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TOTALLY UMBILICAL PSEUDO-SLANT SUBMANIFOLDS OF RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. In the present paper we have study totally umbilical pseudo-slant submanifolds of Riemannian product manifolds via Riemannian curvature tensor and finally obtained a classification for the Totally umbilical pseudo-slant submanifolds of Riemannian product manifolds.

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1. Introduction

The notion of slant submanifolds of an almost Hermitian manifold was introduced by B.Y. Chen [3]. These submanifolds are the generalization of both holomorphic and totally real submanifolds of an almost Hermitian manifold with an almost complex structure J. The notion of semi-slant submanifolds of Kaehler manifolds was initiated by N. Papaghuic [8]. Bi-slant submanifolds of an almost Hermitian manifold were introduced as a natural generalization of semi-slant submanifolds and anti-slant submanifolds by A. Carriazo [1]. The class of bi-slant submanifolds includes complex, totally real and CR-submanifolds. But the name anti-slant seems it has no slant factor, so B. Sahin [4] named these submanifolds as pseudo-slant submanifolds and studied these (pseudo-slant) submanifolds in Kaehler setting for their warped product. B. Sahin [5] studied semi-invariant and totally umbilical semi-invariant submanifolds of Riemannian product manifolds and a step forward M. Atceken [7] defined slant and bi-slant submanifolds in the setting of Riemannian product manifolds and in particular he studied semi-slant submanifolds, since pseudo-slant submanifolds are special cases of bi-slant submanifolds then it will be worthwhile to study the pseudo-slant submanifolds in this setting. The purpose of this paper is to study totally umbilical pseudo-slant submanifolds of Riemannian product manifolds.

2. Preiliminaries

Let (M_1, g_1) and (M_2, g_2) be the Riemannian manifolds with dimension m_1 and m_2 respectively, and $M_1 \times M_2$ be Riemannian product manifold of M_1 and M_2. We denote projection mapping of $T(M_1 \times M_2)$ onto TM_1 and TM_2 by $\sigma_\star$ and $\pi_\star$ respectively. Then we have $\sigma_\star + \pi_\star = I$, $\sigma_\star^2 = \sigma_\star$ $\pi_\star^2 = \pi_\star$ and $\sigma_\star \circ \pi_\star = \pi_\star \circ \sigma_\star = 0$, where $\star$ mean derivatives.

Riemannian metric of the Riemannian product manifold $M = M_1 \times M_2$ is defined by

$$g(X, Y) = g_1(\sigma_\star X, \sigma_\star Y) + g_2(\pi_\star X, \pi_\star Y)$$

for any $X, Y \in T\bar{M}$. If we set $F = \sigma_\star - \pi_\star$ then $F^2 = I$, $F \neq I$ and g satisfies condition

$$g(FX, Y) = g(X, FY)$$

for any $X, Y \in T\bar{M}$ thus F defines an almost Riemannian product structure on $\bar{M}$. We denote Levi-Civita connection on $\bar{M}$ by $\bar{\nabla}$, then the covariant derivative of F is defined as

$$(\bar{\nabla}_X F)Y = \bar{\nabla}_X FY - F\bar{\nabla}_X Y,$$

for any $X, Y \in TM$. We say that F is parallel with respect to the connection $\bar{\nabla}$ if we have $(\bar{\nabla}_X F)Y = 0$. Here from [10], we know that F is parallel, that is, F is Riemannian product structure.

Let $\bar{M}$ be a Riemannian product manifold with Riemannian product structure F and M be a immersed submanifold of $\bar{M}$, we also denote by g the induced metric tensor on M as well as on $\bar{M}$. If $\bar{\nabla}$ is the Levi-civita connection on $\bar{M}$, then the Gauss and Weiengarten formulas are given by respectively, as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V \tag{2}$$

for any $X, Y \in TM$ and $V \in T^\perp M$, where ∇ is the connection on M and $\nabla^\perp$ is the connection in the normal bundle, h is the second fundamental form of M and A_V the shape operator of M. The second fundamental form h and the shape operator A_V are related by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{3}$$

For any $X \in TM$, we can write

$$FX = fX + \omega X, \tag{4}$$

where fX and ωX are the tangential and normal components of FX, respectively and for $V \in T^{\perp}M$

$$FV = tV + nV, \tag{5}$$

where tV and nV are the tangential and normal components of FV, the submanifold M is said to be invariant if ω is identically zero.On the other hand M is said to be an anti-invariant submanifold if f is identically zero.

The covariant derivatives of f, ω, t and n is defined as

$$(\bar{\nabla}_X f)Y = \nabla_X fY - f\nabla_X Y \tag{6}$$

$$(\bar{\nabla}_X \omega)Y = \nabla^{\perp}_X \omega Y - \omega\nabla_X Y \tag{7}$$

$$(\bar{\nabla}_X t)V = \nabla_X tV - t\nabla_X V \tag{8}$$

$$(\bar{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X Y. \tag{9}$$

Using (1),(2) (4) and (6) we get

$$(\bar{\nabla}_X f)Y = A_{\omega Y}X + th(X,Y) \tag{10}$$

Let M be an immersed submanifold of a Riemannian product manifold $\bar{M}$, for each nonzero vector X tangent to M at a point x, we denote by $\theta(x)$ the angle between FX and T_xM. The angle $\theta(x)$ is called the slant angle of immersion.

