Acta Universitatis Apulensis No. 41/2015
ISSN: 1582-5329 pp. 1-9
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.41.01

CONSTRUCTION OF ALL TOPOLOGIES AND ALGEBRAS ON FINITE SETS

Gh. Sadeghi, A. Arefijamaal, E. Zekaee

Abstract. In this paper, we introduce an algorithm to construct all topologies and algebras on finite sets. For a non-empty set X we first provide a Booleian algebra of integers isomorphic to the power set of $P(X)$. Then the problem of finding all topologies (algebras) in X can be discussed explicitly in the Boolean algebra. This upward movement indicates us to obtain an algorithm to generate all topologies (algebras) in X.

2010 *Mathematics Subject Classification*: Primary 54A10; Secondary 94C10.

Keywords: Boolean algebra, topology, algebra.

1. Introduction and preliminaries

There are many kind of different algorithms to find the number of the topologies on finite sets [3, 5, 7]. Sharp [6] showed that the number of topologies and quasi-orders on finite sets are the same. Moreover, a method for constructing one-point expansions of a topology on a finite set is discussed in [1]. However, there is no explicit construction of topologies on a finite set. The main goal of this article is to introduce an algorithm to characterize all topologies on finite sets. Some applications of these processes in the theory of computation can be found in [2, 8]

Let us start with a brief review of some basic facts from the set theory and topology. A topology τ on a set $X \neq \emptyset$ is a subset of $P(X)$, the power set of X, that contains $\emptyset$ and X, and is closed under unions and finite intersections. The elements of τ are called *open sets* and the pair (X, τ) is called a *topological space*. Clearly, the number of topologies on X is exactly the number of sublattices $(P(X), \subseteq)$. A topological space (X, τ) is said to be T_0-space if for every distinct $x, y \in X$ there exists an open set U containing exactly one of them.

We first construct a Boolean algebra isomorphic to the power set of $P(X)$ to generate the set of all topologies on a finite set X. Finding topologies on X can be read as looking for some special subsets of the Boolean algebra. As we have seen, this leads to obtain an algorithm to establish all topologies and algebras on X.

Definition 1. *Let $\mathfrak{B}$ be a nonempty set with a pair of operations $\oplus$ and $\odot$. The triple $(\mathfrak{B}, \oplus, \odot)$ is called a Boolean algebra if for all $a, b, c \in \mathfrak{B}$:*

1. $a \odot b = b \odot a$, $a \oplus b = b \oplus a$.
2. $a \odot (b \odot c) = (a \odot b) \odot c$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
3. $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$, $a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$.
4. *There exist* $1_{\mathfrak{B}}$ *and* $0_{\mathfrak{B}}$ *such that*
$$a \oplus 0_{\mathfrak{B}} = a_{\mathfrak{B}}, \quad a \odot 1_{\mathfrak{B}} = a.$$
5. *For every* $a \in \mathfrak{B}$ *there exist* $\acute{a} \in \mathfrak{B}$ *such that*
$$a \oplus \acute{a} = 1_{\mathfrak{B}}, \quad a \odot \acute{a} = 0_{\mathfrak{B}}.$$

The following examples have a key roll in this note.

Example 1. 1. *Let $\mathfrak{B} = \{0, 1\}$. Define the operations $\oplus$ and $\odot$ as follows:*

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

Then the triple $(\mathfrak{B}, \oplus, \odot)$ is a Boolean algebra.

2. *For each non-empty set X the triple $(P(X), \cup, \cap)$ is a Boolean algebra with $1_{P(X)} = X$ and $0_{P(X)} = \emptyset$.*

According to binary representation of numbers each $j \in M := \{0, 1, 2, \dots, 2^n - 1\}$ can be represented as $\sum_{l=0}^{n-1} \lambda_l 2^l$ where $\lambda_l \in \{0, 1\}$. In this case, we denote j as $(\lambda_{n-1} \cdots \lambda_0)_2$. For two elements $j = \sum_{l=0}^{n-1} \lambda_l 2^l$ and $k = \sum_{l=0}^{n-1} \gamma_l 2^l$ define

$$j \oplus k = \sum_{l=0}^{n-1} (\lambda_l \oplus \gamma_l) 2^l, \qquad j \odot k = \sum_{l=0}^{n-1} (\lambda_l \odot \gamma_l) 2^l.$$

It is easy to see that $(M, \oplus, \odot)$ is a Boolean algebra with $1_M = 2^n - 1$ and $0_M = 0$.

