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NEW SUBCLASSES OF BI-UNIVALENT BAZILEVIC FUNCTIONS OF TYPE ALPHA INVOLVING SALAGEAN DERIVATIVE OPERATOR

A.T. Oladipo

Abstract. The author introduced new subclasses of bi-univalent functions of Bazilevic functions of type α which are defined by means of Salagean derivative operator. Furthermore, the author finds the estimates on the first few coefficients for functions in these new subclasses. Also we use this estimates to determine the relevance connections to classical Fekete-Szego estimate.

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1. Introduction

Let A denote the set of all analytic functions f in the unit disk $D = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

and S the subclasses of functions in A that are univalent in D. The class S is indeed the central object in the study of univalent functions.
The following are some of the important, well known and regularly investigated subclasses of univalent functions class S

$$S^*(\beta) = \left\{ f \in S : Re\left(\frac{zf'z}{f(z)}\right) > \beta, z \in D, 0 \leq \beta < 1 \right\}.$$

$$K(\beta) = \left\{ f \in S : Re\left(1 + \frac{zf''z}{f'(z)}\right) > \beta, z \in D, 0 \leq \beta < 1 \right\}.$$

In 1955, a Russian Mathematician called [2] discovered certain function in D and he defined it by

$$f(z)=\left\{\frac{\alpha}{1+\epsilon^2}\int_0^z \frac{p(v)-i\epsilon}{v\,(1+(i\alpha\epsilon/1+\epsilon^2))}g(v)^{\frac{\alpha}{1+\epsilon^2}}dv\right\}^{\frac{1+i\epsilon}{\alpha}} \tag{2}$$

where $p\in P$ and $g\in S^*$. The number $\alpha>0$ and ϵ are real, and all powers are meant to be principal determination only.
The family of functions in (2) became known as Bazilevic functions and is in this work denoted by $B(\alpha,\epsilon)$. Except that, he, [2] showed that each function $f\in B(\alpha,\epsilon)$ is univalent in D, very little is known regarding the family as a whole.
However, with some simplifications, it may be possible to understand and investigate the family. Indeed, it is easy to verify that, with special choices of the parameters α and ϵ and the function $g(z)$, the family $B(\alpha,\epsilon)$ crack down to some well-known subclasses of univalent functions.
For instance, if we choose $\epsilon=0$ in (2), we have

$$f(z)=\left\{\alpha\int_0^z \frac{p(v)}{v}g(v)^{\alpha}dv\right\}^{\frac{1}{\alpha}}. \tag{3}$$

On differentiating (3) we have

$$\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}=p(z). \tag{4}$$

Or equivalently

$$Re\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}>0 \quad z\in D \tag{5}$$

The subclasses of Bazilevic functions satisfying (5) are called Bazilevic functions of type α and are denoted by $B(\alpha)$ see [3]. In 1973, [4] gave a plausible description of functions of the class $B(\alpha)$ as those functions in S for which $r<1$ and the tangent to the curve $D_\alpha(r)=\left\{\epsilon f\left(re^{i\theta}\right)^{\alpha},0\le\theta<2\pi\right\}$ never turns back on itself as much as π radian. If α is taking as 1, the class $B(\alpha)$ reduces to the family of close-to-convex function. That is,

$$Re\frac{zf'(z)}{g(z)}>0 \quad z\in D. \tag{6}$$

Suppose we replace $g(z)$ by $f(z)$ in (6) then we have

$$Re\frac{zf'(z)}{f(z)}>0 \quad z\in D, \tag{7}$$

which implies that $f(z)$ is starlike (see for details [[1],[5],[6],[7]]).
Furthermore, in 1992, [8] introduced a generalization of functions satisfying (5) by putting $g(z)^\alpha \equiv z^\alpha$ as

$$Re\frac{D^n f(z)^\alpha}{z^\alpha} > 0 \quad z \in D, \tag{8}$$

which are largely non-univalent in the unit disk, but by proving the inclusion

$$B_{n+1}(\alpha) \subset B_n(\alpha), \tag{9}$$

[8] was able to show that for all $n \in N$, each function of the $B_1(\alpha)$ is univalent in D.
In 1994, [9], and also [1] gave a more generalized form of [8] geometric condition (5) with some little modification in [1],by defining a class $T_n^\alpha(\beta)$ whose functions satisfying

$$Re\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta \quad z \in D, \tag{10}$$

where $\alpha > 0$ is real $0 \leq \beta < 1$ and D^n is the [10] derivative operator defined as follows

