Acta Universitatis Apulensis No. 43/2015
ISSN: 1582-5329 pp. 209-228
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.43.17

SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR ABSOLUTELY CONTINUOUS FUNCTIONS (II)

S. S. Dragomir

Abstract. In this paper, further perturbed Ostrowski type inequalities for absolutely continuous functions are established.

2010 *Mathematics Subject Classification*: 25D15, 25D10

Keywords: Ostrowski's inequality, Integral inequalities, Perturbed Ostrowski Type Inequalities.

1. Introduction

In order to obtain various perturbed Ostrowski type inequalities, in the earlier paper [24] we established the following equality:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt,$$

where the integrals in the right hand side are taken in the Lebesgue sense.

The following equality in terms of one parameter holds:

Corollary 2. *With the assumption in Lemma 1, we have for any $\lambda(x)\in\mathbb{C}$ that*

$$f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda(x)\right]dt.$$

Remark 1. *If we take $\lambda(x) = 0$ in (2), then we get Montgomery's identity for absolutely continuous functions, namely*

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt, \tag{3}$$

for $x \in [a,b]$.

We have the following midpoint representation:

Corollary 3. *With the assumption in Lemma 1, we have for any λ_1, $\lambda_2 \in \mathbb{C}$ that*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_2 - \lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \lambda_1\right] dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \lambda_2\right] dt. \tag{4}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \lambda\right] dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \lambda\right] d. \tag{5}$$

The identity (1) has many particular cases of interest.

If $x \in (a,b)$ is a point of differentiability for the absolutely continuous function $f : [a,b] \to \mathbb{C}$, then we have the equality:

$$f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(x)\right] dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - f'(x)\right] dt. \tag{6}$$

In particular we have

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right] dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right] dt \tag{7}$$

provided $f'\left(\frac{a+b}{2}\right)$ exists and is finite.

For $x \in (a,b)$, if we take in (1)

$$\lambda_1(x) = \frac{f(x)-f(a)}{x-a} \text{ and } \lambda_2(x) = \frac{f(b)-f(x)}{b-x},$$

then we get, after some elementary calculations,

$$\begin{aligned} &\frac{1}{2}\left[f(x) + \frac{(b-x)f(b)+(x-a)f(a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \frac{f(x)-f(a)}{x-a}\right]dt \\ &+ \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \frac{f(b)-f(x)}{b-x}\right]dt. \end{aligned} \tag{8}$$

In particular, we have

$$\begin{aligned} &\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{b-a}{2}}\right]dt \\ &+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \frac{f(b)-f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]dt. \end{aligned} \tag{9}$$

If we assume that the lateral derivatives $f'_+(a)$ and $f'_-(b)$ exist and are finite, then we have from (1) for $\lambda_1(x) = f'_+(a)$ and $\lambda_2(x) = f'_-(b)$

$$\begin{aligned} &f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'_+(a)\right]dt \\ &+ \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - f'_-(b)\right]dt, \end{aligned} \tag{10}$$

for all $x \in [a,b]$.

In particular, we have

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{11}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-f'_{+}(a)\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-f'_{-}(b)\right]dt.$$

If we take in (1) $\lambda_2(x)=\lambda_2(x)=f'\left(\frac{a+b}{2}\right)$, provided this derivative exists and is finite, then we get

$$f(x)+\left(\frac{a+b}{2}-x\right)f'\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{12}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt$$
$$+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt,$$

for all $x\in[a,b]$.

In [24] we obtained the following perturbed Ostrowski type inequalities:

Theorem 4. *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b]\subset\mathring{I}$. If the* *derivative* $f' : \mathring{I} \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, *then*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ &\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\ &\leq \frac{1}{4}\left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f')\right](b-a) \\ &\leq \frac{1}{4}(b-a) \\ &\times \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q + \left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f') \end{cases} \end{aligned} \tag{13}$$

for any $x \in [a,b]$.

