Acta Universitatis Apulensis No. 42/2015
ISSN: 1582-5329 pp. 171-180
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.42.11

SOME NEW DISTORTION THEOREMS FOR STARLIKE HARMONIC FUNCTIONS OF ORDER ALPHA

H. E. Özkan Uçar, M. Aydoğan, Y. Polatoğlu

Abstract. Let $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{U}$. If $f(z)$ satisfies the condition $(|h'(z)|^2 - |g'(z)|^2) > 0$, then $f(z)$ is called sense-preserving harmonic univalent function and denoted by $\mathcal{S}_H$. We also note that $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_H$ if and only if $g'(z) = \omega(z)h'(z)$ where $\omega(z)$ is second dilatation of $f(z)$. Moreover, let $H(\mathbb{U})$ be the linear space of all analytic functions defined on the simply connected domain $\mathbb{U} \subset \mathbb{C}$. A log-harmonic mapping F is a solution of the non-linear elliptic partial differential equation $\dfrac{\overline{F_{\overline{z}}}}{\overline{F}} = \omega_1(z)\dfrac{F_z}{F}$, where the second dilatation function $\omega_1(z) \in H(\mathbb{U})$ is such that $|\omega_1(z)| < 1$ for all $z \in \mathbb{U}$. It has been shown that if F is non-vanishing log-harmonic mapping, then F can be expressed on $F = H(z)\overline{G(z)}$, where $H(z)$ and $G(z)$ are analytic functions in $\mathbb{U}$ with the normalization $H(0) = G(0) = 1$, and the class of non-vanishing log-harmonic functions is denoted by $\mathcal{S}^*_{LH}$.
The aim of this paper is to give the relation between the classes $\mathcal{S}^*_H$ and $\mathcal{S}^*_{LH}$ the new distortion theorems of starlike harmony univalent functions of LH order α.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Harmonic functions, Log-harmonic functions, Univalent functions, Distortion theorem.

1. Introduction

Let $\mathcal{S}^*(\alpha)$ denote the class of functions $s(z) = z + a_2z^2 + \dots$ which are analytic in the open unit disc $\mathbb{U} = \{z : \ |z| < 1\}$ and satisfy

$$Re\left(z\frac{s'(z)}{s(z)}\right) > \alpha \tag{1}$$

for all $z \in \mathbb{U}$.

Next, let Ω be the family of functions $\phi(z)$ which are analytic in $\mathbb{U}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{U}$. Let $\mathcal{P}$ denote the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ which are regular and satisfy the conditions $Re\, p(z) > \alpha$, $p(0) = 1$ for all $z \in \mathbb{U}$, and we note that $p(z) \in \mathcal{P}$ if and only if

$$p(z) = \frac{1 + (1 - 2\alpha)\phi(z)}{1 - \phi(z)} \tag{2}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{U}$, see [4].

Moreover, let $f_1(z) = z + d_2 z^2 + \ldots$ and $f_2(z) = z + e_2 z^2 + \ldots$ be analytic functions in $\mathbb{U}$. If there exists a function $\phi(z) \in \Omega$ such that $f_1(z) = f_2(\phi(z))$, we then say that $f_1(z)$ is subordinate to $f_2(z)$ and we write $f_1(z) \prec f_2(z)$.

Finally, a function f is said to be a complex valued harmonic function in $\mathbb{U}$ if both $Re\, f$ and $Im\, f$ are real harmonic in $\mathbb{U}$. Every such f can be uniquely represented by $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic with the normalization $h(0) = g(0) = 0$, $h'(0) = 1$. A complex-valued harmonic function f which is not identically constant and satisfies $f = h(z) + g(z)$ is said to be sense-preserving in $\mathbb{U}$ if it satisfies the equation

$$g'(z) = \omega(z) h'(z) \tag{3}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with $|\omega(z)| < 1$ for every $z \in \mathbb{U}$ and $\omega(z)$ is called the second dilatation of f. The Jacobian of f is defined by

