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PRESERVING PROPERTIES AND ESTIMATIONS OF THE COEFFICIENTS FOR TWO SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. In this paper we study the preserving properties and bounds of the coefficients for two subclasses of analytic functions $UCSPT(\alpha,\beta)$ and $SP_PT(\alpha,\beta)$.

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1. Introduction

Let A denote the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are regular in the unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$. The function $f \in A$ is spirallike if $Re\ \left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0$ for all $z \in U$ and for some α with $|\alpha| < \frac{\pi}{2}$. Also $f(z)$ is convex spirallike if $zf'(z)$ is spirallike.

Let T denote the class consisting of functions f of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where a_n is a non-negative real number.

Definition 1. *[1] Let I_A be the Alexander integral operator defined as:*

$$I_A : A \to A,\ I_A(F) = f,\ \text{where}$$

$$f(z) = \int_0^z \frac{F(t)}{t} dt. \tag{2}$$

Definition 2. *[1] Let I_a be the Bernardi integral operator defined as:*

$$I_a : A \to A,\ I_a(F) = f,\ a = 1, 2, 3, \ldots,\ where$$

$$f(z) = \frac{a+1}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{3}$$

Definition 3. *[1] Let $I_{c+\delta}$ be the integral operator defined as: $I_{c+\delta} : A \to A$, $0 < u \leq 1$, $1 \leq \delta < \infty$, $0 < c < \infty$,*

$$f(z) = I_{c+\delta}(F)(z) = (c+\delta) \int_0^1 u^{c+\delta-2} F(uz) du. \tag{4}$$

Remark 1. *[1] For $\delta = 1$ and c=1,2,..., from the integral operator $I_{c+\delta}$ we obtain the Bernardi integral operator defined by (3).*

Definition 4. *[1] Let $F \in A$, $F(z) = z + b_2 z^2 + \cdots + b_n z^n + \ldots$, and $a \in \mathbb{R}^*$. We define the integral operator $L : A \to A$ by*

$$f(z) = L(F)(z) = \frac{1+a}{z^a} \int_0^z F(t) \left(t^{a-1} + t^{a+1}\right) dt\ . \tag{5}$$

2. Preliminary results

We now defined $UCSPT(\alpha, \beta)$ and $SP_P T(\alpha, \beta)$.

Definition 5. *[2] Let $UCSPT(\alpha, \beta)$ be the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ which satisfy the condition*

$$Re\, e^{-i\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right| + \beta,$$

$|\alpha| < \frac{\pi}{2},\ 0 \leq \beta < 1.$

Definition 6. *[2] Let $SP_P T(\alpha, \beta)$ be the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ which satisfy the condition*

$$Re\, e^{-i\alpha} \frac{zf'(z)}{f(z)} \geq \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta,$$

$|\alpha| < \frac{\pi}{2},\ 0 \leq \beta < 1.$

Lemma 1. *[3] Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0$. *Then*

$$\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta)\, n a_n \leq \cos\alpha - \beta, \tag{6}$$

if and only if $f(z)$ *is in* $UCSPT(\alpha, \beta)$.

Lemma 2. *[3] The function* f *given by* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0$ *is in* $SP_PT(\alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta)\, a_n \leq \cos\alpha - \beta. \tag{7}$$

Corollary 3. *[4] Let the function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0$ *be in the class* $UCSPT(\alpha, \beta)$, $|\alpha| < \frac{\pi}{2},\ 0 \leq \beta < 1$, *then*

$$a_n \leq \frac{\cos\alpha - \beta}{n\,(2n - \cos\alpha - \beta)}\ , \qquad n \geq 2. \tag{8}$$

Corollary 4. *[4] Let the function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0$ *be in the class* $SP_PT(\alpha, \beta)$, $|\alpha| < \frac{\pi}{2},\ 0 \leq \beta < 1$, *then*

$$a_n \leq \frac{\cos\alpha - \beta}{2n - \cos\alpha - \beta}\ , \qquad n \geq 2. \tag{9}$$

3. Main results

In what follows allong this article we consider $|\alpha| < \frac{\pi}{2}$ and $0 \leq \beta < 1$ such that $cos\alpha - \beta > 0$.

