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ABOUT A CLASS OF LINEAR AND POSITIVE STANCU-TYPE OPERATORS

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Abstract. The objective of this paper is to introduce a class of Stancu-type operators with the property that the test functions e_0 and e_1 are reproduced. Also, in our approach, two theorems of error approximation and two Voronovskaja-type theorems for this operators are obtained.

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1. Introduction

Let $\mathbb{N}$ be a set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and α, β positive real numbers. We denote by e_j the monomial of j degree, $j \in \mathbb{N}_0$. In 1969 D.D.Stancu [9], for any $m \in \mathbb{N}$ and $0 \leq \alpha \leq \beta$, has introduced the linear positive operator

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.1}$$

defined for any $f \in C([0,1])$ and $x \in [0,1]$. The author proved that if $f \in C([0,1])$ then $P_m^{(\alpha,\beta)}(f) \longrightarrow f$ uniform on $[0,1]$. Note that the operator from (1.1) preserves only the test function e_0. Following the ideas from [3], [4], [5], [6] and [8], in this paper we introduce a general class which preserves the test functions e_0 and e_1. For our approach two convergence theorems and two Voronovskaja-type theorems are obtained. The paper is organized as follows: in Section 2 we recall some results obtained in [7] which are essentially for obtaining the main results of this paper; Section 3 is devoted to the properties that the class of linear and positive operators that preserves the test functions e_0 and e_1, it has; finally, in Section 4, we plot on the same graph the images generated for exponential function by our operator and by the classical Stancu operator.

2. Preliminaries

In this section, we recall some notions and results which we will use in what follows.

We consider I, J real intervals with the property $I \cap J \neq \emptyset$ and we shall use the function sets: $E(I), F(J)$ which are certain subsets of the set of real valued functions defined on I, respectively J,

$$B(I) = \{f | f : I \to \mathbb{R}, f \textit{ bounded on } I\},$$

$$C(I) = \{f | f : I \to \mathbb{R}, f \textit{ continuous on } I\}$$

and

$$C_B(I) = B(I) \cap C(I).$$

For $x \in I$, we consider the function $\psi_x : I \to \mathbb{R}, \psi_x(t) = t - x$. For any $m \in \mathbb{N}$ we consider the functions $\varphi_{m,k} : J \to \mathbb{R}$, with the property $\varphi_{m,k}(x) \geq 0$, for any $x \in J, k \in \{0, 1, ..., m\}$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$, $k \in \{0, 1, ..., m\}$. For $m \in \mathbb{N}$ we define the operators $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f). \tag{2.1}$$

Remark 1. The operators $(L_m)_{m\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For any $f \in E(I), x \in I \cap J$ and for $i \in \mathbb{N}_0$, we define $T_{m,i}$ by

$$(T_{m,i}L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi_x^i) \tag{2.2}$$

In the following, let s be a fixed even natural number, we suppose that the operators $L_m, m \in \mathbb{N}$ verifies the conditions: there exists the smallest $\alpha_s, \alpha_{s+1} \in [0, \infty)$ such that

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,j}L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{2.3}$$

for any $x \in I \cap J, j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{2.4}$$

If $I \subset \mathbb{R}$ is a given interval and $f \in C_B(I)$, then the first order modulus of smoothness of f is the function $\omega(f; \cdot) : [0, +\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by $\omega(f, \delta) = sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\}$.

Theorem 1. ([7]) *Let $f : I \longrightarrow \mathbb{R}$ be a function. If $x \in I \cap J$ and f is a s times derivable function in x, the function $f^{(s)}$ is continuous in x, then*

$$\lim_{m \longrightarrow \infty} m^{s-\alpha_s}\Big((L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T_{m,i}L_m)(x)\Big) = 0. \tag{2.5}$$

If f is a s times differentiable function on I, the function $f^{(s)}$ is continuous on I and there exists $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ such that for any natural number $m \geq m(s)$ and for any $x \in I \cap J$ we have

$$\frac{(T_{m,j}L_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{2.6}$$

where $j \in \{s, s+2\}$, then the convergence given in (2.5) is uniformly on $I \cap J$ and

$$m^{s-\alpha_s}\Big|(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T_{m,i}L_m)(x)\Big| \leq \tag{2.7}$$

$$\leq \frac{1}{s!}(k_s + k_{s+2})\omega\Big(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\Big),$$

for any $x \in I \cap J$ and $m \geq m(s)$.

