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EXISTENCE OF SOLUTION TO A SEMILINEAR DISCRETE PROBLEM INVOLVING P−LAPLACIAN

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Abstract. In this work, under appropriate assumptions, we present a result on the existence of solutions of difference equation

$$-\Delta\Big(|\Delta u(k-1)|^{p-2}\Delta u(k-1)\Big) = f(k,u(k)) + g(k)|u(k)|^{q-2}u, \quad k \in [1,T].$$

Our method is based on the Ekeland's variational principle.

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1. Introduction

Many papers devoted to study of the discrete equations have increased in recent years, by using various methods and techniques as fixed point theorems, lower and upper solutions, and variational approach, see for example [1, 2, 3, 4, 5, 6, 7, 10, 11, 12]...

Let $T \geq 2$ be a fixed positive integer, $[a,b]$ be the discrete interval $\{a, a+1, ..., b\}$ where a and b are integers and $a < b$.

Motivated by the above mentioned papers, we deal with the following discrete boundary value problem

$$\begin{gathered}-\Delta\Big(|\Delta u(k-1)|^{p-2}\Delta u(k-1)\Big) = f(k,u(k)) + g(k)|u(k)|^{q-2}u, \quad k \in [1,T]. \\ u(0) = u(T+1) = 0,\end{gathered} \tag{1}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $1 < q < p < \infty$ while $f : \ [1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Denoting by $F : \ \ [1,T] \to \mathbb{R}$ the primitive of f i.e.,

$$F(k,u) = \int_0^u f(k,s)ds, \quad u \in \mathbb{R}.$$

There is by now a rich literature on problems like (1), which began to receive much attention and are known to be mathematical models of various phenomena arising in the study of elastic mechanics, control systems, artificial or biological neural networks, computing, electrical circuit analysis, dynamical systems, economics,...For more detail, we may refer to [8, 13].

Here, we are concerned with investigating nonlinear discrete boundary value problems by using variational approach.

The rest of this paper is arranged as follows. In section 2, we recall some basic definitions and tools in order to prove our main result and at after all we give an example.

2. Preliminaries and statement of main results

In this section, we recall some basic properties which will be used in the proof of the precise result.

Solutions to (1) will be investigated in a space

$$W = \Big\{ u : [0, T+1] \to \mathbb{R} \text{ s.t } u(0) = u(T+1) = 0 \Big\},$$

which is a T-dimensional Hilbert space, see [2], with the inner product

$$(u, v) = \sum_{k=1}^{T+1} \Delta u(k-1) \Delta v(k-1), \qquad \text{for all } u, v \in W.$$

The associated norm is defined by

$$\|u\| = \Big(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p \Big)^{\frac{1}{p}}.$$

Moreover, it is useful to introduce other norms on W, namely

$$|u|_m = \Big(\sum_{k=1}^{T} |u(k)|^m \Big)^{\frac{1}{m}}, \quad \forall u \in W \text{ and } m \geq 2.$$

It can be verified (see [6]) that

$$T^{\frac{2-m}{2m}} |u|_2 \leq |u|_m \leq T^{\frac{1}{m}} |u|_2, \quad \forall u \in W \text{ and } m \geq 2. \tag{2}$$

Next, we list some inequalities that will be are used later.

Lemma 1. *([10]) For every $u \in W$, we have*

$$(a) \quad \sum_{k=1}^{T} |u(k)|^m \leq T(T+1)^{m-1} \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall m \geq 2.$$

$$(b) \quad \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq 2^m \sum_{k=1}^{T} |u(k)|^m, \quad \forall m > 1.$$

We say that $u \in W$ is a weak solution of problem (1) if

$$\sum_{k=1}^{T+1} \Big(|\Delta u(k-1)|^{p(k-1)-2} \Delta \varphi(k-1) \Big) = \sum_{k=1}^{T} f(k, u(k))\varphi(k) + \mu \sum_{k=1}^{T} g(k)|u(k)|^p \varphi(k),$$

for any $\varphi \in W$.

Related to problem (1), define the functional $\phi : \; W \to \mathbb{R}$ given by

$$\phi(u) = \sum_{k=1}^{T+1} \Big(\frac{1}{p} |\Delta u(k-1)|^p \Big) - \sum_{k=1}^{T} F(k, u(k)) - \frac{1}{q} \sum_{k=1}^{T} g(k)|u(k)|^q.$$

We assume the following conditions.

