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PROPERTIES OF AN INTEGRAL OPERATOR

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Abstract. In this paper we define an integral operator for analytic functions in the open unit disk and we determine some properties of this integral operator.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$ which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We consider $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$.

We denote by $\mathcal{P}$ the class of functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

which are analytic in open unit disk $\mathcal{U}$, with $Re\, p(z) > 0$, for all $z \in \mathcal{U}$.

Let $\mathcal{H}(\mathcal{U})$ be the space of holomorphic functions in $\mathcal{U}$ and let

$$\mathcal{A}_n = \left\{ f \in \mathcal{H}(\mathcal{U}), f(z) = z + a_{n+1} z^{n+1} + \cdots, z \in \mathcal{U} \right\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In this work we introduce a new integral operator $J_{\alpha,\beta} : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U})$ defined by

$$J_{\alpha,\beta}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} \left(g(t)\right)^{\beta} dt,\ z \in \mathcal{U}, g \in \mathcal{H}(\mathcal{U}), \tag{1}$$

for $\alpha,\ \beta$ be complex numbers, $\alpha \neq 0,\ \beta \neq 0$.

We have the next remarks:

i_1) For $\beta = 1,\ \alpha = 1,$
we have the integral operator Alexander [1], $A : \mathcal{A} \to \mathcal{A}$,

$$A(z) = \int_0^z \frac{g(t)}{t} dt, \quad z \in \mathcal{U}. \tag{2}$$

i_2) For $\beta = 1,\ \alpha = \frac{1}{2}$,
we obtain the integral operator Libera [4], $L : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U})$,

$$L(z) = \frac{2}{z} \int_0^z g(t) dt, \quad z \in \mathcal{U}. \tag{3}$$

i_3) If $\beta = 1,\ \alpha = \frac{1}{n},\ n \in \mathbb{N}^*$,
we get the integral operator Bernardi [3], $L_n : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U})$,

$$L_n(z) = \frac{n}{z^{n-1}} \int_0^z t^{n-2} g(t) dt, \quad z \in \mathcal{U}. \tag{4}$$

i_4) For $\beta = 1,\ \alpha \in \mathbb{R},\ 0 < \alpha \leq 1$,
we obtain the integral operator Pascu [6], $L_\alpha : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U})$,

$$L_\alpha(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} g(t) dt, \quad z \in \mathcal{U}. \tag{5}$$

These integral operators are integral operators of type Libera.

To discuss our problem for integral operator $J_{\alpha,\beta}$, we need the following theorem.

Theorem 1. *(Becker [2]). If $f(z) = z + a_2 z^2 + \cdots$ is analytic in $\mathcal{U}$ and*

$$(1 - |z|^2) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \tag{6}$$

for all $z \in \mathcal{U}$, then the function $f(z)$ is univalent in $\mathcal{U}$.

2. Main results

Theorem 2. *Let $\alpha,\ \beta$ be complex numbers, $\alpha \neq 0,\ \beta \neq 0$, and the function $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + \cdots$.*

If

$$\left| \frac{1}{\alpha} + \beta - 2 \right| < 1 \tag{7}$$

and

$$\left|\frac{zg'(z)}{g(z)}-1\right| \leq \frac{1-\left|\frac{1}{\alpha}+\beta-2\right|}{|\beta|}, \quad z\in\mathcal{U}, \tag{8}$$

then $z^{\frac{1}{\alpha}-1}J_{\alpha,\beta}(z)\in\mathcal{S}$ *and* $J_{\alpha,\beta}(z)$ *has the form*

$$J_{\alpha,\beta}(z)=z^{2-\frac{1}{\alpha}}+b_2z^{3-\frac{1}{\alpha}}+\cdots, \quad z\in\mathcal{U}. \tag{9}$$

Proof. We have

$$J_{\alpha,\beta}(z)=\frac{z^{1-\frac{1}{\alpha}}}{\alpha}\int_0^z t^{\frac{1}{\alpha}+\beta-2}\left(\frac{g(t)}{t}\right)^{\beta}dt, \tag{10}$$

for all $z\in\mathcal{U}$.