We say that $v : [a,b] \to \mathbb{C}$ is *Lipschitzian* with the constant $L > 0$, if it satisfies the condition

$$|v(t) - v(s)| \leq L\,|t-s| \text{ for any } t,s \in [a,b].$$

Theorem 5. *Let* $f : I \to \mathbb{C}$ *be a differentiable function on* $\mathring{I}$ *and* $[a,b] \subset \mathring{I}$. *Let* $x \in (a,b)$. *If the derivative* $f' : \mathring{I} \to \mathbb{C}$ *is Lipschitzian with the constant* $K_1(x)$ *on* $[a,x]$ *and constant* $K_2(x)$ *on* $[x,b]$, *then*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ &\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\ &\leq \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^3 K_1(x) + \left(\frac{b-x}{b-a}\right)^3 K_2(x)\right](b-a)^2 \end{aligned} \tag{14}$$

$$\leq \frac{1}{8}(b-a)^2 \times \begin{cases} \left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right]\max\{K_1(x),K_2(x)\}, \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[K_1^q(x)+K_2^q(x)\right]^{1/q} \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3\left[K_1(x)+K_2(x)\right]. \end{cases}$$

For other Ostrowski type inequalities see [1]-[19] and [21]-[43].

Motivated by the above results, we establish in this paper other perturbed Ostrowski type inequalities for complex valued differentiable functions.

2. Inequalities for Derivatives of Bounded Variation

Assume that the function $f : I \to \mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b] \subset \mathring{I}$. Then, from (6) we have the equality

$$f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{15}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'(x)\right]dt$$

for any $x \in [a,b]$.

We have the following result:

Theorem 6. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the* *derivative* $f':\mathring{I}\to\mathbb{C}$ *is of bounded variation on* $[a,b]$, *then*

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{16}$$

$$\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_t^x (f')\,dt+\int_x^b (b-t)\bigvee_x^t (f')\,dt\right]$$

$$\leq \frac{1}{2}(b-a)\left[\left(\frac{x-a}{b-a}\right)^2\bigvee_a^x (f')\,dt+\left(\frac{b-x}{b-a}\right)^2\bigvee_x^b (f')\right]$$

$$\leq \frac{1}{2}(b-a)$$

$$\times\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f')+\frac{1}{2}\left|\bigvee_a^x (f')-\bigvee_x^b (f')\right|\right], \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q+\left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f'),\end{cases}$$

for any $x\in[a,b]$.

Proof. Taking the modulus in (15) we have

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{17}$$
$$\leq \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t)-f'(x)\right]dt\right|$$
$$+\frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t)-f'(x)\right]dt\right|$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\left|f'(t)-f'(x)\right|dt$$
$$+\frac{1}{b-a}\int_x^b (b-t)\left|f'(t)-f'(x)\right|dt.$$

Since the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a, x]$ and $[x, b]$, then

$$\left|f'(t) - f'(x)\right| \le \bigvee_t^x (f') \text{ for } t \in [a, x]$$

and

$$\left|f'(t) - f'(x)\right| \le \bigvee_x^t (f') \text{ for } t \in [x, b].$$

Therefore

$$\begin{aligned}\int_a^x (t-a)\left|f'(t) - f'(x)\right| dt &\le \int_a^x (t-a) \bigvee_t^x (f')\, dt \\ &\le \frac{1}{2}(x-a)^2 \bigvee_a^x (f')\, dt\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left|f'(t) - f'(x)\right| dt &\le \int_x^b (b-t) \bigvee_x^t (f')\, dt \\ &\le \frac{1}{2}(b-x)^2 \bigvee_x^b (f'),\end{aligned}$$

which, by (17) produce the first two inequalities in (16).

The last part follows by Hölder's inequality

$$mn + pq \le (m^\alpha + p^\alpha)^{1/\alpha} \left(n^\beta + q^\beta\right)^{1/\beta},$$

where $m, n, p, q \ge 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. ∎

Corollary 7. *With the assumptions of Theorem 6, we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \qquad (18) \\ &\le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} (t-a) \bigvee_t^{\frac{a+b}{2}} (f')\, dt + \int_{\frac{a+b}{2}}^b (b-t) \bigvee_{\frac{a+b}{2}}^t (f')\, dt\right] \\ &\le \frac{1}{8}(b-a) \bigvee_a^b (f')\, dt.\end{aligned}$$