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z),$$

$$D^n f(z) = D(D^{n-1}f(z)) = z(D^{n-1}f(z))' = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{11}$$

Notable contributors, the likes of [1], [11], [3], [12], [13], [14] and the present author [[5],[6],[7]] just to mention but few, had earlier considered various special cases of the parameter n and α of (10) and many interesting and useful results were obtained.
Before we discuss further we wish to quickly say here that from (1) we can write that

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha. \tag{12}$$

Using binomial expansion on (12), we obtain

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}, \tag{13}$$

also applying (11) to (13) we have

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = 1 + \sum_{k=2}^{\infty} \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k(\alpha) z^{\alpha+k-1}, \tag{14}$$

where α, n and D^n are as earlier defined and that all powers are meant to be principal determination only.
It is well known that every function $f \in S$ has an inverse f^{-1} defined by

$$f^{-1}\left(f(z)\right) = z \quad z \in D, \tag{15}$$

and

$$f\left(f^{-1}(\omega)\right) = \omega, \left(|\omega| < r_0(f); r_0(f) \geq \frac{1}{4}\right). \tag{16}$$

It is easily seen from (15) and (16) that

$$f^{-\alpha}\left(f(z)\right)^\alpha = z^\alpha, \qquad z \in D \qquad \alpha > 0, \alpha(is \ \ real)$$

and

$$f^{\alpha}\left(f(\omega)\right)^{-\alpha} = \omega^\alpha, \left(|\omega| < r_0(f); r_0(f) \geq \frac{1}{4}\right) \tag{17}$$

where

$$(f(z))^\alpha = 1 + \alpha_1 a_2 z + \left[\alpha_1 a_3 + \alpha_2 a_2^2\right] z^2 + \left[\alpha_1 a_4 + 2\alpha a_2 a_3 + \alpha_3 a_2^3\right] z^3 + ... \tag{18}$$

where $\alpha_1 = \alpha, \alpha_2 = \frac{\alpha(\alpha-1)}{2}, \alpha_3 = \frac{\alpha(\alpha-1)(\alpha-2)}{3!}$ and

$$(f(z))^{-\alpha} = 1 - \alpha_1 a_2 z + \left[\alpha_2 a_2^2 - \alpha_1 a_3\right] z^2 + \left[2\alpha a_2 a_3 - \alpha_1 a_4 - \alpha_3 a_2^3\right] z^3 + ... \tag{19}$$

where $\alpha_1 = \alpha, \alpha_2 = \frac{\alpha(\alpha+1)}{2}, \alpha_3 = \frac{\alpha(\alpha+1)(\alpha+2)}{3!}$.
A function $f(z) \in A$ is said to be bi- univalent in D if both $f(z)$ and f^{-1} are univalent in D. Here we denote the class of bi-univalent function in D by $\sum$.
The object of the present work is to introduce new subclass of bi-univalence of Bazilevic functions of type α and to determine the first few coefficient bounds and their relevant connection to Fekete-Szego estimates. Our techniques shall depend on the earlier ones used by [15],[16] and [15].
For the purpose of the present investigation, the following lemma and definitions shall be necessary.