Theorem 5. *The Alexander integral operator defined by (2) preserves the class* $UCSPT(\alpha, \beta)$, *that is: If* $F \in UCSPT(\alpha, \beta)$, *then* $f(z) = I_A F(z) \in UCSPT(\alpha, \beta)$, *for* $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$.

Proof. Let $F \subset T$, $F(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0$. Then

$$f(z) = I_A F(z) = \int_0^z \frac{F(t)}{t} dt =$$

$$= \int_0^z \frac{1}{t} \left(t - \sum_{n=2}^{\infty} a_n\, t^n \right) dt =$$

$$= z - \sum_{n=2}^{\infty} \frac{a_n}{n} z^n$$

$$= z - \sum_{n=2}^{\infty} b_n z^n, \text{ with}$$

$b_n = \dfrac{a_n}{n} \geq 0$, $n \geq 2$. It follows that $f \in T$. We have now to prove that $f \in UCSPT(\alpha, \beta)$. Using Lemma 1 we need to prove that:

$$\sum_{n=2}^{\infty} (2n - cos\,\alpha - \beta)\, n b_n \leq cos\,\alpha - \beta, \tag{10}$$

for $n \geq 2$, $|\alpha| < \dfrac{\pi}{2}$, $0 \leq \beta < 1$. This means:

$$\sum_{n=2}^{\infty} (2n - cos\,\alpha - \beta)\, n \frac{a_n}{n} \leq cos\,\alpha - \beta. \tag{11}$$

But we have $\dfrac{a_n}{n} \leq a_n$, for n $\geq$ 2, and by using (6) and (11), we observe that inequality (10) is fulfilled.This means that $f \in UCSPT(\alpha, \beta)$.

In a similarly way we obtain:

Theorem 6. *The Alexander integral operator defined by (2) preserves the class* $SP_PT(\alpha, \beta)$, *that is: If* $F \in SP_PT(\alpha, \beta)$, *then* $f(z) = I_A F(z) \in SP_PT(\alpha, \beta)$, *for* $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$.

Theorem 7. *The integral operator* $I_{c+\delta}$ *defined by (4) preserves the class* $UCSPT(\alpha, \beta)$, *that is: If* $F \in UCSPT(\alpha, \beta)$, *then* $f(z) = I_{c+\delta}(F)(z) \in UCSPT(\alpha, \beta)$, *for* $F(z) = z - \sum_{n=2}^{\infty} a_n\, z^n$, $a_n \geq 0$.

Proof. Let $F \in UCSPT(\alpha,\beta),\ F(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0.$
We have, from Lemma 1

$$\sum_{n=2}^{\infty} (2n - cos\,\alpha - \beta)\, na_n \leq cos\,\alpha - \beta. \tag{12}$$

From (4) we obtain $f(z) = I_{c+\delta}(F)(z) = z - \sum_{n=2}^{\infty} \frac{c+\delta}{c+n+\delta-1}\, a_n\, z^n, where\ \ 0 < c < \infty,\ 1 \leq \delta < \infty.$
We also remark that for $0 < c < \infty,\ n \geq 2$ and $1 \leq \delta < \infty,$ we have

$$0 < \frac{c+\delta}{c+n+\delta-1} < 1 \tag{13}$$

Thus $f \in T$ and by using Lemma 1 we have only to prove that.

$$\sum_{n=2}^{\infty} (2n - cos\,\alpha - \beta)\, n \frac{c+\delta}{c+n+\delta-1} a_n \leq cos\,\alpha - \beta, \tag{14}$$

where $|\alpha| < \frac{\pi}{2},\ 0 \leq \beta < 1,\ 0 < c < \infty$ and $1 \leq \delta < \infty$.
By using the relation (13) we have

$$\frac{c+\delta}{c+n+\delta-1} \cdot a_n < a_n,$$

for $0 < c < \infty,\ n \geq 2,\ 1 \leq \delta < \infty$, and thus from (12) we conclude that the condition (14) take place and thus the proof it is complete.