2. Main Results

Throughout this paper we assume that $n \in \mathbb{N}$ is a fix natural number and $X = \{x_0, x_1, \ldots, x_{n-1}\}$ is a finite set with n distinct elements. Although the elements of a set are not ordered but we will consider the elements of X only by this form.

Theorem 1. *Let $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $M := \{0, 1, 2, \ldots, 2^n - 1\}$. Then $(P(X), \cup, \cap)$ and $(M, \oplus, \odot)$ are Boolean algebra isomorphic.*

Proof. Define $\varphi : P(X) \to M$ by

$$\varphi(A) = \sum_{l=0}^{n-1} \chi_A(x_l) 2^l.$$

It is easy to see that φ is a bijective map, $\varphi(\emptyset) = 0$ and $\varphi(X) = 2^n - 1$. Moreover, for $A, B \subseteq X$ we have

$$\begin{aligned}
\varphi(A \cup B) &= \sum_{l=0}^{n-1} \chi_{(A \cup B)}(x_l) 2^l \\
&= \sum_{l=0}^{n-1} \chi_A(x_l) 2^l \oplus \sum_{l=0}^{n-1} \chi_B(x_l) 2^l \\
&= \varphi(A) \oplus \varphi(A),
\end{aligned}$$

and

$$\begin{aligned}
\varphi(A \cap B) &= \sum_{l=0}^{n-1} \chi_{(A \cap B)}(x_l) 2^l \\
&= \sum_{l=0}^{n-1} \chi_A(x_l) 2^l \odot \sum_{l=0}^{n-1} \chi_B(x_l) 2^l \\
&= \varphi(A) \odot \varphi(A).
\end{aligned}$$

Let $P(X)$ be the power set of X. To apply this theorem for $P(X)$ instead of X we need to consider an order on $P(X)$. In fact, we rewrite the power set of X as $P(X) = \{A_0, A_1, \ldots, A_{2^n-1}\}$ where $A_j = \varphi^{-1}(j)$ and denote the obtained Boolean algebra isomorphic by Φ. Thus,

$$\Phi : P(P(X)) \to N, \quad \Phi(\mathcal{A}) = \sum_{i=0}^{2^n-1} \chi_{\mathcal{A}}(A_i) 2^i, \tag{1}$$

where $N := \{0, 1, 2, \ldots, 2^{2^n} - 1\}$. In particular, $\Phi(\emptyset) = 0$ and $\Phi(P(X)) = 2^{2^n} - 1$. Some properties of Φ are given in the following proposition.

Proposition 1. *Let $X = \{x_0, x_1, \ldots, x_{n-1}\}$, $\mathcal{A} \in P(P(X))$ and $0 \leq j \leq 2^n - 1$.*

1. *$\Phi(\{\varphi^{-1}(j)\}) = 2^j$*
2. *$\varphi^{-1}(j) \in \mathcal{A}$ if and only if $2^j \odot \Phi(\mathcal{A}) = 2^j$.*

It is well-known that each topology τ on X must contain the empty set and X itself. This fact can be expressed as follows:

$$1 \odot \Phi(\tau) = 1, \qquad 2^{2^n - 1} \odot \Phi(\tau) = 2^{2^n - 1}.$$

The natural question is how to find $\Phi(\tau)$ for a topology τ. Let us now give an example. Choose $t = 2^{2^n - 1} + 1$ and take $\tau = \Phi^{-1}(t)$. Consider $(\overline{\lambda_{2^n - 1} \cdots \lambda_0})_2$ as the binary representation of t, then $\lambda_j = 0$ for all $0 < j < 2^n - 2$. So $\tau = \{\emptyset, X\}$ is the non-discrete topology on X. Indeed, $\varphi^{-1}(j) \notin \tau$ for such j by Proposition 1. A similar argument shows that $\tau = \Phi^{-1}(2^{2^n} - 1)$ is the discrete topology.

The following theorem describes all topologies on X.