We consider the function

$$G_{\alpha,\beta}(z)=\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}+\beta-2}\left(\frac{g(t)}{t}\right)^{\beta}dt,\ z\in\mathcal{U}. \tag{11}$$

From (11) we obtain

$$G'_{\alpha,\beta}(z)=\frac{1}{\alpha}z^{\frac{1}{\alpha}+\beta-2}\left(\frac{g(z)}{z}\right)^{\beta}$$

and hence

$$\begin{aligned} G''_{\alpha,\beta}(z) &= \frac{1}{\alpha}\left(\frac{1}{\alpha}+\beta-2\right)z^{\frac{1}{\alpha}+\beta-3}\left(\frac{g(z)}{z}\right)^{\beta}+ \\ &+ \frac{1}{\alpha}z^{\frac{1}{\alpha}+\beta-2}\cdot\beta\left(\frac{g(z)}{z}\right)^{\beta-1}\frac{zg'(z)-g(z)}{z^2}. \end{aligned}$$

We obtain

$$(1-|z|^2)\left|\frac{zG''_{\alpha,\beta}(z)}{G'_{\alpha,\beta}(z)}\right| \leq \left|\frac{1}{\alpha}+\beta-2\right|+|\beta|\left|\frac{zg'(z)}{g(z)}-1\right|, \tag{12}$$

for all $z\in\mathcal{U}$.

From (8) and (12) we have

$$(1-|z|^2)\left|\frac{zG''_{\alpha,\beta}(z)}{G'_{\alpha,\beta}(z)}\right| \leq 1, \quad z\in\mathcal{U}. \tag{13}$$

From (13), using the Theorem 1 we obtain $G_{\alpha,\beta}(z) \in \mathcal{S}$ and hence $z^{\frac{1}{\alpha}-1}J_{\alpha,\beta}(z) \in \mathcal{S}$.

We have

$$z^{\frac{1}{\alpha}-1}J_{\alpha,\beta}(z) = z + b_2z^2 + \cdots \tag{14}$$

and we obtain $J_{\alpha,\beta}(z)$ is of the form

$$J_{\alpha,\beta}(z) = z^{2-\frac{1}{\alpha}} + b_2z^{3-\frac{1}{\alpha}} + \cdots, \quad z \in \mathcal{U}. \tag{15}$$

From the Theorem 2 we have the next corollaries.

Corollary 3. *Let the function $g \in \mathcal{A}$, $g(z) = z + a_2z^2 + \cdots$*

If

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1, \quad z \in \mathcal{U}, \tag{16}$$

then $A(z) \in \mathcal{S}$ and A is of the form

$$A(z) = z + b_2z^2 + \cdots, \quad z \in \mathcal{U}. \tag{17}$$

Proof. For $\alpha = 1,\ \beta = 1$, from Theorem 2 we obtain the Corollary 3.

Corollary 4. *Let the function $g \in \mathcal{A}$, $g(z) = z + a_2z^2 + \cdots$ and β be a complex number, $|\beta| < 1$.*

If

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq \frac{1-|\beta|}{|\beta|}, \quad z \in \mathcal{U}, \tag{18}$$

then $J_{\frac{1}{2},\beta}(z)$ is of the form

$$J_{\frac{1}{2},\beta}(z) = 1 + b_2z + \cdots, \quad z \in \mathcal{U}. \tag{19}$$

If Re $J_{\frac{1}{2},\beta}(z) > 0$, then $J_{\frac{1}{2},\beta}(z) \in \mathcal{P}$.

Proof. For $\alpha = \frac{1}{2}$, from Theorem 2 we have the Corollary 4.

Corollary 5. *Let α be real number, $\alpha \in (\frac{1}{2}, 1]$ and the function $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + \cdots$.*

If

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq 2 - \frac{1}{\alpha}, \quad z \in \mathcal{U}, \tag{20}$$

then the integral operator Pascu satisfies the properties:

$$z^{\frac{1}{\alpha}-1} J_{\alpha,1}(z) \in \mathcal{S}, \quad z \in \mathcal{U}. \tag{21}$$

and

$$J_{\alpha,1}(z) = z^{2-\frac{1}{\alpha}} + b_2 z^{3-\frac{1}{\alpha}} + \cdots, \quad z \in \mathcal{U}. \tag{22}$$

Proof. For $\beta = 1$, from Theorem 2 we obtain the Corollary 5.

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