3. The properties of the class of linear and positive operators

Let α be a real number, $\alpha \geq 0$. We impose the condition that $m_0 \geq [\alpha]+1$, then we have $\frac{\alpha}{m_0} < 1$ for $m_0 \in \mathbb{N}$. If $m \geq m_0$ then $\left[\frac{\alpha}{m_0}; 1\right] \subset \left[\frac{\alpha}{m}; 1\right]$ Let $\mathbb{N}_1 = \{m \in \mathbb{N}_0 | m \geq m_0\}$. In the above conditions we introduce the operators

$$\left(Q_m^{*\alpha} f\right)(x) = \frac{1}{m^m}\sum_{k=0}^{m}\binom{m}{k}(mx-\alpha)^k(m+\alpha-mx)^{m-k} f\left(\frac{k+\alpha}{m}\right) \tag{3.1}$$

for any $f \in C([0,1]), m \in \mathbb{N}_1$ and any $x \in \left[\frac{\alpha}{m_0}; 1\right]$.

Proposition 1. The operators $(Q_m^{*\alpha})_{m \geq m_0}$ are linear and positive.

Proof. It follows from (3.1).

Remark 2. For $\alpha = 0$, in (3.1), we obtain Berstein's operators.

Lemma 1. For $m \in \mathbb{N}_1$ and $x \in \left[\frac{\alpha}{m_0};1\right]$ we have

$$(Q_m^{*\alpha} e_0)(x) = 1, \tag{3.2}$$

$$(Q_m^{*\alpha} e_1)(x) = x, \tag{3.3}$$

$$(Q_m^{*\alpha} e_2)(x) = \frac{m-1}{m}x^2 + \frac{m+2\alpha}{m^2}x - \frac{\alpha(m+\alpha)}{m^3}. \tag{3.4}$$

$$(Q_m^{*\alpha} e_3)(x) = \frac{(m-1)(m-2)}{m^2}x^3 + \frac{3(m-1)(m+2\alpha)}{m^3}x^2 + \tag{3.5}$$

$$+\frac{(6m-3m^2)\alpha^2 + (6m^2-3m^3)\alpha + m^3}{m^5}x + \frac{-2m\alpha^3 + (3m^3-6m^2)\alpha^2 - m^3\alpha}{m^6}$$

$$(Q_m^{*\alpha} e_4)(x) = \frac{(m-1)(m-2)(m-3)}{m^3}x^4 + \frac{6(m-1)(m-2)(m+2\alpha)}{m^4}x^3 + \tag{3.6}$$

$$+\frac{m-1}{m^3}\left(\frac{6\alpha^2(m-2)(m-3)}{m^2} - \frac{3\alpha(6+4\alpha)(m-2)}{m} + 7 + 12\alpha + 6\alpha^2\right)x^2$$

$$+\frac{1}{m^3}\left(\frac{-4\alpha^3(m-1)(m-2)(m-3)}{m^3} + \frac{3\alpha^2(6+4\alpha)(m-1)(m-2)}{m^2} - \right.$$

$$\left. - \frac{2\alpha(7+12\alpha+6\alpha^2)(m-1)}{m} + 1 + 4\alpha + 6\alpha^2 + 4\alpha^3\right)x +$$

$$+\frac{\alpha}{m^4}\left(\frac{(m-1)(m-2)(m-3)}{m^3}\alpha^3 - \frac{(m-1)(m-2)\alpha^2(6+4\alpha)}{m^2} + \right.$$

$$\left. + \frac{\alpha(7+12\alpha+6\alpha^2)(m-1)}{m} - (1+4\alpha+6\alpha^2+3\alpha^3)\right).$$

Proof. After some calculus we obtain these results.

Lemma 2. For $m \in \mathbb{N}_1$ and $x \in \left[\frac{\alpha}{m_0};1\right]$, the following identities

$$(T_{m,0} Q_m^{*\alpha})(x) = 1, \tag{3.7}$$

$$(T_{m,1} Q_m^{*\alpha})(x) = 0, \tag{3.8}$$

$$(T_{m,2} Q_m^{*\alpha})(x) = -mx^2 + (m+2\alpha)x - \frac{\alpha(m+\alpha)}{m}, \tag{3.9}$$

$$(T_{m,3} Q_m^{*\alpha})(x) = 2mx^3 + \left(2m^2(1+2\alpha) - 3m - 6\alpha\right)x^2 + \tag{3.10}$$

$$+\frac{m^3 + 6m^2\alpha + 6m\alpha^2}{m^2}x + \frac{m^3(3\alpha^2-\alpha) - 6m^2\alpha^2 - 2m\alpha^3}{m^3},$$