2. Preliminary Lemma and Definitions.

Lemma 1. *[16] If $h \in P$, then $|c_k| \leq 2$ for each $k \geq 1$, where P is the family of all functions h analytic in D for which $Reh(z) > 0$, $h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ...$ for $z \in D$. Unless otherwise stated we assume throughout this work that $\alpha > 0$ α (is real), $n \in N = N \cup \{0\}$, $0 < \beta \leq 1$ and D^n is the Salagean derivative operator and that all powers are understood as principal values.*

Definition 1. *A function $f(z)^\alpha$ given by (13) is said to be in the class $T_n^{\sum,\alpha}(\beta)$ if it satisfies the following condition*

$$\left|arg\frac{D^nf(z)^\alpha}{\alpha^nz^\alpha}\right| < \frac{\beta\pi}{2} \quad and \quad \left|arg\frac{g(\omega)^\alpha}{\omega^\alpha}\right| < \frac{\beta\pi}{2} \tag{20}$$

where $0 < \beta \leq 1, n \in N_0, \alpha > 0$ (is real) and D^n is the Salagean derivative operator

Definition 2. *A function $f(z)^\alpha$ given by (13) is said to be in the class $T_n^{\sum,\alpha}(\beta)$ if it satisfies the following condition*

$$Re\left\{\frac{D^nf(z)^\alpha}{\alpha^nz^\alpha}\right\} > \beta \quad and \quad Re\left\{\frac{g(\omega)^\alpha}{\alpha^n\omega^\alpha}\right\} > \beta \tag{21}$$

where $0 \leq \beta < 1$ and all other parameters are as earlier defined and that all powers are meant to be principal values.

3. Main Result

Theorem 2. *Let $f(z)$ be given by (13) be in the class $T_n^{\sum,\alpha}(\beta)$ $\alpha > 0, n \in N_0, 0 < \beta \leq$,then*

$$|a_2| \leq \frac{2\alpha^{n-1}\beta}{\sqrt{\alpha^n\beta(\alpha+2)^n} + \sqrt{(1-\beta)}(\alpha+1)^n} \tag{22}$$

$$|a_3| \leq \frac{2\alpha^{n-1}\beta}{(\alpha+2)^n} + \frac{2\alpha^{2(n-1)}\beta^2}{(\alpha+1)^{2n}} \tag{23}$$

Proof. It follows from definition 1 that,

$$\frac{D^nf(z)^\alpha}{\alpha^nz^\alpha} = [p(z)]^2$$

and

$$\frac{g(\omega)^\alpha}{\alpha^n\omega^\alpha} = [q(\omega)]^2 \tag{24}$$

where $p(z)$ and $q(\omega)$ in P have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 ...and \;\; q(\omega) = 1 + q_1(\omega) + q_2(\omega)^2 + ... \tag{25}$$

Now equating the coefficients (24), by means of (18) and (19), (11) and (13) we obtain

$$\frac{(\alpha+1)^n}{\alpha^{n-1}} a_2 = \beta p_1 \tag{26}$$

$$\frac{(\alpha+2)^n}{\alpha^{n-1}} a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{2\alpha^n} a_2^2 = \beta p_2 + \frac{\beta(\beta-1)}{2} p_1^2 \tag{27}$$

$$-\frac{(\alpha+1)^n}{\alpha^{n-1}} a_2 = \beta q_1 \tag{28}$$

$$\frac{\alpha(\alpha+1)(\alpha+2)}{2\alpha^n} a_2^2 - \frac{(\alpha+2)^n}{\alpha^{n-1}} a_3 = \beta q_2 + \frac{\beta(\beta-1)}{2} q_1^2 \tag{29}$$

From (26) and (28) we have

$$p_1 = -q_1, \tag{30}$$

and that

$$\frac{2(\alpha+1)^{2n}}{\alpha^{2n-2}} a_2^2 = \beta^2(p_1^2 + q_1^2) \tag{31}$$

Now from (27), (29) and (31) we obtain

$$\begin{aligned} \frac{(\alpha+2)^{2n}}{\alpha^{n-2}} a_2^2 &= \beta(p_2 - q_2) + \frac{\beta(\beta-1)}{2}(p_1^2 + q_1^2) \\ &= \beta(p_2 - q_2) + \frac{(\alpha+1)^{2n}(\beta-1)}{\beta\alpha^{2n-2}} a_2^2 \end{aligned} \tag{32}$$