Let M be an immersed submanifold of a Riemannian product manifold $\bar{M}$. M is said to be slant submanifold of Riemannian product manifold $\bar{M}$ if the slant angle $\theta(x)$ is constant which is independent of choice of $x \in M$ and $X \in TM$.

Invariant and anti-invariant submanifolds are particular cases of slant submanifolds with angles $\theta = 0$ and $\theta = \pi/2$. respectively, a slant submanifold which is neither invariant nor anti-invariant is called proper slant submanifold. The following characterization of slant submanifolds of Riemannian product manifolds is proved by M. Atceken [7].

Theorem 1. *Let M be an immersed submanifold of a Riemannian product manifold $\bar{M}$. Then M is a slant submanifold if and only if there exist a constant $\lambda \in [0,1]$ such that $f^2 = \lambda I$.*
Moreover if θ is the slant angle of M, then it satisfies $\lambda = \cos^2\theta$.

Hence, for a slant submanifold we have the following relations which are consequences of above theorem

$$g(fX, fY) = \cos^2\theta g(X,Y) \tag{11}$$

$$g(\omega X, \omega Y) = \sin^2\theta g(X,Y). \tag{12}$$

for any $X, Y \in TM$.

Now, we define the pseudo-slant submanifold of Riemannian product manifold $\bar{M}$ as follows

Definition 1. *A Submanifold M of a Riemannian product manifold $\bar{M}$ is said to be pseudo-slant submanifold if there exist two orthogonal complemantry distribution D_θ and $D^\perp$ satisfying*

(i) $TM = D_\theta \oplus D^\perp$

(ii) D_θ *is a slant distribution with slant angle* $\theta \neq \pi/2$

(iii) $D^\perp$ *is anti-invariant distribution i.e.,* $FD^\perp \subseteq T^\perp M$.

If μ is invariant subspace under F of the normal bundle $T^\perp M$, then in the case of pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = \mu \oplus \omega D_\theta + \omega D^\perp.$$

A pseudo-slant submanifold M is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H \tag{13}$$

where $H = \frac{1}{n}(\text{trace } h)$, called the mean curvature vector. For the totally umbilical pseudo-slant submanifold M, the equation (1)and (2) take the form

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)H \tag{14}$$

$$\bar{\nabla}_X V = -g(H,V)X + \nabla^\perp_X V. \tag{15}$$

The Riemannian curvature tensor is defined as

$$R(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{16}$$

The equation of Coddazi for totally umbilical pseudo-slant submanifold M is given by

$$\bar{R}(X,Y,Z,V) = g(Y,Z)g(\nabla^\perp_X H, V) - g(X,Z)g(\nabla^\perp_Y H, V), \tag{17}$$

where $\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$ and X,Y,Z are vector fields on M and $V \in T^\perp M$.

It is easy to see that the Riemannian curvature tensor for Riemannian product manifolds satisfies the following properties

$$(a)\bar{R}(FX,FY)Z = \bar{R}(X,Y)Z \qquad (b)F\bar{R}(X,Y)Z = \bar{R}(X,Y)FZ \tag{18}$$

By an externsic sphere we mean a submanifold of an arbitrary Riemannian manifold which is totally umbilical and has nonzero parallel mean curvature vector [9].

3. Totally Umbilical Pseudo-Slant Submanifolds

In this section, we will study a special class of pseudo-slant submanifolds which are totally umbilical. Throughout the section we consider M as a totally umbilical pseudo-slant submanifold of a Riemannian product manifold. Now we have the following theorem

Theorem 2. *Let M be a totally umbilical pseudo-slant submanifold of a Riemannian product manifold $\bar{M}$ such that the mean curvature vector $H \in \mu$. Then one of the following statement is true*

(i) M is totally geodesic.

(ii) M is semi-invariant submanifold.

Proof. For $V \in FD^{\perp}$ and $X \in D_{\theta}$, we have

$$\bar{\nabla}_X FV = F\bar{\nabla}_X V \tag{19}$$

using equations (14) and (15) the above equation becomes

$$\nabla_X FV + g(X, FV)H = -FXg(X, V) + F\nabla^{\perp}_X V.$$

Then by orthogonality of two distributions and the assumption $H \in \mu$ the above equation takes the form

$$\nabla_X FV = F\nabla^{\perp}_X V \tag{20}$$

which implies that $\nabla^{\perp}_X V \in FD^{\perp}$, for any $V \in FD^{\perp}$. Also we have $g(V, H) = 0$, for $V \in FD^{\perp}$, then using this fact we derive

$$g(\nabla^{\perp}_X V, H) = -g(V, \nabla^{\perp}_X H) = 0. \tag{21}$$

The equation (21) gives $\nabla^{\perp}_X H \in \mu \oplus \omega D_{\theta}$.
Now, for any $X \in D_{\theta}$, we have

$$\bar{\nabla}_X FH = F\bar{\nabla}_X H,$$

using equation (15), we obtain

$$-Xg(H, FH) + \nabla^{\perp}_X FH = -FXg(H, H) + F\nabla^{\perp}_X H,$$

using the equation (4) above equation takes the form

$$\nabla^{\perp}_X FH = -fXg(H, H) - \omega Xg(H, H) + F\nabla^{\perp}_X H,$$

taking Inner product with $\omega X \in \omega D_\theta$ and using the equation (12)

$$g(\nabla_X FH, \omega X) = -\sin^2\theta \|H\|^2 \|X\|^2 + g(\omega \nabla^\perp_X H, \omega X).$$