Remark 2. *If $p \in (a,b)$ is a median point in bounded variation for the derivative, i.e. $\bigvee_a^p (f') = \bigvee_p^b (f')$, then under the assumptions of Theorem 6 we have*

$$\left| f(p) + \left(\frac{a+b}{2} - p \right) f'(p) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{19}$$
$$\leq \frac{1}{b-a} \left[\int_a^p (t-a) \bigvee_t^p (f')\,dt + \int_p^b (b-t) \bigvee_p^t (f')\,dt \right]$$
$$\leq \frac{1}{4}(b-a) \left[\frac{1}{4} + \left(\frac{p - \frac{a+b}{2}}{b-a} \right)^2 \right] \bigvee_a^b (f').$$

3. Inequalities for Lipschitzian Derivatives

We start with the following result.

Theorem 8. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If $\alpha_i > -1$ and $L_{\alpha_i} > 0$ with $i = 1,2$ are such that*

$$\left| f'(t) - f'(x) \right| \leq L_{\alpha_1} (x-t)^{\alpha_1} \text{ for any } t \in [a,x) \tag{20}$$

and

$$\left| f'(t) - f'(x) \right| \leq L_{\alpha_2} (t-x)^{\alpha_2} \text{ for any } t \in (x,b], \tag{21}$$

then we have

$$\left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{22}$$
$$\leq \frac{1}{b-a} \left[\frac{L_{\alpha_1}}{(\alpha_1+1)(\alpha_1+2)} (x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{(\alpha_2+1)(\alpha_2+2)} (b-x)^{\alpha_2+2} \right].$$

Proof. Taking the modulus in (15) we have

$$\left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{23}$$
$$\leq \frac{1}{b-a} \int_a^x (t-a) \left| f'(t) - f'(x) \right| dt$$
$$+ \frac{1}{b-a} \int_x^b (b-t) \left| f'(t) - f'(x) \right| dt.$$

Using the properties (20) and (21) we have

$$\begin{aligned}\int_a^x (t-a)\left|f'(t)-f'(x)\right| dt &\le L_{\alpha_1}\int_a^x (t-a)(x-t)^{\alpha_1}\,dt\\ &= L_{\alpha_1}(x-a)^{\alpha_1+2}\int_0^1 u(1-u)^{\alpha_1}\,du\\ &= L_{\alpha_1}(x-a)^{\alpha_1+2}\int_0^1 u^{\alpha_1}(1-u)\,du\\ &= \frac{1}{(\alpha_1+1)(\alpha_1+2)}L_{\alpha_1}(x-a)^{\alpha_1+2}\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left|f'(t)-f'(x)\right| dt &\le L_{\alpha_2}\int_x^b (b-t)(t-x)^{\alpha_2}\,dt\\ &= \frac{1}{(\alpha_2+1)(\alpha_2+2)}L_{\alpha_2}(b-x)^{\alpha_2+2}.\end{aligned}$$

Utilising (23) we get the desired result (22). ∎

Corollary 9. *Let $f: I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative is f' of r-H-Hölder type on $[a,b]$, i.e. we have the condition*

$$\left|f'(t)-f'(s)\right| \le H\left|t-s\right|^r$$

for any $t,s \in [a,b]$, where $r \in (0,1]$ and $H > 0$ are given, then

$$\begin{aligned}&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{H}{(r+1)(r+2)}\left[\left(\frac{x-a}{b-a}\right)^{r+2}+\left(\frac{b-x}{b-a}\right)^{r+2}\right](b-a)^{r+1},\end{aligned} \tag{24}$$

for any $x \in [a,b]$.

In particular, if f' is Lipschitzian with the constant $L > 0$, then

$$\begin{aligned}&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{6}L\left[\left(\frac{x-a}{b-a}\right)^{3}+\left(\frac{b-x}{b-a}\right)^{3}\right](b-a)^{2},\end{aligned} \tag{25}$$

for any $x \in [a,b]$.