Therefore we have

$$a_2^2 = \frac{\beta^2\alpha^{2n-2}(p_2+q_2)}{\beta\alpha^n(\alpha+2)^n + (1-\beta)(\alpha+1)^{2n}}. \tag{33}$$

Applying Lemma 1 for the coefficients p_2 and q_2 we obtain

$$|a_2| \leq \frac{2\alpha^{n-1}\beta}{\sqrt{\alpha^n\beta(\alpha+2)^n} + \sqrt{(1-\beta)}(\alpha+1)^n}. \tag{34}$$

This gives the bound on $|a_2|$ as asserted in (22).
Next, in order to find the bound on $|a_3|$, we subtract (29) from (27) and using (30) we obtain

$$\frac{2(\alpha+2)^n}{\alpha^{n-1}}a_3 - \frac{(\alpha+2)^n}{\alpha^{n-1}}a_2^2 = \beta p_2 + \frac{\beta(\beta-1)}{2}p_1^2 - \left(\beta q_2 + \frac{\beta(\beta-1)}{2}q_1^2\right) = \beta(p_2 - q_2). \quad (35)$$

Then it follows from (31) and (35) that

$$\frac{2(\alpha+2)^n}{\alpha^{n-1}}a_3 = \beta(p_2 - q_2) + \frac{\beta^2\alpha^{n-1}(\alpha+2)^n(p_1^2+q_1^2)}{2(\alpha+1)^{2n}}$$

which yields

$$a_3 = \frac{\alpha^{n-1}\beta(p_2-q_2)}{2(\alpha+2)^n} + \frac{\beta^2\alpha^{2n-2}(p_1^2+q_1^2)}{4(\alpha+1)^{2n}}$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1, q_2 we quickly have

$$|a_3| \leq \frac{2\alpha^{n-1}\beta}{(\alpha+2)^n} + \frac{2\alpha^{(2n-1)}\beta^2}{(\alpha+1)^{2n}}.$$

This complete the proof of Theorem 2.

Let $\alpha = 1$ in Theorem 2 then we have

Corollary 3. *Let $f(z)$ be given by (13) be in the class $T_n^{1,\sum}$ then*

$$|a_2| \leq \frac{2\beta}{\sqrt{3^n\beta + 2^n\sqrt{(1-\beta)}}}. \quad (36)$$

$$|a_3| \leq \frac{2\beta}{3^n} + \frac{2\beta^2}{2^{2n}} \quad (37)$$

Theorem 4. *Let $f^\alpha(z)$ be given by (13) be in the class $T_n^{\sum,\alpha}(\beta)\alpha > 0, \alpha(is\ real), n \in N, 0 \leq \beta < 1,$ then*

$$|a_2| \leq \frac{2\sqrt{\alpha^{n-2}(1-\beta)}}{\sqrt{(\alpha+2)^n}} \quad (38)$$

$$|a_3| \leq \frac{2\alpha^{2n-2}(1-\beta)^2}{(\alpha+1)^n} + \frac{2\alpha^{n-1}(1-\beta)}{(\alpha+2)^n}. \quad (39)$$

Proof. It follows from Definition 2 that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \beta + (1-\beta)[p(z)]$$

$$\frac{D^n g(\omega)^\alpha}{\alpha^n \omega^\alpha} = \beta + (1-\beta)[q(\omega)] \tag{40}$$

where $p(z)$ and $q(\omega)$ in P have the forms (25) respectively.
By following the proof of Theorem 2, and suitably comparing coefficient in (38) we have

$$\frac{(\alpha+1)^n}{\alpha^{n-1}(1-\beta)} a_2 = p_1, \tag{41}$$

$$\left(\alpha_1 a_3 + \alpha_2 a_2^2\right) \frac{(\alpha+2)^n}{\alpha^n(1-\beta)} = p_2, \tag{42}$$

$$-\frac{(\alpha+1)^n}{\alpha^{n-1}(1-\beta)} a_2 = q_1, \tag{43}$$

$$\left(\alpha_2 a_2^2 - \alpha_1 a_3\right) \frac{(\alpha+2)^n}{\alpha^n(1-\beta)} = q_2. \tag{44}$$