$$(T_{m,4}Q_m^{*\alpha})(x) = (3m^2-6m)x^4 + (-6m^2+12m(1-\alpha)+24\alpha)x^3 + \tag{3.11}$$
$$+\frac{3m^3+m^2(18\alpha-7)+m(18\alpha^2-36\alpha)-36\alpha^2}{m}x^2+$$
$$+\frac{m^3(-12\alpha^2-6\alpha+1)+m^2(-6\alpha^2+14\alpha)+m(-12\alpha^3+36\alpha^2)+24\alpha^3}{m^2}x+$$
$$+\frac{m^3(-6\alpha^4-10\alpha^3+2\alpha^2+11\alpha+7)+m^2(18\alpha^4+18\alpha^3-6\alpha^2-12\alpha-7)}{m^3}+$$
$$+\frac{m(3\alpha^4-12\alpha^3)-6\alpha^4}{m^3}$$

hold.

Proof. We take (2.2) and Lemma 1 into account.

Coming back to Theorem 1, for our operator (3.1) we have $I=[0,1+\alpha]$ and $E([0,1+\alpha]) = C([0,1+\alpha])$ and from (3.7)-(3.11) we obtain $k_0 = 2$, $k_2 = \frac{5}{4}$, $k_4 = \frac{19}{16}$, $\alpha_0 = 0$, $\alpha_2 = 1$ and $\alpha_4 = 2$.

Theorem 2. *Let $f : [0,1+\alpha] \longrightarrow \mathbb{R}$ be a continuous function s times differentiable on $[0,1+\alpha]$, having the s-order derivative continuous on $[0,1+\alpha]$. For $s=0$ we have*

$$\lim_{m\longrightarrow\infty} Q_m^{*\alpha} = f \tag{3.12}$$

uniformly on $J=\left[\frac{\alpha}{m_0};1\right]$, there exists $m^* = max(m_0, m(0), m(2))$ such that

$$|(Q_m^{*\alpha}f)(x) - f(x)| \le \frac{13}{4}\omega\left(f;\frac{1}{\sqrt{m}}\right), \tag{3.13}$$

for any $x\in\left[\frac{\alpha}{m_0};1\right], m\in\mathbb{N}, m\ge m^*$. For $s=2$ we have

$$\lim_{m\longrightarrow\infty} m((Q_m^{*\alpha}f)(x) - f(x)) = \frac{x(1-x)}{2}f^{(2)}(x) \tag{3.14}$$

uniformly on $J=\left[\frac{\alpha}{m_0};1\right]$, there exists $m_1 = max(m^*, m(4))$ such that

$$m|(Q_m^{*\alpha}f)(x) - f(x) - \frac{f^{(2)}(x)}{2m^2}(T_{m,2}Q_m^{*\alpha})(x)| \le \frac{39}{32}\omega\left(f^{(2)};\frac{1}{\sqrt{m}}\right), \tag{3.15}$$

for any $x\in\left[\frac{\alpha}{m_0};1\right], m\in\mathbb{N}, m\ge m_1$. For $s=4$ we have

$$\lim_{m\longrightarrow\infty} m^2\left((Q_m^{*\alpha}f)(x) - f(x) - \frac{x(1-x)}{2m}f^{(2)}(x) - \frac{(2\alpha+1)x^2}{3m}f^{(3)}(x)\right) = \tag{3.16}$$

$$= \frac{\alpha(2x-1)}{2} f^{(2)}(x) + \frac{x(1-x)(1-2x)}{6} f^{(3)}(x) + \frac{x^2(1-x)^2}{8} f^{(4)}(x)$$

Proof. We use Theorem 1, Lemma 2 and the relations (3.7)-(3.11).

The relations (3.14) and (3.16) are Voronovskaja-type theorems.

4. Application

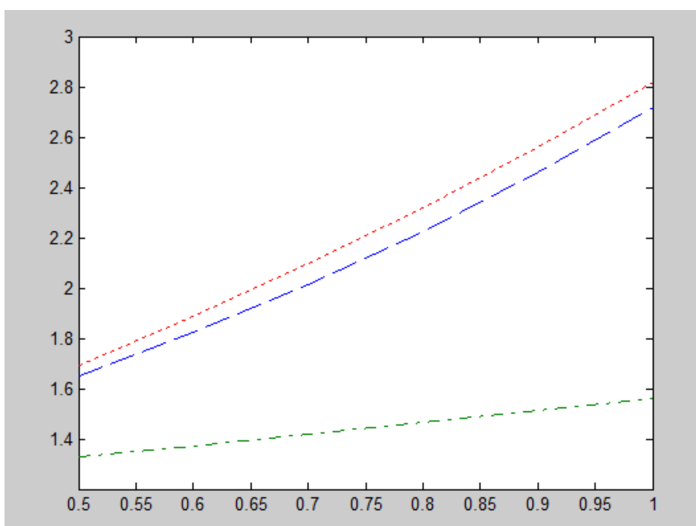
Next, using graphical representation, we will plot some graphs for this type of polynomials. We choose $1 = \alpha \leq \beta = 6$ and we compare the following polynomials: the classical Stancu operator

$$\left(P_m^{1,6} f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k+1}{m+6}\right), x \in [0,1], \tag{4.1}$$