4. Inequalities for Differentiable Convex Functions

The case of convex functions is as follows:

Theorem 10. *Let $f : I \to \mathbb{C}$ be a differentiable convex function on $\mathring{I}$ and $[a, b] \subset \mathring{I}$. Then for any $x \in [a, b]$ we have*

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f(x) - \left(\frac{a+b}{2} - x\right) f'(x) \leq \begin{cases} I_1(x) \\ \\ I_2(x) \\ \\ I_3(x) \end{cases} \tag{26}$$

where

$$I_1(x) := \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - f(x) - 2f'(x)\left(\frac{a+b}{2} - x\right),$$

$$I_2(x) := \frac{1}{2}\frac{f'(b)(b-x)^2 - f'(a)(x-a)^2}{b-a} - f'(x)\left(\frac{a+b}{2} - x\right)$$

and

$$I_3(x) := \frac{1}{2}\left[\frac{f(b)(b-x) + f(a)(x-a)}{b-a} - f(x)\right] - f'(x)\left(\frac{a+b}{2} - x\right)$$

Proof. We have the equality

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt - f(x) - \left(\frac{a+b}{2} - x\right) f'(x) \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(x) - f'(t)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(t) - f'(x)\right]dt \end{aligned} \tag{27}$$

for any $x \in [a, b]$.

Since f is a differentiable convex function on $\mathring{I}$, then f' is monotonic nondecreasing on $\mathring{I}$ and then

$$\int_a^x (t-a)\left[f'(x) - f'(t)\right]dt \geq 0$$

and

$$\int_x^b (b-t)\left[f'(t) - f'(x)\right]dt \geq 0,$$

which proves the first inequality in (26).

We have

$$\begin{aligned}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt &\le (x-a)\int_a^x \left[f'(x)-f'(t)\right]dt \\ &= (x-a)\left[f'(x)(x-a)-f(x)+f(a)\right]\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt &\le (b-x)\int_x^b \left[f'(t)-f'(x)\right]dt \\ &= (b-x)\left[f(b)-f(x)-f'(x)(b-x)\right].\end{aligned}$$

Adding these inequalities we get

$$\begin{aligned}&\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt+\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt \\ &\le (x-a)\left[f'(x)(x-a)-f(x)+f(a)\right] \\ &+(b-x)\left[f(b)-f(x)-f'(x)(b-x)\right] \\ &=(b-x)f(b)+(x-a)f(a)-(b-a)f(x) \\ &+f'(x)\left[2x-(a+b)\right](b-a)\end{aligned}$$

and by (27) we get the second inequality for $I_1(x)$.

We also have

$$\begin{aligned}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt &\le \int_a^x (t-a)\left[f'(x)-f'(a)\right]dt \\ &= \frac{1}{2}\left[f'(x)-f'(a)\right](x-a)^2\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt &\le \int_x^b (b-t)\left[f'(b)-f'(x)\right]dt \\ &= \frac{1}{2}\left[f'(b)-f'(x)\right](b-x)^2.\end{aligned}$$

Adding these inequalities we get

$$\begin{aligned}&\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt+\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt \\ &\le \frac{1}{2}\left[f'(x)-f'(a)\right](x-a)^2+\frac{1}{2}\left[f'(b)-f'(x)\right](b-x)^2 \\ &=\frac{1}{2}\left[f'(b)(b-x)^2-f'(a)(x-a)^2+f'(x)(b-a)\left[2x-(a+b)\right]\right]\end{aligned}$$

and by (27) we get the second inequality for $I_2(x)$.

Further, we use the Čebyšev inequality for asynchronous functions (functions of opposite monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt \le \frac{1}{d-c}\int_c^d g(t)\,dt \cdot \frac{1}{d-c}\int_c^d h(t)\,dt.$$

Therefore

$$\begin{aligned}
&\frac{1}{x-a}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt \\
&\le \frac{1}{x-a}\int_a^x (t-a)\,dt \cdot \frac{1}{x-a}\int_a^x \left[f'(x)-f'(t)\right]dt \\
&= \frac{(x-a)^2}{2(x-a)} \cdot \frac{f'(x)(x-a)-f(x)+f(a)}{x-a} \\
&= \frac{1}{2}\left[f'(x)(x-a)-f(x)+f(a)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{b-x}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt \\
&\le \frac{1}{b-x}\int_x^b (b-t)\,dt \cdot \frac{1}{b-x}\int_x^b \left[f'(t)-f'(x)\right]dt \\
&= \frac{(b-x)^2}{2(b-x)} \cdot \frac{f(b)-f(x)-f'(x)(b-x)}{b-x} \\
&= \frac{1}{2}\left[f(b)-f(x)-f'(x)(b-x)\right].
\end{aligned}$$