(g_1) $g \in L^\infty([1,T]), \quad g \geq (\not\equiv) 0.$

(f_1) $f(x,t) \in C([1,k] \times \mathbb{R}), f \not\equiv 0.$

(f_2) $0 \leq l_1 := \lim\limits_{t \to 0} \dfrac{f(k,t)}{|t|^{p-1}} < \lambda_1$ and $\lambda_1 < l_2 := \lim\limits_{t \to \infty} \dfrac{f(k,t)}{t^{p-1}} < \infty$
uniformly in $[1,T]$ where $\lambda_1 > 0$ is the smallest positive eigenvalue of

$$\begin{aligned} -\Delta \Big(|\Delta u(k-1)|^{p-2} \Delta u(k-1) \Big) &= \lambda |u(k)|^{p-2} u, \quad k \in [1,T], \\ u(0) = u(T+1) &= 0, \end{aligned} \tag{3}$$

see [2] .

The following theorem is the main result of this paper.

Theorem 2. *If $(g_1), (f_1)$ and (f_2) hold true, then the problem (1) has at least one nontrivial weak solution.*

Proof of Theorem 2. It is clear that the functional ϕ is differentiable and its derivative is given by

$$\phi'(u).v = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p-2} \Delta u(k-1) \Delta v(k-1) - \sum_{k=1}^{T} f(k, u(k)) v(k) - \sum_{k=1}^{T} g(k)|u(k)|^{p-2} u(k) v(k),$$

for all $u, v \in W$. For simplicity, we give an auxiliary result about f:
In view of (f_1) and (f_2) with $l_2 < \infty$ we have

$$\lim_{t\to\infty} \frac{f(k,t)}{t^{r-1}} = 0, \quad \text{for} \quad p < r < \infty, \quad \text{uniformly in} \quad [1,T].$$

Therefore, for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that
$f(k,t) \le (l_1+\varepsilon)t^{p-1} + C(\varepsilon)t^{r-1}$ for all $t \ge 0, k \in [1,T]$,
and thus
$F(k,t) \le \frac{l_1+\varepsilon}{p}t^p + \frac{C(\varepsilon)}{r}t^r$, for all $t \ge 0, k \in [1,T]$.
On the other hand, according to (f_2) and the fact that $l_1 < \lambda_1$, we may find $\varepsilon_1 > 0$ such that

$$l_1 + \varepsilon_1 < \lambda_1,$$

$$F(k,t) \le \frac{l_1+\varepsilon_1}{p}t^p + C_0 t^r, \quad \text{for all} \quad t \ge 0, k \in [1,T]$$

with $C_0 > 0$ is nonnegative constant.

We present the following Lemma:

Lemma 3. *There exist $\rho > 0$ and $\alpha > 0$ such that*

$$\phi(u) \ge \alpha > 0, \quad \textit{for every} \quad u \in W, \quad \textit{with} \quad \|u\| = \rho.$$

Proof of Lemma 3. By using of Lemma 1 we have

$$\begin{aligned}
\phi(u) &= \sum_{k=1}^{T+1}\Big(\frac{1}{p}|\Delta u(k-1)|^p\Big) - \sum_{k=1}^{T} F(k,u(k)) - \frac{1}{q}\sum_{k=1}^{T} g(k)|u(k)|^q \\
&\ge \frac{1}{p}\|u\|^p - \frac{l_1+\varepsilon}{p}\sum_{k=1}^{T}|u(k)|^p - \frac{C(\varepsilon)}{r}\sum_{k=1}^{T}|u(k)|^r - \frac{|g|_\infty}{q}\sum_{k=1}^{T}|u(k)|^q \\
&\ge \frac{1}{p}\|u\|^p - \frac{|g|_\infty}{q}T(T+1)^{q-1}\sum_{k=1}^{T+1}|\Delta u(k-1)|^q - \frac{l_1+\varepsilon_1}{p}T(T+1)^{p-1}\sum_{k=1}^{T+1}|\Delta u(k-1)|^p \\
&\quad -C(\varepsilon)T(T+1)^{r-1}\sum_{k=1}^{T+1}|\Delta u(k-1)|^r \\
&\ge C_1\|u\|^p - |g|_\infty C_2\|u\|^q - C_3\|u\|^r > 0, \qquad (4)
\end{aligned}$$

for $\|u\|$ is small enough, with $C_1 = \frac{1}{p}\Big(1 - \frac{l_1+\varepsilon_1}{\lambda_1}\Big) > 0$ and C_2, C_3 are positive constants. $\square$