Theorem 2. *Suppose $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $\Phi : P(P(X)) \to N$ is given by (1).*

1. *$\Phi(\tau) > 2^{2^n - 1}$ is an odd number for each topology τ on X.*
2. *$\tau \subseteq P(X)$ is a topology on X if and only if $2^{2^n - 1} < \Phi(\tau) < 2^{2^n}$ is an odd number and for each $0 < i < j < 2^n - 1$ with $2^i \odot \Phi(\tau) = 2^i$ and $2^j \odot \Phi(\tau) = 2^j$ we have*
$$2^{i \odot j} \odot \Phi(\tau) = 2^{i \odot j}, \quad 2^{i \oplus j} \odot \Phi(\tau) = 2^{i \oplus j}. \tag{2}$$

Proof. Let τ be a topology on X. Then

$$1 \odot \Phi(\tau) = \Phi(\{\emptyset\} \cap \tau) = \Phi(\{\emptyset\}) = 1,$$

Now the definition of $\odot$ shows that the last number of $\Phi(\tau)$, in the binary representation, must be 1. Therefore, $\Phi(\tau)$ is an odd number. Moreover, by Proposition 1 we obtain $2^{2^n - 1} \odot \Phi(\tau) = 2^{2^n - 1}$. By attention to the binary representation we have $\Phi(\tau) > 2^{2^n - 1}$.

To prove (2), let $2^{2^n - 1} < t < 2^n$ is odd and $\tau = \Phi^{-1}(t)$. Apply Proposition 1 we obtain

$$\begin{aligned} 1 &= \Phi(\{\emptyset\}) = \Phi(\{\emptyset\} \cap \tau) = 1 \odot \Phi(\tau) \\ 2^{2^n - 1} &= \Phi(\{\varphi^{-1}(2^n - 1)\}) = \Phi(\{X\} \cap \tau) = 2^{2^n - 1} \odot \Phi(\tau). \end{aligned}$$

Consequently $\{\emptyset, X\} \subseteq \tau$. Suppose that $A, B \subseteq X$ are non-trivial, by Theorem 1 there exist $0 < i < j < 2^n - 1$ such that $\varphi(A) = i$, $\varphi(B) = j$. By Proposition 1 if $A, B \in \tau$, then

$$2^i \odot \Phi(\tau) = 2^i, \quad 2^j \odot \Phi(\tau) = 2^j.$$

Combining (2) with the fact that $A \cap B = \varphi^{-1}(i \odot j)$ and $A \cup B = \varphi^{-1}(i \oplus j)$ easily follows that $A \cup B$ and $A \cap B$ belong to τ. This shows that τ is a topology on X. The converse of (2) follows immediately.

Recall that a collection $\mathcal{F}$ of subsets of a finite set X is called an algebra (σ-algebra) if it contains $\emptyset$ and X, and is closed under unions and complements. Obviously, every algebra is a topology. By Theorem 2 we can find a characterization of all algebras on X as follows.

Corollary 3. *Suppose $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and τ is a topology on X. The family τ is an algebra on X if and only if*

$$2^i \odot \Phi(\tau) = 2^i \Longleftrightarrow 2^{2^n-1-i} \odot \Phi(\tau) = 2^{2^n-1-i}, \quad (0 < i < 2^n - 1).$$

3. Algorithm

Our algorithm to construct all topologies on a finite set X is a based on the binary representation of numbers. We first establish an one-to-one correspondence between $P(X)$ and $M = \{0, 1, \ldots, 2^n - 1\}$. More precisely, if $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $A = \{x_{n_1}, x_{n_2}, \ldots, x_{n_m}\}$ is a subset of X, then the associated integer is $j = (\overline{\lambda_{n_m-1}\lambda_{n_m-2}\cdots\lambda_0})_2$ where

$$\lambda_l = \begin{cases} 1 & l \in \{n_1, n_2, \ldots, n_m\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we can extend this correspondence to $P(P(X))$ and $N = \{0, 1, \ldots, 2^{2^n} - 1\}$. The problem of finding all topologies (algebras) in X can be discussed explicitly in the Boolean algebra $(N, \oplus, \odot)$. It is enough to examine only odd numbers $2^{2^n-1} < t < 2^{2^n}$ such that

$$2^{i \odot j} \odot t = 2^{i \odot j}, \quad 2^{i \oplus j} \odot t = 2^{i \oplus j},$$

when $0 < i < j < 2^n - 1$ with $2^i \odot t = 2^i$ and $2^j \odot t = 2^j$, see Theorem 2. These two facts give us an algorithm for computing all topologies X (Algorithm 1). Table 1 presents all topologies on a three-member set. We observe that there are only five algebras out of those 29 topologies.