In a similarly way we obtain:

Theorem 8. *The integral operator $I_{c+\delta}$ defined by (4) preserves the class $SP_PT(\alpha,\beta)$, that is: If $F \in SP_PT(\alpha,\beta)$, then $f(z) = I_{c+\delta}(F)(z) \in SP_PT(\alpha,\beta)$, for $F(z) = z - \sum_{n=2}^{\infty} a_n\, z^n,\ a_n \geq 0$.*

The following two results are proved by using the Remark 1:

Corollary 9. *The Bernardi integral operator defined by (3) preserves the class $UCSPT(\alpha,\beta)$, that is: If $F \in UCSPT(\alpha,\beta)$, then $f(z) = I_a F(z) \in UCSPT(\alpha,\beta)$, for $F(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0.$*

Corollary 10. *The Bernardi integral operator defined by (3) preserves the class* $SP_PT(\alpha,\beta)$*, that is: If* $F \in SP_PT(\alpha,\beta)$*, then* $f(z) = I_aF(z) \in SP_PT(\alpha,\beta)$*, for* $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$.

Theorem 11. *Let* $F \in UCSPT(\alpha,\beta)$ *with* $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$*. For* $f(z) = I_a(F)(z)$, $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, $z \in U$*, where the integral operator* I_a *it is defined by (3), we have:*

$$a_n \leq \frac{(a+1)(\cos\alpha - \beta)}{n(a+n)\,(2n - \cos\alpha - \beta)}\ , \qquad n \geq 2.$$

Proof. For $f = I_a(F)(z)$ with $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$ and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ we have

$$a_n = b_n \cdot \frac{a+1}{a+n},$$

where $a = 1, 2, 3, \ldots$, $n \geq 2$.
The coefficient bounds for the functions belonging to the class $UCSPT(\alpha,\beta)$ are

$$b_n \leq \frac{\cos\alpha - \beta}{n\,(2n - \cos\alpha - \beta)}.$$

For $n \geq 2$ we obtain

$$a_n = b_n \cdot \frac{a+1}{a+n} \leq$$
$$\leq \frac{\cos\alpha - \beta}{n\,(2n - \cos\alpha - \beta)} \cdot \frac{a+1}{a+n} =$$
$$= \frac{(a+1)(\cos\alpha - \beta)}{n(a+n)\,(2n - \cos\alpha - \beta)}$$

Hence the theorem is proved.

Theorem 12. *Let* $F \in UCSPT(\alpha,\beta)$ *with* $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$*. For* $f(z) = L(F)(z)$, $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, $z \in U$*, where the integral* *operator L it is defined by (5), we have:*

$$a_2 \leq \frac{(a+1)(cos\,\alpha - \beta)}{2(a+2)\,(4 - cos\,\alpha - \beta)}\ ,$$

$$a_3 \leq \frac{(a+1)(18 - 2cos\,\alpha - 4\beta)}{3(a+3)\,(6 - cos\,\alpha - \beta)}\ ,$$

$$a_n \leq \frac{1}{(n-2)}\left(\frac{cos\,\alpha - \beta}{2n - cos\,\alpha - \beta} + \frac{cos\,\alpha - \beta}{2n - 4 - cos\,\alpha - \beta}\right) \cdot \frac{a+1}{a+n}.$$