Adding these inequalities, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt \\
&\le \frac{1}{2}\frac{\left[f'(x)(x-a)-f(x)+f(a)\right](x-a)}{b-a} \\
&+ \frac{1}{2}\frac{\left[f(b)-f(x)-f'(x)(b-x)\right](b-x)}{b-a} \\
&= \frac{1}{2(b-a)}\left[\left[f'(x)(x-a)-f(x)+f(a)\right](x-a)\right] \\
&+ \frac{1}{2(b-a)}\left[\left[f(b)-f(x)-f'(x)(b-x)\right](b-x)\right] \\
&= \frac{1}{2}\left[\frac{f(b)(b-x)+f(a)(x-a)}{b-a}-f(x)\right]+f'(x)\left(x-\frac{a+b}{2}\right)
\end{aligned}$$

which proves the inequality for $I_3(x)$. ∎

Remark 3. *From the first inequality in (26) we have*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{(b-x)\,f(b)+(x-a)\,f(a)}{b-a} - f'(x)\left(\frac{a+b}{2}-x\right) \tag{28}$$

for any $x \in [a,b]$.

From the second inequality in (26) we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f(x) \leq \frac{1}{2}\cdot\frac{f'(b)\,(b-x)^2 - f'(a)\,(x-a)^2}{b-a} \tag{29}$$

for any $x \in [a,b]$.

From the third inequality in (26) we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left[\frac{f(b)\,(b-x)+f(a)\,(x-a)}{b-a} + f(x)\right] \tag{30}$$

for any $x \in [a,b]$.

5. Inequalities for Absolutely Continuous Derivatives

We use the *Lebesgue p-norms* defined as follows:

$$\|g\|_{[c,d],p} := \left(\int_c^d |g(s)|^p\,dt\right)^{1/p}, \ g \in L_p[c,d], \ p \geq 1$$

and

$$\|g\|_{[c,d],\infty} := ess \sup_{s\in[c,d]} |g(s)|, \ g \in L_\infty[c,d].$$

The case of absolutely continuous derivatives is as follows:

Theorem 11. *Let* $f : I \to \mathbb{C}$ *be a differentiable function on* $\mathring{I}$ *and* $[a,b] \subset \mathring{I}$. *If the* *derivative f' is absolutely continuous on $[a,b]$, then for any $x \in [a,b]$*

$$\left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{31}$$

$$\leq \frac{1}{b-a} \times \begin{cases} \frac{1}{6}(x-a)^3 \|f''\|_{[a,x],\infty}, \\ \\ \frac{q}{(q+1)(q+2)}(x-a)^{1/q+2} \|f''\|_{[a,x],p}, \\ \\ \frac{1}{2}(x-a)^2 \|f''\|_{[a,x],1}, \end{cases}$$

$$+ \frac{1}{b-a} \times \begin{cases} \frac{1}{6}(b-x)^3 \|f''\|_{[x,b],\infty}, \\ \\ \frac{q}{(q+1)(q+2)}(b-x)^{1/q+2} \|f''\|_{[x,b],p}, \\ \\ \frac{1}{2}(b-x)^2 \|f''\|_{[x,b],1}, \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking the modulus in (15) we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt + \left(\frac{a+b}{2} - x \right) f'(x) \right| \qquad (32) \\ &\leq \frac{1}{b-a} \int_a^x (t-a) \left| f'(t) - f'(x) \right| dt + \frac{1}{b-a} \int_x^b (b-t) \left| f'(t) - f'(x) \right| dt \\ &= \frac{1}{b-a} \int_a^x (t-a) \left| \int_x^t f''(s)\,ds \right| + \frac{1}{b-a} \int_x^b (b-t) \left| \int_x^t f''(s)\,ds \right| \\ &\leq \frac{1}{b-a} \int_a^x (t-a) \int_t^x \left| f''(s) \right| ds + \frac{1}{b-a} \int_x^b (b-t) \int_x^t \left| f''(s) \right| ds. \end{aligned}$$