Then from equation (12), the last term of right hand side is identically zero, thus the above equation becomes

$$g(\nabla_X FH, \omega X) + \sin^2\theta \|H\|^2 \|X\|^2 = 0. \tag{22}$$

Therefore equation (22) has a solution if either $H = 0$ i.e., M is totally geodesic or the angle of slant distribution D_θ is zero i.e, M is semi-invariant submanifold.

Now for any $Z \in D^\perp$, by equation (10)

$$-f\nabla_Z Z = A_{\omega Z} Z + th(Z, Z).$$

Taking Inner product with $W \in D^\perp$ the above equation takes the form

$$-g(f\nabla_Z Z, W) = g(A_{\omega Z} Z, W) + g(th(Z, Z), W).$$

As M is totally umbilical pseudo-slant submanifold, then above equation becomes

$$g(Z, W)g(H, FZ) + g(tH, W)\|Z\|^2 = 0. \tag{23}$$

The above equation has a solution if either $H \in \mu$ or $\dim D^\perp = 1$.

Now, in the following theorem we will see the impact of parallelism of ω on M.

Theorem 3. *Let M be a totally umbilical pseudo-slant submanifold of a Riemannian product manifold $\bar{M}$ such that dimension of slant distribution $D_\theta \geq 4$ and ω is parallel, then M is either*

(i) extrinsic sphere.

(ii) or anti-invariant submanifold.

Proof. Since dimension of slant distribution $D_\theta \geq 4$, then we can choose a set of orthogonal vectors $X, Y \in D_\theta$, such that $g(X, Y) = 0$. Now from equation (18)(b), we have

$$F\bar{R}(X, Y)Z = \bar{R}(X, Y)FZ$$

for any $X, Y, Z \in D_\theta$. Replacing Z by fY, we obtain

$$F\bar{R}(X,Y)fY = \bar{R}(X,Y)FfY.$$

Using equation (4) and Theorem (2.1), the above equation takes the form

$$F\bar{R}(X,Y)fY = \cos^2\theta\bar{R}(X,Y)Y + \bar{R}(X,Y)\omega fY. \tag{24}$$

On the other hand, since ω is parallel, then we have

$$\bar{R}(X,Y)\omega fY = \omega\bar{R}(X,Y)fY. \tag{25}$$

Then by equations (24) and (25) we have

$$F\bar{R}(X,Y)fY = \cos^2\theta\bar{R}(X,Y)Y + \omega\bar{R}(X,Y)fY. \tag{26}$$

Taking Inner product in equation (27) by $N \in T^\perp M$, we get

$$g(F\bar{R}(X,Y)fY, N) = \cos^2\theta g(\bar{R}(X,Y)Y, N) + g(\omega\bar{R}(X,Y)fY, N),$$

using equation (4) the above equation reduced to

$$\cos^2\theta g(\bar{R}(X,Y,Y,N) = 0. \tag{27}$$

Then, from equation (17), we derive

$$\cos^2\theta g(Y,Y)g(\nabla^\perp_X H, N) - g(X,Y)g(\nabla^\perp_Y H, N) = 0.$$

Since X and Y are orthogonal vectors, then the above equation gives

$$\cos^2\theta g(\nabla^\perp_X H, N)\|Y\|^2 = 0. \tag{28}$$

The equation (28) has a solution either $\theta = \pi/2$ i.e., M is anti-invariant or $\nabla^\perp_X H = 0$ $\forall X \in D_\theta$. By similar calculation for any $X \in D^\perp$ we can obtain $\nabla^\perp_X H = 0$, hence $\nabla^\perp_X H = 0$ for all $X \in TM$ i.e., the mean curvature vector H is parallel to submanifold, i.e., M is extrinsic sphere.

Now we are in position to prove our main theorem:

Theorem 4. *Let M be a totally umbilical pseudo-slant submanifold of a Riemannian product manifold $\bar{M}$. Then M is either*

(i) Totally geodesic or

(ii) Semi-invariant or

(iii) dim $D^{\perp} = 1$ *or*

(iv) Extrinsic sphere.

case (iv) holds if ω *is parallel and dim* $M \geq 5$*(odd)*

Proof. If $H \in \mu$ then by Theorem 3.1 M is either totally geodesic or semi-invariant submanifolds which are case (i) and (ii). If $H \notin \mu$, then equation (24) has a solution if dim $D^{\perp} = 1$ which is case (iii) and moreover if $H \notin \mu$ and ω is parallel on M then by Theorem 3.2 M is extrinsic sphere which proves the theorem completely.

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