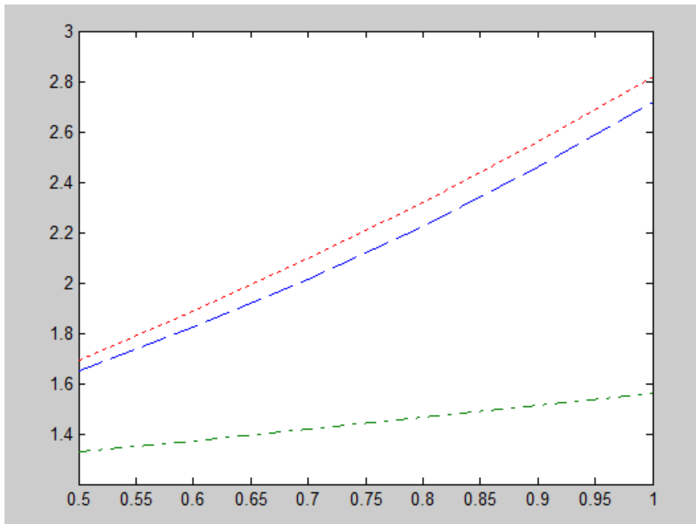
with the particular class of Stancu-type operators

$$(Q_m^{*1} f)(x) = \frac{1}{m^m} \sum_{k=0}^{m} \binom{m}{k} (mx-1)^k (m+1-mx)^{m-k} f\left(\frac{k+1}{m}\right), x \in \left[\frac{1}{m_0}, 1\right]. \tag{4.2}$$

We fix $m_0 = 2$. For $m = 3$, we plot - with dashed line $f : [0,1] \longrightarrow \mathbb{R}, f(x) = exp(x)$ - with dash-dot line $(P_3^{(1,6)} exp(\cdot))(x) = (1-x)^3 exp\left(\frac{1}{9}\right) + 3x(1-x)^2 exp\left(\frac{2}{9}\right) + 3x^2(1-x) exp\left(\frac{3}{9}\right) + x^3 exp\left(\frac{4}{9}\right)$, for $x \in [0,1]$ - with dotted line $(Q_3^{*1} exp(\cdot))(x) = \frac{1}{27}(4-3x)^3 exp\left(\frac{1}{3}\right) + \frac{3}{27}(3x-1)(4-3x)^2 exp\left(\frac{2}{3}\right) + \frac{3}{27}(3x-1)^2(4-3x) exp\left(\frac{3}{3}\right) + \frac{1}{27}(3x-1)^3 exp\left(\frac{4}{3}\right)$, for $x \in \left[\frac{1}{2}, 1\right]$.



We fix $m_0 = 2$. For $m = 4$ we plot - with dashed line $f : [0,1] \longrightarrow \mathbb{R}, f(x) = exp(x)$ - with dash-dot line $(P_4^{(1,6)} exp(\cdot))(x) = (1-x)^4 exp\left(\frac{1}{10}\right) + 4x(1-x)^3 exp\left(\frac{2}{10}\right) + 6x^2(1-x)^2 exp\left(\frac{3}{10}\right) + 4x^3(1-x) exp\left(\frac{4}{10}\right) + x^4 exp\left(\frac{5}{10}\right)$, for $x \in [0,1]$ - with dotted line $(Q_4^{*1} exp(\cdot))(x) = \frac{1}{256}(5-4x)^4 exp\left(\frac{1}{4}\right) + \frac{4}{256}(4x-1)(5-4x)^3 exp\left(\frac{2}{4}\right) + \frac{6}{256}(4x-1)^2(5-4x)^2 exp\left(\frac{3}{4}\right) + \frac{4}{256}(4x-1)^3(5-4x) exp\left(\frac{4}{4}\right) + \frac{1}{256}(4x-1)^4 exp\left(\frac{5}{4}\right)$, for $x \in \left[\frac{1}{2}, 1\right]$.



Note that our operator in the interval $\left[\frac{1}{2}, 1\right]$ for this function, approximates better than the classical Stancu operator.

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