input : n; cardinal of X
output: List of all $2^{2^n-1}+1 \leq t \leq 2^{2^n}-1$ such that $\tau := \Phi^{-1}(t)$ is a topology on X

1 **begin**
2     $a \longleftarrow 2^{2^n-1}+1$
3     $b \longleftarrow 2^{2^n}-1$
4   **for** t **from** a **by** 2 **to** b **do**
5   **for** $\lambda \in \{0, 1, \dots, 2^n-2\}$ **do**
6   **for** $\mu \in \{\lambda+1, \dots, 2^2-2\}$ **do**
7   **if** $2^\lambda \odot t \neq 0 \wedge 2^\mu \odot t \neq 0$ **then**
8   **if** $2^{\lambda\oplus\mu} \odot t = 0 \vee 2^{\lambda\odot\mu} \odot t = 0$ **then**
9   **go to** *15*
10   **end**
11   **end**
12   **end**
13   **end**
14   **print** τ *is a topology on* X
15   **continue**
16   **end**
17 **end**

Algorithm 1: Finding all topologies on X.

Corollary 3 help us to obtain all algebras on finite set $X = \{x_0, x_1, x_2, x_3\}$, see Table 2. Moreover, the number of algebras in some finite sets are shown in Table 3.

Before we end this section, let us remark some problems for interested readers:

- How many topologies with m elements are there on an n-element set?
- Let X be a set with n elements, and let $m \in \mathbb{N}$ such that $\frac{3}{4}2^n < m < 2^n$. Is there a topology on X with m elements?

Acknowledgements. The authors would like to thank Prof. M. Mirzavaziri for helpful comments and suggestions. Thanks are also due to the referees.

Odd number $2^{2^3-1} < t < 2^{2^3}$	All topologies and algebras on three-member set	
	Topology $\Phi^{-1}(t)$	Is also an algebra?
129	$\{\emptyset, X\}$	✓
131	$\{\emptyset, X, \{x_0\}\}$	-
133	$\{\emptyset, X, \{x_1\}\}$	-
137	$\{\emptyset, X, \{x_0, x_1\}\}$	-
139	$\{\emptyset, X, \{x_0\}, \{x_0, x_1\}\}$	-
141	$\{\emptyset, X, \{x_1\}\}, \{x_0, x_1\}\}$	-
143	$\{\emptyset, X, \{x_0\}, \{x_1\}\}, \{x_0, x_1\}\}$	-
145	$\{\emptyset, X, \{x_2\}\}$	-
153	$\{\emptyset, X, \{x_0, x_1\}, \{x_2\}\}$	✓
161	$\{\emptyset, X, \{x_0, x_2\}\}$	-
163	$\{\emptyset, X, \{x_0\}, \{x_0, x_2\}\}$	-
165	$\{\emptyset, X, \{x_1\}, \{x_0, x_2\}\}$	✓
171	$\{\emptyset, X, \{x_0\}, \{x_0, x_1\}, \{x_0, x_2\}\}$	-
175	$\{\emptyset, X, \{x_0\}, \{x_1\}, \{x_0, x_1\}, \{x_0, x_2\}\}$	-
177	$\{\emptyset, X, \{x_2\}, \{x_0, x_2\}\}$	-
179	$\{\emptyset, X, \{x_0\}, \{x_2\}, \{x_0, x_2\}\}$	-
187	$\{\emptyset, X, \{x_0\}, \{x_0, x_1\}, \{x_2\}, \{x_0, x_2\}\}$	-
193	$\{\emptyset, X, \{x_1, x_2\}\}$	-
195	$\{\emptyset, X, \{x_0\}, \{x_1, x_2\}\}$	✓
197	$\{\emptyset, X, \{x_1\}, \{x_1, x_2\}\}$	-
205	$\{\emptyset, X, \{x_1\}, \{x_0, x_1\}, \{x_1, x_2\}\}$	-
207	$\{\emptyset, X, \{x_0\}, \{x_1\}, \{x_0, x_1\}, \{x_1, x_2\}\}$	-
209	$\{\emptyset, X, \{x_2\}, \{x_1, x_2\}\}$	-
213	$\{\emptyset, X, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$	-
221	$\{\emptyset, X, \{x_1\}, \{x_0, x_1\}, \{x_2\}, \{x_0, x_2\}\}$	-
241	$\{\emptyset, X, \{x_2\}, \{x_0, x_2\}, \{x_1, x_2\}\}$	-
243	$\{\emptyset, X, \{x_0\}, \{x_2\}, \{x_0, x_2\}, \{x_1, x_2\}\}$	-
245	$\{\emptyset, X, \{x_1\}, \{x_2\}, \{x_0, x_2\}, \{x_1, x_2\}\}$	-
255	$P(X)$	✓