Using Hölder's integral inequality we have for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$\int_a^x (t-a)\int_t^x \left|f''(s)\right| ds \leq \begin{cases} \int_a^x (t-a)(x-t)\|f''\|_{[t,x],\infty}\, dt \\ \\ \int_a^x (t-a)(x-t)^{1/q}\|f''\|_{[t,x],p}\, dt \\ \\ \int_a^x (t-a)\|f''\|_{[t,x],1}\, dt \end{cases}$$

$$\leq \begin{cases} \|f''\|_{[a,x],\infty}\int_a^x (t-a)(x-t)\, dt \\ \\ \|f''\|_{[a,x],p}\int_a^x (t-a)(x-t)^{1/q}\, dt \\ \\ \|f''\|_{[a,x],1}\int_a^x (t-a)\, dt \end{cases}$$

$$= \begin{cases} \frac{1}{6}(x-a)^3\|f''\|_{[a,x],\infty} \\ \\ \frac{q}{(q+1)(q+2)}(x-a)^{1/q+2}\|f''\|_{[a,x],p} \\ \\ \frac{1}{2}(x-a)^2\|f''\|_{[a,x],1} \end{cases}$$

and, similarly

$$\int_x^b (b-t)\int_x^t \left|f''(s)\right| ds \leq \begin{cases} \frac{1}{6}(b-x)^3\|f''\|_{[x,b],\infty} \\ \\ \frac{q}{(q+1)(q+2)}(b-x)^{1/q+2}\|f''\|_{[x,b],p} \\ \\ \frac{1}{2}(b-x)^2\|f''\|_{[x,b],1}\, . \end{cases}$$

Utilizing the inequality (32) we get the desired result (31). ∎

Remark 4. *Since*

$$\begin{aligned} &\frac{1}{6}(x-a)^3\left\|f''\right\|_{[a,x],\infty}+\frac{1}{6}(b-x)^3\left\|f''\right\|_{[x,b],\infty} \\ &\leq \frac{1}{6}\left[(x-a)^3+(b-x)^3\right]\max\left\{\left\|f''\right\|_{[a,x],\infty},\left\|f''\right\|_{[x,b],\infty}\right\} \\ &= \frac{1}{6}(b-a)\left[(x-a)^2-(x-a)(b-x)+(b-x)^2\right]\left\|f''\right\|_{[a,b],\infty}, \end{aligned}$$

then by (31) we get

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{33}$$
$$\le \frac{1}{6}\left[\left(\frac{x-a}{b-a}\right)^2-\left(\frac{x-a}{b-a}\right)\left(\frac{b-x}{b-a}\right)+\left(\frac{b-x}{b-a}\right)^2\right]$$
$$\times (b-a)^2\left\|f''\right\|_{[a,b],\infty},$$

for any $x\in[a,b]$.

Since

$$(x-a)^{1/q+2}\left\|f''\right\|_{[a,x],p}+(b-x)^{1/q+2}\left\|f''\right\|_{[x,b],p}$$
$$\le\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{1/q}\left[\left\|f''\right\|^p_{[a,x],p}+\left\|f''\right\|_{[x,b],p}\right]^{1/p}$$
$$=\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{1/q}\left\|f''\right\|_{[a,b],p},$$

then by (31) we get

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{34}$$
$$\le\frac{q}{(q+1)(q+2)}\left[\left(\frac{x-a}{b-a}\right)^{2q+1}+\left(\frac{b-x}{b-a}\right)^{2q+1}\right]^{1/q}$$
$$\times(b-a)^{1+1/q}\left\|f''\right\|_{[a,b],p},$$

for any $x\in[a,b]$.