From (41) and (43) we have that

$$p_1 = -q_1, \tag{45}$$

that is,

$$\frac{2(\alpha+1)^{2n}}{\alpha^{2n-2}(1-\beta)^2} a_2^2 = (p_1^2 + q_1^2). \tag{46}$$

Also, from (42) and (44) we get

$$\frac{(\alpha+2)^n}{\alpha^{n-2}(1-\beta)} a_2^2 = p_2 + q_2. \tag{47}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^{n-2}(1-\beta)}{(\alpha+2)^n}(p_2+q_2). \tag{48}$$

Applying Lemma 1 for the coefficients p_2 and q_2 we obtain

$$|a_2| \leq \frac{2\sqrt{\alpha^{n-2}(1-\beta)}}{\sqrt{(\alpha+2)^n}}.$$

This gives bound on $|a_2|$ as asserted in (38).
Next is to find the bound on $|a_3|$, by subtracting (44) from (42) we obtain

$$\frac{2(\alpha+2)^n}{\alpha^{n-1}(1-\beta)}a_3 - \frac{(\alpha+2)^n}{\alpha^{n-1}(1-\beta)}a_2^2 \tag{49}$$

or equivalently

$$a_3 = \frac{1}{2}a_2^2 + \frac{\alpha^{n-1}(1-\beta)(p_2-q_2)}{2(\alpha+2)^n} \tag{50}$$

and, then from (46) we find that

$$a_3 = \frac{\alpha^{2n-2}(1-\beta)^2(p_1^2+q_1^2)}{4(\alpha+1)^{2n}} + \frac{\alpha^{n-1}(1-\beta)(p_2-q_2)}{2(\alpha+2)^n}. \tag{51}$$

Applying Lemma 1 on (51) for the coefficients of p_1, p_2, q_1 and q_2, we readily obtain

$$|a_3| \leq \frac{2\alpha^{2n-2}(1-\beta)^2}{(\alpha+1)^n} + \frac{2\alpha^{n-1}(1-\beta)}{(\alpha+2)^n}. \tag{52}$$

This complete the proof of Theorem 4.

Theorem 5. *Let f^α be given by (13) be in the class $T_n^{\sum,\alpha}(\beta)$, $\alpha > 0, \alpha$ (is real) $0 < \beta \leq 1, n \in N$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{2\alpha^{n-1}\beta\left((\alpha+1)^n + 2\alpha^{n-1}\beta\right)}{(\alpha+1)^{2n}} - \frac{4\beta^2\alpha^{2n-2}}{\beta\alpha^n(\alpha+2)^n + (1-\beta)(\alpha+1)^{2n}}. \tag{53}$$

Theorem 6. *Let f^α be given by (13) be in the class $T_n^{\sum,\alpha}(\beta)$, $\alpha > 0, \alpha$ (is real) $0 \leq \beta < 1, n \in N$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{2\alpha^{n-2}(1-\beta)^2}{(\alpha+1)^n} + \frac{2\alpha^{n-2}(1-\beta)(\alpha-1)}{(\alpha+2)^n} \tag{54}$$

conclusively, with various special choices of the parameters involved, many new results could be derived that could be suitable bounds for some of the cited literatures in the direction of bi-univalency.

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Abiodun Tinuoye Oladipo
Department of Pure and Applied Mathematics,
Ladoke Akintola University of Technology, Ogbomoso
P. M. B. 4000,
Ogbomoso, Nigeria.
email: atlab_3@yahoo.com