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 \tag{4}$$

Let $H(\mathbb{U})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{U}$. A log-harmonic mapping F is the solution of the non-linear elliptic partial differential equation

$$\frac{F_z}{F} = \omega(z) \frac{F_z}{F} \tag{5}$$

where $\omega(z)$ is the second dilatation of F and $\omega(z) \in H(\mathbb{U})$, $|\omega(z)| < 1$ for every $z \in \mathbb{U}$. It has been show that if F is a non-vanishing log-harmonic function, then F can be expressed as

$$F = H(z) \cdot \overline{G(z)} \tag{6}$$

where $H(z)$ and $G(z)$ are analytic in $\mathbb{U}$ with the normalization $H(0) = G(0) = 1$. The class of non-vanishing log-harmonic functions is denoted by $\mathcal{S}^0_{LH}$. Also,the class of log-harmonic functions is denoted by $\mathcal{S}_{LH}$. For details, see [1], [2], and [3].

In [5], Jack's lemma states that for the (non-constant) function $\omega(z)$ which is analytic in $\mathbb{U}$ with $\omega(0) = 0$, if $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$,then $z_0 \omega'(z_0) = k\omega(z_0)$,where k is a real number and $k \geq 1$.

2. Main Results

Theorem 1. $f = h(z) + \overline{g(z)} \in \mathcal{S}_H^* \Longleftrightarrow F = H(z)\overline{G(z)} = e^{h(z)+\overline{g}(z)} \in \mathcal{S}_{LH}^0$.

Proof. Let $f = h(z) + \overline{g(z)} \in \mathcal{S}_H$. Then we have

$$\omega(z) = \frac{g'(z)}{h'(z)}. \tag{7}$$

Now we define the function

$$\begin{cases} H(z) = e^{h(z)} \\ G(z) = e^{g(z)} \end{cases} \Longrightarrow F = H(z) \cdot \overline{G(z)} = e^{h(z)+\overline{g(z)}}, \tag{8}$$

then we have

$$\begin{cases} \log H(z) = h(z) \Longrightarrow h'(z) = \dfrac{H'(z)}{H(z)} \\ \log G(z) = g(z) \Longrightarrow g'(z) = \dfrac{G'(z)}{H'(z)} \end{cases} \tag{9}$$

$$\begin{cases} H(0) = e^{h(0)} = e^0 = 1 \\ G(0) = e^{g(0)} = e^0 = 1 \end{cases} \Longrightarrow F(0) = H(0)\overline{G(0)} = 1, \tag{10}$$

$$\omega(z) = \frac{g'(z)}{h'(z)} = \frac{G'(z)/G(z)}{H'(z)/H(z} \Longleftrightarrow \frac{\overline{F_{\overline{z}}}}{\overline{F}} = \omega(z)\frac{F_z}{F}. \tag{11}$$

Therefore, $F = H(z)\overline{G(z)} \in \mathcal{S}_{LH}^0$.

Conversely, let $F = H(z)\overline{G(z)} \in \mathcal{S}_{LH}^0$. Then we define the following functions

$$\begin{cases} \log H(z) = h(z) \\ \log G(z) = g(z) \end{cases} \tag{12}$$

Then,

$$\begin{cases} h(0) = \log H(0) = log 1 = 0 \\ g(0) = \log G(0) = log 1 = 0 \end{cases}$$

$h(z)$ and $g(z)$ are analytic in $\mathbb{U}$ and also we have (11). Using (9) in (11) we obtain $\omega(z) = \dfrac{g'(z)}{h'(z)}$ this shows that $f = h(z) + g(z) \in \mathcal{S}_H$.

Lemma 2. *The starlike condition of* $F = H(z).\overline{G(z)} = e^{h(z)+\overline{g(z)}}$ *is*

$$Re(zh'(z) - zg'(z)) > 0. \tag{13}$$

Proof.

$$\begin{aligned}
F &= H(z).\overline{G(z)} = e^{h(z)+\overline{g(z)}}\\
\Rightarrow F_z &= h'(z).e^{h(z)+\overline{g(z)}} \Rightarrow zF_z = zh'(z).e^{h(z)+\overline{g(z)}}\\
F_{\overline{z}} &= \overline{g'(z)}.e^{h(z)+\overline{g(z)}} \Rightarrow \overline{z} \Rightarrow F_{\overline{z}} = \overline{z}\overline{g'(z)}e^{h(z)+\overline{g(z)}}\\
\Rightarrow \frac{zF_z - \overline{z}F_{\overline{z}}}{F} &= \frac{e^{h(z)+\overline{g(z)}} \cdot [zh'(z) - \overline{z}\overline{g'(z)}]}{e^{h(z)+\overline{g(z)}}} = zh'(z) - \overline{z}\overline{g'(z)}\\
\Rightarrow Re\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F}\right) &= Re(zh'(z) - \overline{z}\overline{g'(z)}) = Re(zh'(z) - zg'(z)) > 0.
\end{aligned}$$