Since

$$(x-a)^2\left\|f''\right\|_{[a,x],1}+(b-x)^2\left\|f''\right\|_{[x,b],1}$$
$$\le\max\left\{(x-a)^2,(b-x)^2\right\}\left[\left\|f''\right\|_{[a,x],1}+\left\|f''\right\|_{[x,b],1}\right]$$
$$=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2\left\|f''\right\|_{[a,b],1},$$

then by (31) we get

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\le\frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a)\left\|f''\right\|_{[a,b],1}$$

for any $x\in[a,b]$.

References

[1] A. M. Acu, A. Baboş and F. Sofonea, *The mean value theorems and inequalities of Ostrowski type.* Sci. Stud. Res. Ser. Math. Inform. 21 (2011), no. 1, 5–16.

[2] A. M. Acu and F. Sofonea, *On an inequality of Ostrowski type.* J. Sci. Arts 2011, no. 3(16), 281–287.

[3] F. Ahmad, N. S. Barnett and S. S. Dragomir, *New weighted Ostrowski and Čebyšev type inequalities.* Nonlinear Anal. 71 (2009), no. 12, e1408–e1412.

[4] M. W. Alomari, *A companion of Ostrowski's inequality with applications.* Transylv. J. Math. Mech. 3 (2011), no. 1, 9–14.

[5] M. W. Aalomari, M. Darus, S. S. Dragomir and P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense.* Appl. Math. Lett. 23 (2010), no. 9, 1071–1076.

[6] G. A. Anastassiou, *Ostrowski type inequalities.* Proc. Amer. Math. Soc. 123 (1995), No. 12, 3775–3781.

[7] G. A. Anastassiou, *Univariate Ostrowski inequalities, revisited.* Monatsh. Math. 135 (2002), No. 3, 175–189.

[8] G. A. Anastassiou, *Ostrowski inequalities for cosine and sine operator functions.* Mat. Vesnik 64 (2012), no. 4, 336–346.

[9] G. A. Anastassiou, *Multivariate right fractional Ostrowski inequalities.* J. Appl. Math. Inform. 30 (2012), no. 3-4, 445–454.

[10] G. A. Anastassiou, *Univariate right fractional Ostrowski inequalities.* Cubo 14 (2012), no. 1, 1–7.

[11] N. S. Barnett, S. S. Dragomir and I. Gomm, A *companion for the Ostrowski and the generalised trapezoid inequalities.* Math. Comput. Modelling 50 (2009), no. 1-2, 179–187.

[12] P. Cerone, W.-S. Cheung and S. S. Dragomir, *On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation.* Comput. Math. Appl. 54 (2007), No. 2, 183–191.

[13] P. Cerone and S. S. Dragomir, *Midpoint-type rules from an inequalities point of view.* Handbook of analytic-computational methods in applied mathematics, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[14] P. Cerone and S. S. Dragomir, *Trapezoidal-type rules from an inequalities point of view.* Handbook of analytic-computational methods in applied mathematics, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[15] P. Cerone, S. S. Dragomir and C. E. M. Pearce, *A generalised trapezoid inequality for functions of bounded variation,* Turk. J. Math., 24 (2000), 147-163.

[16] S. S. Dragomir, *The Ostrowski inequality for mappings of bounded variation,* Bull. Austral. Math. Soc., 60 (1999), 495-826.

[17] S. S. Dragomir, *On the mid-point quadrature formula for mappings with bounded variation and applications,* Kragujevac J. Math., 22 (2000), 13-19.

[18] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications,* Math. Ineq. & Appl., 4(1) (2001), 33-40. Preprint, *RGMIA Res. Rep. Coll.* 2(1999), No. 1, Article 7. [Online: http://rgmia.vu.edu.au/v2n1.html].

[19] S. S. Dragomir, *On the trapezoid quadrature formula and applications,* Kragujevac J. Math., 23 (2001), 25-36.

[20] S. S. Dragomir, *Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative.* Tamkang J. Math. 34 (2003), no. 3, 213–222.

[21] S. S. Dragomir, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation.* Arch. Math. (Basel) 91 (2008), no. 5, 450–460.

