in U, i \in \{1, \ldots, n\}. \tag{31}$$

If

$$\mathrm{Re}\zeta \geq \sum_{\mathrm{i}=1}^{\mathrm{n}} |\alpha_{\mathrm{i}}| \left(3 + \mathrm{M}_{\mathrm{i}} + \mathrm{P}_{\mathrm{i}}\right)$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\zeta} \sum_{i=1}^{n} |\alpha_i| \left(3 + M_i + P_i\right)$$

for all $i \in \{1, \dots, n\}$,

$$J_1(z) = \left(\zeta \int_0^z t^{\zeta-1} \prod_{i=1}^{n} \left[f_i'(t) e^{h_i(t)}\right]^{\alpha_i} dt\right)^{\frac{1}{\zeta}}, \tag{32}$$

is in the class S.

Remark 1. *Many other interesting corollaries of Theorems 4 and 7 can be obtained by suitably specializing the parameters and the functions involved.*

References

[1] D. Breaz, S. Owa and N. Breaz, *A new integral univalent operator,* Acta Universitatis Apulensis, 16 (2008).

[2] R. Bucur, L. Andrei and D. Breaz, *Geometric Properties of a New Integral Operator,* Abstract and Applied Analysis, Vol. 2015, Art. ID 430197 (2015).

[3] R. Bucur, L. Andrei and D. Breaz, *Univalence criterion, starlikeness and convexity for a new integral operator,* International Electronic Journal of Pure and Applied Mathematics, Vol.9, No.3 (2015), 215-223.

[4] R. Bucur, L. Andrei and D. Breaz, *Some Results of a New Integral Operator,* Journal of Computational Analysis and Applications, No. 194 (2016), to appear.

[5] R. Bucur and D. Breaz, *Univalence Conditions and Properties of a New General Integral Operator,* Carpathian J. Math., to appear.

[6] R. Bucur, D. Breaz, *Properties of a general integral operator,* submitted.

[7] B. A. Frasin, D. Breaz, *Univalence conditions of general integral operator,* Matematiqki Vesnik 65, 3 (2013), 394–402.

[8] B. A. Frasin and J. Jahangiri, *A new and comprehensive class of analytic functions,* Analele Univ. Oradea, Fasc. Math. XV (2008), 59-62.

[9] B. A. Frasin, and M. Darus, *On certain analytic univalent functions,* Int. J. Math. and Math. Sci. 25, 5 (2001), 305-310.

[10] Y. J. Kim and E. P. Merkes, *On an integral of powers of a spirallike function*, Kyungpook Math. J. 12, 2 (1972), 249-252.

[11] O. Mayer, *The functions theory of one variable complex,* Bucuresti, 1981.

[12] N. Ularu and D. Breaz, *Univalence criterion and convexity for an integral operator,* Applied Mathematics Letter, 25 (2012), 658-661.

[13] N. Ularu and D. Breaz, *Univalence condition and properties for two integral operators,* 15 (2013), 658-661.

[14] A. Oprea and D. Breaz, *Univalence conditions for two general integral operators,* Advances in Pure Mathematics 4, 8 (2014), 487-493.

[15] N. N. Pascu, *An improvement of Becker's univalence criterion,* Proceedings of the Commemorative Session Simion Stoilow, Brasov, Preprint (1987), 43-48.

[16] N. N. Pascu and V. Pescar, *On the integral operators of Kim-Merkes and Pfaltzgraff*, Mathematica, Universitatis Babes-Bolyai Cluj-Napoca 32, 2 (1990), 185-192.

[17] V. Pescar, *A new generalization of Ahlfors's and Becker's criterion of univalence,* Bull. Malaysian Math. Soc., Ser. 2, 19 (1996), 53-54.

[18] V. Pescar, *Some Integral Operators and Their Univalence,* The Journal of Analysis, 5 (1997), 157-162.

[19] V. Pescar, *New univalence criteria for some integral operators,* Stud. Univ. Babes-Bolyai Math. 59, 2 (2014), 167-176.

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