Lemma 3. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_H^*$. Then,*

$$Re(zh'(z) - zg'(z)) = r\frac{\partial}{\partial r}\log\left|e^{h(z)-\overline{g(z)}}\right| \tag{14}$$

Proof.

$$\begin{aligned}
&e^{h(re^{i\theta})-\overline{g(re^{i\theta})}} = \left|e^{h(re^{i\theta})-g(re^{i\theta})}\right| e^{i\theta}\\
&\Rightarrow \log(e^{h(re^{i\theta})-\overline{g(re^{i\theta})}}) = \log|e^{h(re^{i\theta})-\overline{g(re^{i\theta})}}|e^{i\theta}\\
&\Rightarrow h(re^{i\theta}) - \overline{g(re^{i\theta}} = \log|e^{h(re^{i\theta})-\overline{g(re^{i\theta})}}| + i\theta\log e = \log|e^{h(re^{i\theta})-\overline{g(re^{i\theta})}}| + i\theta\\
&\Rightarrow e^{i\theta}\cdot h'(re^{i\theta}) - \overline{e^{i\theta}\cdot g(re^{i\theta})} = \frac{\partial}{\partial r}\log|e^{h(re^{i\theta})-\overline{g(re^{i\theta})}}|\\
&\Rightarrow re^{i\theta}\cdot h'(re^{i\theta}) - \overline{re^{i\theta}\cdot g'(re^{i\theta})} = r\frac{\partial}{\partial r}\log|e^{h(re^{i\theta})-\overline{g(re^{i\theta})}}|\\
&\Rightarrow zh'(z) - \overline{zg'(z)} = r\frac{\partial}{\partial r}\log|e^{h(z)-\overline{g(z)}}|\\
&\Rightarrow Re(zh'(z) - \overline{zg'(z)}) = r\frac{\partial}{\partial r}\log|e^{h(z)-\overline{g(z)}}|\\
&\Rightarrow Re(zh'(z) - zg'(z)) = r\frac{\partial}{\partial r}\log|e^{h(z)-\overline{g(z)}}|.
\end{aligned}$$

Theorem 4. *Let $f = h(z) + g(z)$ be an element of $\mathcal{S}_H^*$. The function f satisfies the condition*

$$zh'(z) - zg'(z) \prec \frac{2(1-\alpha)z}{1-z} \tag{15}$$

if and only if $F = ze^{h(z)+\overline{g(z)}} \in \mathcal{S}_{LH}^*(\alpha)$.

Proof. Let f satisfies (15). We define the function $\phi(z) \in \Omega$ by

$$e^{h(z)-g(z)} = (1-\phi(z))^{-2(1-\alpha)}, \tag{16}$$

where $(1-\phi(z))^{-2(1-\alpha)}$ has the value 1 at $z=0$ (we consider the corresponding Riemann branch). Then $\phi(z)$ is analytic and $\phi(0)=0$. If we take the logarithmic derivative of (16) and after the brief calculations we get

$$h'(z) - g'(z) = \frac{-2(1-\alpha)(-\phi'(z))}{1-\phi(z)}$$

and then

$$zh'(z) - zg'(z) = \frac{2(1-\alpha)z\phi'(z)}{1-\phi(z)}. \tag{17}$$

On the other hand, the function $w := \dfrac{2(1-\alpha)z}{1-z}$ maps $|z|=r$ onto the circle with the radius $\rho = \rho(r) = \dfrac{2(1-\alpha)r}{1-r^2}$ and the center $c = c(r) = \left(\dfrac{2(1-\alpha)r^2}{1-r^2}, 0\right)$. Now it is easy to realize that the subordination (15) is equivalent to $|\phi(z)| <1$ for all $z \in \mathbb{U}$. Indeed, let us assume to the contrary. Then there is a $z_1 \in \mathbb{U}$ such that $|\phi(z_1)| = 1$. By Jack?s Lemma, $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$,so for such z_1 we have