[22] S. S. Dragomir, *Some inequalities for continuous functions of selfadjoint operators in Hilbert spaces,* Acta Math. Vietnamica, to appear. Preprint *RGMIA Res. Rep. Coll.* 15(2012), Art. 16. http://rgmia.org/v15.php.

[23] S. S. Dragomir, *Refinements of the Ostrowski inequality in terms of the cumulative variation and applications,* Preprint RGMIA Res. Rep. Coll. 16 (2013), Art. 29, pp. 15 [http://rgmia.org/papers/v16/v16a29.pdf].

[24] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for absolutely continuous functions (I)*, Preprint RGMIA Res. Rep. Coll. 16 (2013), Art. 94.

[25] Z. Liu, *Some inequalities of perturbed trapezoid type.* J. Inequal. Pure Appl. Math. 7 (2006), no. 2, Article 47, 9 pp.

[26] Z. Liu, *A note on Ostrowski type inequalities related to some s-convex functions in the second sense.* Bull. Korean Math. Soc. 49 (2012), no. 4, 775–785.

[27] Z. Liu, *A sharp general Ostrowski type inequality.* Bull. Aust. Math. Soc. 83 (2011), no. 2, 189–209.

[28] Z. Liu, *New sharp bound for a general Ostrowski type inequality.* Tamsui Oxf. J. Math. Sci. 26 (2010), no. 1, 53–59.

[29] Z. Liu, *Some Ostrowski type inequalities and applications.* Vietnam J. Math. 37 (2009), no. 1, 15–22.

[30] Z. Liu, *Some companions of an Ostrowski type inequality and applications.* J. Inequal. Pure Appl. Math. 10 (2009), no. 2, Article 52, 12 pp.

[31] M. Masjed-Jamei and S. S. Dragomir, *A new generalization of the Ostrowski inequality and applications.* Filomat 25 (2011), no. 1, 115–123.

[32] B. G. Pachpatte, *A note on a trapezoid type integral inequality.* Bull. Greek Math. Soc. 49 (2004), 85–90.

[33] J. Park, *On the Ostrowskilike type integral inequalities for mappings whose second derivatives are s^*-convex.* Far East J. Math. Sci. (FJMS) 67 (2012), no. 1, 21–35.

[34] J. Park, *Some Ostrowskilike type inequalities for differentiable real* (α, m)-*convex mappings.* Far East J. Math. Sci. (FJMS) 61 (2012), no. 1, 75–91

[35] M. Z. Sarikaya, *On the Ostrowski type integral inequality.* Acta Math. Univ. Comenian. (N.S.) 79 (2010), no. 1, 129–134.

[36] W. T. Sulaiman, *Some new Ostrowski type inequalities.* J. Appl. Funct. Anal. 7 (2012), no. 1-2, 102–107.

[37] K.-L. Tseng, *Improvements of the Ostrowski integral inequality for mappings of bounded variation II.* Appl. Math. Comput. 218 (2012), no. 10, 5841–5847.

[38] K.-L. Tseng, S.-R. Hwang, G.-S. Yang, and Y.-M. Chou, *Improvements of the Ostrowski integral inequality for mappings of bounded variation I.* Appl. Math. Comput. 217 (2010), no. 6, 2348–2355.

[39] N. Ujević, *Error inequalities for a generalized trapezoid rule.* Appl. Math. Lett. 19 (2006), no. 1, 32–37.

[40] S. W. Vong, *A note on some Ostrowskilike type inequalities.* Comput. Math. Appl. 62 (2011), no. 1, 532–535.

[41] Y. Wu and Y. Wang, *On the optimal constants of Ostrowskilike inequalities involving n knots.* Appl. Math. Comput. 219 (2013), no. 14, 7789–7794.

[42] Q. Wu and S. Yang, *A note to Ujević's generalization of Ostrowski's inequality.* Appl. Math. Lett. 18 (2005), no. 6, 657–665.

[43] Y.-X. Xiao, *Remarks on Ostrowskilike inequalities.* Appl. Math. Comput. 219 (2012), no. 3, 1158–1162.

Silvestru Sever Dragomir
Mathematics, College of Engineering and Science,
Victoria University,
Melbourne, Victoria, Australia
email: *sever.dragomir@vu.edu.au*