Proof. For $f = L(F)(z)$ with $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$ and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ we have:

$$a_2 = b_2 \cdot \frac{a+1}{a+2}\ ,$$

$$a_3 = (b_3 + 1) \cdot \frac{a+1}{a+3}\ ,$$

$$a_n = (b_n + b_{n-2}) \cdot \frac{a+1}{a+n}\ ,$$

where $a \in \mathbb{R}^*,\ n \geq 4$.
The coefficient bounds for the functions belonging to the class $UCSPT(\alpha, \beta)$ are :

$$b_n \leq \frac{cos\,\alpha - \beta}{n\,(2n - cos\,\alpha - \beta)}.$$

For $n \geq 4$ we obtain:

$$a_n = (b_n + b_{n-2}) \cdot \frac{a+1}{a+n} \leq$$

$$\leq \frac{cos\,\alpha - \beta}{n\,(2n - cos\,\alpha - \beta)} \cdot \frac{a+1}{a+n} +$$

$$+ \frac{cos\,\alpha - \beta}{(n-2)\,(2n - 4 - cos\,\alpha - \beta)} \cdot \frac{a+1}{a+n}\ ,$$

$$a_n \leq \frac{1}{(n-2)}\left(\frac{cos\,\alpha - \beta}{2n - cos\,\alpha - \beta} + \frac{cos\,\alpha - \beta}{2n - 4 - cos\,\alpha - \beta}\right) \cdot \frac{a+1}{a+n}.$$

For $n = 2$ we have:

$$a_2 = b_2 \cdot \frac{a+1}{a+2} \leq$$

$$\leq \frac{\cos\alpha - \beta}{2\,(4 - \cos\alpha - \beta)} \cdot \frac{a+1}{a+2} =$$

$$= \frac{(a+1)(\cos\alpha - \beta)}{2(a+2)\,(4 - \cos\alpha - \beta)}$$

Similarly for $n = 3$ we have:

$$a_3 \leq \left(\frac{\cos\alpha - \beta}{3(6 - \cos\alpha - \beta)} + 1 \right) \cdot \frac{a+1}{a+3}\ ,$$

$$a_3 \leq \frac{(a+1)(18 - 2\cos\alpha - 4\beta)}{3(a+3)\,(6 - \cos\alpha - \beta)}.$$

Hence the theorem is proved.

In a similarly way we obtain:

Theorem 13. *Let $F \in SP_PT(\alpha, \beta)$ with $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$. For $f(z) = I_a(F)(z)$, $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, $z \in U$, where the integral operator I_a it is defined by (3), we have:*

$$a_n \leq \frac{(a+1)(\cos\alpha - \beta)}{(a+n)\,(2n - \cos\alpha - \beta)}\ , \qquad n \geq 2.$$

Theorem 14. *Let $F \in SP_PT(\alpha, \beta)$ with $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$. For $f(z) = L(F)(z)$, $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, $z \in U$, where the integral operator L it is defined by (5), we have:*

$$a_2 \leq \frac{(a+1)(\cos\alpha - \beta)}{(a+2)\,(4 - \cos\alpha - \beta)}\ ,$$

$$a_3 \leq \frac{(a+1)(18 - 2\cos\alpha - 4\beta)}{(a+3)\,(6 - \cos\alpha - \beta)}\ ,$$

$$a_n \leq \left(\frac{\cos\alpha - \beta}{2n - \cos\alpha - \beta} + \frac{\cos\alpha - \beta}{2n - 4 - \cos\alpha - \beta} \right) \cdot \frac{a+1}{a+n}.$$

References

[1] R.Diaconu, *Special classes of analytic functions and their transformations by means of integral operators*, Doctorat thesis, Piteşti 2014.

[2] C. Selvaraj, R. Geetha, *On Certain subclasses of holomorphicfunctions defined on the unit disk*, Acta Universitatis Apulensis, ISSN: 1582-5329, No. 36/2013, pp. 225-233.

[3] C. Selvaraj, R. Geetha, *On subclasses of uniformly convex spirallike functions and corresponding class of spirallike functions,* International Journal of Contemporary Mathematical Sciences, 5, 37-40 (2010).

[4] C. Selvaraj, R. Geetha, *On uniformly spirallike functions and a corresponding subclass of spirallike functions,* Global Journal of Science Frontier Research, 10,5 (2010), 36-41.

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