$$z_1h'(z_1) - z_1g'(z_1) = \frac{2(1-\alpha)k\phi(z_1)}{1-\phi(z_1)} = kw(\phi(z_1)) \notin \mathcal{S}(\mathbb{U})$$

but this contradicts to (15); so our assumption is wrong, i.e., $|\phi(z)| < 1$ for every $z \in \mathbb{U}$. By using the condition (15) we get

$$1 + zh'(z) - zg'(z) = \frac{1+(1-2\alpha)\phi(z)}{1-\phi(z)}. \tag{18}$$

On the other hand, using Theorem 1, Lemma 2, and Lemma 3 and after simple calculations we get

$$F = zH(z).\overline{G(z)} = ze^{h(z)+\overline{g(z)}} \in \mathcal{S}^*_{LH}$$

$$\Rightarrow \log F = \log z + \log H(z) + \log \overline{G(z)} = \log z + \log h(z) + \log \overline{g(z)}$$

$$\Rightarrow \begin{cases} \dfrac{F_z}{F} = \dfrac{1}{z} + \dfrac{H'(z)}{H(z)} = \dfrac{1}{z} + h'(z) \Rightarrow \dfrac{zF_z}{F} = 1 + z\dfrac{H'(z)}{H(z)} = 1 + zh'(z) \\ \dfrac{F_{\overline{z}}}{F} = \dfrac{\overline{G'(z)}}{\overline{G(z)}} = \overline{g'(z)} \Rightarrow \dfrac{\overline{z}F_{\overline{z}}}{F} = \overline{z}\dfrac{\overline{G'(z)}}{\overline{G(z)}} = \overline{z}\overline{g'(z)} \end{cases}$$

$$\Rightarrow Re\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)=Re\left(1+z\frac{H'(z)}{H(z)}-\overline{z}\frac{\overline{G'(z)}}{\overline{G(z)}}\right)=Re(1+zh'(z)-\overline{z}\overline{g'(z)}). \quad (19)$$

Considering (18) and (19) together we obtain the desired result.
For the converse, let $F=ze^{h(z)+\overline{g(z)}}$ be an element of $\mathcal{S}^*_{LH}(\alpha)$. It follows that $Re\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)>\alpha$ and

$$\frac{zF_z-\overline{z}F_{\overline{z}}}{F}=\frac{1+(1-2\alpha)\phi(z)}{1-\phi(z)}.$$

On the other hand,

$$\frac{zF_z-\overline{z}F_{\overline{z}}}{F}=1+zh'(z)-\overline{z}\overline{g'(z)}$$

$$\Rightarrow Re\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)=Re(1+zh'(z)-\overline{z}\overline{g'(z)})>\alpha$$

$$\Rightarrow 1+zh'(z)-zg'(z)=\frac{1+(1-2\alpha)\phi(z)}{1-\phi(z)}$$

$$\Rightarrow zh'(z)-zg'(z)=\frac{2(1-\alpha)\phi(z)}{1-\phi(z)}.$$

This shows that $zh'(z)-zg'(z)\prec\frac{2(1-\alpha)z}{1-z}$.

Theorem 5. *Let $f(z)=h(z)+\overline{g(z)}$ be an element of $\mathcal{S}^*_H(\alpha)$. Then,*

$$\frac{(1+r)^{2\alpha-3}}{r(1-r)}\leq|e^{h(z)-\overline{g(z)}}|\leq\frac{(1-r)^{2\alpha-3}}{r(1+r)}.$$

This inequality is sharp because if we consider the following simple calcula- ??tions

$$\begin{aligned}
&h(z)-g(z)=\log(1-z)^{-2(1-\alpha)}\\
&\Longrightarrow h(z)-g(z)=-2(1-\alpha)\log(1-z)\\
&\Longrightarrow h'(z)-g'(z)=\frac{2(1-\alpha)}{1-z}\\
&\Longrightarrow zh'(z)-zg'(z)=\frac{2(1-\alpha)z}{1-z}\\
&\Longrightarrow 1+zh'(z)-zg'(z)=1+\frac{2(1-\alpha)z}{1-z}=\frac{1+(1-2\alpha)z}{1-z}
\end{aligned}$$

then the extremal function is the solution of the following differential equation

$$h(z) - g(z) = \log(1-z)^{-2(1-\alpha)}$$
$$g_{\overline{z}} = \overline{f}_z - \overline{h}_z = 0.$$