Define

$$\overline{B}_\rho = \{u \in E \ : \ \|u\| \le \rho\}$$

endowed with metric

$$d(u,v) = \|u-v\|, \quad u,v \in \overline{B}_\rho,$$

and

$$\partial B_\rho = \{u \in W \ : \ \|u\| = \rho\}.$$

By the previous Lemma 3, we know that

$$\phi(u)/\partial B_\rho \geq \alpha > 0$$

and it is easy to see that the functional ϕ is bounded from below.
Let $c_* = \min_{\overline{B}_\rho} \phi(u)$, so we have $c_* < 0$. In fact, fix a nonnegative function $v \in W \setminus \{0\}$, and note that for $t > 0$ we have

$$\begin{aligned}
\phi(tv) &= \sum_{k=1}^{T+1} \Big(\frac{1}{p}|\Delta tv(k-1)|^p\Big) - \sum_{k=1}^{T} F(k, tv(k)) - \frac{1}{q}\sum_{k=1}^{T} g(k)|tv(k)|^q \\
&\leq \frac{t^p}{p}\|v\|^p - \frac{l_1 t^p}{p}\sum_{k=1}^{T} v(k)^p - \frac{t^q}{q}\sum_{k=1}^{T} g(k)v(k)^q.
\end{aligned}$$

Since $q < p$, the last inequality implies that for some $t_* > 0$ sufficiently small,

$$\phi(t_* v) < 0, \text{and} \ \ t_* v \in \overline{B}_\rho(0).$$

By the Ekeland's variational principle in [9], we may find a sequence $(u_n)_n \subset \overline{B}_\rho(0)$ such that

$\phi(u_n) \to c_*$ i.e $c_* \leq \phi(u_n) \leq c_* + \frac{1}{n}, \forall n > 0$

and

$$\phi(v) \geq \phi(u_n) - \frac{1}{n}\|u_n - v\|, \quad \forall v \neq u_n.$$

Once that ϕ is differentiable, it follows from the last inequality that

$$\phi'(u_n) \to 0.$$

Indeed, let $u \in W$ such that $\|u\| = 1$, putting $\omega_n = u_n + \lambda u$. Fix $n > 1$ we have

$$\|\omega_n\| \leq \|u_n\| + \lambda < \rho$$

with $\lambda > 0$ is small enough. It yields

$$\phi(u_n + \lambda u) \geq \phi(u_n) - \frac{\lambda}{n}\|u\|$$

then

$$\frac{\phi(u_n + \lambda u) - \phi(u_n)}{\lambda} \geq -\frac{1}{n}\|u\|,$$

tending $\lambda \to 0$ so we get

$$\phi'(u_n).u \geq -\frac{1}{n}$$

that is,

$$|\phi'(u_n).u| \leq \frac{1}{n}$$

with $\|u\| = 1$. So $\phi'(u_n) \to 0$.

Since the sequence (u_n) is bounded in W, there exists $u_1 \in W$ such that, up to a subsequence, (u_n) converges to u_1 in W. Therefore,

$$\phi(u_1) = c_*, \quad \text{and} \quad \phi'(u_1) = 0,$$

and thus u_1 is a non trivial solution of problem (1).

Example 1. *Taking $\beta > 0$ and $l_2 > \lambda_1$, the function f such that*
$f(x,t) = \frac{l_2 t^{\beta+1}}{1+t^\beta}$ *when $t \geq 0$ and $f(x,t) = 0$ if $t \leq 0$,*
satisfies the conditions (f_1) and (f_2) provided $l_2 < \infty$.

Remark 1. *If we assume that $\limsup_{|t|\to\infty} \frac{f(k,t)}{|t|^{p-2}t} < \lambda_1$, $k \in [1,T]$ then, it is easy to check that there exists $0 \leq \lambda < \lambda_1$ such that*

$$\phi(u) \geq \frac{1}{p}\Big(1 - \frac{\lambda}{\lambda_1}\Big)\|u\|^p - C_4\|u\| - C_5\|u\|^q, \tag{5}$$

where C_4 and C_5 are positive constants.
Thus ϕ is coercive and bounded from below so it has a global minimizer.

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