Proof. The set of the values of the function $\frac{2(1-\alpha)z}{1-z}$ is the closed disc with the center c and the radius ρ, where

$$c = c(r) = \left(\frac{??2(1-\alpha)r^2}{1-r^2}, 0\right), \qquad \rho = \rho(r) = \frac{2(1-\alpha)r}{1-r^2}.$$

Using the subordination, we can write

$$\left|(zh'(z) - zg'(z) + 1) - \frac{2(1-\alpha)r^2}{1-r^2}\right| \le \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow \left|(zh'(z) - zg'(z)) + 1 - \frac{2(1-\alpha)r^2}{1-r^2}\right| \le \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow \left|(zh'(z) - zg'(z)) - \left(\frac{2(1-\alpha)r^2}{1-r^2} - 1\right)\right| \le \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow \left|(zh'(z) - zg'(z)) - \left(\frac{2(1-\alpha)r^2 - 1 + r^2}{1-r^2} - 1\right)\right| \le \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow \left|(zh'(z) - zg'(z)) - \left(\frac{(2(1-\alpha)+1)r^2 - 1}{1-r^2} - 1\right)\right| \le \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow \left|(zh'(z) - zg'(z)) - \left(\frac{(3-2\alpha)r^2 - 1}{1-r^2} - 1\right)\right| \le \frac{2(1-\alpha)r}{1-r^2}$$
$$-\frac{2(1-\alpha)r}{1-r^2} \le -\left|(zh'(z) - zg'(z)) - \left(\frac{(3-2\alpha)r^2 - 1}{1-r^2} - 1\right)\right|$$

$$\le Re\left[(zh'(z) - zg'(z)) - \frac{(3-2\alpha)r^2 - 1}{1-r^2}\right]$$
$$\left|(zh'(z) - zg'(z)) - \left(\frac{(3-2\alpha)r^2 - 1}{1-r^2} - 1\right)\right| \le \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow -\frac{2(1-\alpha)r}{1-r^2} \le Re[zh'(z) - zg'(z)] - \frac{(3-2\alpha)r^2 - 1}{1-r^2} \le \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow \frac{(3-2\alpha)r^2 - 1}{1-r^2} - \frac{2(1-\alpha)r}{1-r^2} \le Re[zh'(z) - zg'(z)] \le \frac{(3-2\alpha)r^2 - 1}{1-r^2} + \frac{2(1-\alpha)r}{1-r^2}$$
$$\Rightarrow \frac{(3-2\alpha)r^2 - 2(1-\alpha)r - 1}{1-r^2} v \le Re[zh'(z) - zg'(z)] \le \frac{(3-2\alpha)r^2 + 2(1-\alpha)r - 1}{1-r^2} \tag{20}$$

On the other hand, from Lemma 3 we have

$$Re[zh'(z) - zg'(z)] = r\frac{\partial}{\partial r}\log\left|e^{h(z)-\overline{g(z)}}\right|. \tag{21}$$

Considering (20) and (21) together, then the inequality (20) can be written in the following form

$$\begin{aligned}\frac{(3-2\alpha)r^2-2(1-\alpha)r-1}{1-r^2} \leq r\frac{\partial}{\partial r}\log\left|e^{h(z)-\overline{g(z)}}\right| \leq \frac{(3-2\alpha)r^2+2(1-\alpha)r-1}{1-r^2}\\ \frac{(3-2\alpha)r^2-2(1-\alpha)r-1}{r(1-r^2)} \leq \frac{\partial}{\partial r}\log\left|e^{h(z)-\overline{g(z)}}\right| \leq \frac{(3-2\alpha)r^2+2(1-\alpha)r-1}{r(1-r^2)}\end{aligned} \tag{22}$$

Since

$$\frac{(3-2\alpha)r^2-2(1-\alpha)r-1}{r(1-r^2)} = -\frac{1}{r} + \frac{1}{1-r} + \frac{2\alpha-3}{1+r},$$

It follows that

$$\int \frac{(3-2\alpha)r^2-2(1-\alpha)r-1}{r(1-r^2)}dr = \log\frac{(1+r)^{2\alpha-3}}{r(1+r)} \tag{23}$$

Similarly, since

$$\frac{(3-2\alpha)r^2+2(1-\alpha)r-1}{r(1-r^2)} = -\frac{1}{r} - \frac{1}{1-r} + \frac{3-2\alpha}{1-r},$$

it follows that

$$\int \frac{(3-2\alpha)r^2+2(1-\alpha)r-1}{r(1-r^2)}dr = \log\frac{(1-r)^{2\alpha-3}}{r(1+r)}. \tag{24}$$

Considering (22), (23), (24) and integrating both sides of (22) we obtain

$$\frac{(1+r)^{2\alpha-3}}{r(1-r)} \leq\leq \left|e^{h(z)-\overline{g(z)}}\right| \leq \frac{(1-r)^{2\alpha-3}}{r(1+r)}.$$

Corollary 6. *Let $f(z) = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_H^*(\alpha)$. Then,*

$$\left|(e^{h(z)-g(z)})^{\frac{1}{2(1-\alpha)}} - 1\right| < 1.$$

This inequality is the Marx-Strohhacker inequality [4] for the starlike har- monic univalent functions of order α.

Proof. Using Theorem 4, we have

$$e^{h(z)-g(z)} = (1-\phi(z))^{-2(1-\alpha)}.$$

This equality shows that

$$e^{h(z)-g(z)} = \frac{1}{(1-\phi(z))^{-2(1-\alpha)}} \Rightarrow \left| (e^{h(z)-g(z)})^{\overline{\dfrac{1}{2(1-\alpha)}}} - 1 \right| = |-\phi(z)| < 1.$$

Corollary 7. *Let $f(z) = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_H^*(\alpha)$. Then,*

$$|h'(z) - g'(z)| < \frac{2(1-\alpha)}{1-r}.$$

Proof. Let $s(z) := (e^{h(z)-g(z)})^{\frac{1}{2(1-\alpha)}} - 1$. Then by Corollary 7 and (16), we have $s(0) = 0$, $|s(z)| < 1$ and $s(z) = z\phi(z)$. Since

$$z\phi(z) = (e^{h(z)-g(z)})^{\overline{\dfrac{1}{2(1-\alpha)}}} - 1$$

?we have

$$h(z) - g(z) = 2(1-\alpha)\log(1 + z\phi(z)).$$

So,

$$h'(z) - g'(z) = \frac{2(1-\alpha)(\phi(z) + z\phi'(z))}{1 + z\phi(z)}$$

and hence

$$|h'(z) - g'(z)| \leq \frac{2(1-\alpha)}{1-r}.$$

Acknowledgements The work presented here was partially supported by Işık University Scientific Research Funding Agency under Grant Number: BAP-14B102.

References

[1] Z. Abdulhadi and Y. Abu Muhanna, *Starlike log-harmonic mapping of order α*, JIPAM, Vol. 7, issue 4 Article 123 (2006).

[2] Z. Abdulhadi and D. Bshouty, *Univalent functions $H.\overline{H}(D)$.*, Trans. Amer. MAth. Soc., 305(1988), 841-849.

[3] P. Duren, *Harmonic Mappings in the Plane*, Vol. 156 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge UK, (2004).

[4] A. W. Goodman, *Univalent Functions*, Volume I, Mariner publishing Company INC, Tampa Florida, (1983).

[5] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. 3, (1971), 369-374.

H. Esra Özkan Uçar
Department of Mathematics and Computer Science
İstanbul Kültür University,
İstanbul, Turkey
email: *e.ozkan@iku.edu.tr*

Melike Aydoğan
Department of Mathematics,
Işık University,
İstanbul, Turkey
email: *melike.aydogan@isikun.edu.tr*

Yaşar Polatoğlu
Department of Mathematics and Computer Science
İstanbul Kültür University,
İstanbul, Turkey
email: *y.polatoglu@iku.edu.tr*