Table 1: All topologies and algebras on finite set $X = \{x_0, x_1, x_2\}$.

Odd number $2^{2^3-1} < t < 2^{2^3}$	All algebras on four-member set
	Algebra $\Phi^{-1}(t)$
32769	$\{\emptyset, X\}$
33153	$\{\emptyset, \{x_0, x_1, x_2\}, \{x_3\}, X\}$
33345	$\{\emptyset, \{x_1, x_2\}, \{x_0, x_3\}, X\}$
33825	$\{\emptyset, \{x_0, x_2\}, \{x_1, x_3\}, X\}$
34833	$\{\emptyset, \{x_2\}, \{x_0, x_1, x_3\}, X\}$
36873	$\{\emptyset, \{x_0, x_1\}, \{x_2, x_3\}, X\}$
39321	$\{\emptyset, \{x_0, x_1\}, \{x_2\}, \{x_0, x_1, x_2\}, \{x_3\}, \{x_0, x_1, x_3\}, \{x_2, x_3\}, X\}$
40965	$\{\emptyset, \{x_1\}, \{x_0, x_2, x_3\}, X\}$
42405	$\{\emptyset, \{x_1\}, \{x_0, x_2\}, \{x_0, x_1, x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_0, x_2, x_3\}, X\}$
43605	$\{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_0, x_3\}, \{x_0, x_1, x_3\}, \{x_0, x_2, x_3\}, X\}$
49155	$\{\emptyset, \{x_0\}, \{x_1, x_2, x_3\}, X\}$
50115	$\{\emptyset, \{x_0\}, \{x_1, x_2\}, \{x_0, x_1, x_2\}, \{x_3\}, \{x_0, x_3\}, \{x_1, x_2, x_3\}, X\}$
52275	$\{\emptyset, \{x_0\}, \{x_2\}, \{x_0, x_2\}, \{x_1, x_3\}, \{x_0, x_1, x_3\}, \{x_1, x_2, x_3\}, X\}$
61455	$\{\emptyset, \{x_0\}, \{x_1\}, \{x_0, x_1\}, \{x_2, x_3\}, \{x_0, x_2, x_3\}, \{x_1, x_2, x_3\}, X\}$
65535	$P(X)$

Table 2: All algebras on finite set $X = \{x_0, x_1, x_2, x_3\}$.

	All algebras on four-member set
The number of elements of X	The number of algebras on X
1	1
2	2
3	5
4	15
5	52

Table 3: The number of algebras on finite sets.

References

[1] V. I. Arnautov, A. V. Kochina, *Method for constructing one-point expansions of a topology on a finite set and its applications*, Bul. Acad. Ştiinţe Repub. Mold. Mat. 3,64 (2010), 67–76.

[2] J. A. Barmak, *Algebraic Topology of Finite Topological Spaces and Applications*, Springer, 2011.

[3] M. Benoumhani, *The number of topologies on a finite set*, J. Integer Seq. 9 (2006), 1–9.

[4] K. Hensel, *Über eine neue Begründung der Theorie der algebraischen Zahlen*, Jahresber. Deutsch. Math.-Verein. 6 (1897), 83–88.

[5] D. Kleitman and B. L. Rothschild, *The number of finite topologies*, Proc. Amer. Math. Soc. 25 (1970), 276–282.

[6] H. Sharp. Jr., *Quasi-orderings and topologies on finite sets*, Proc. Amer. Math. Soc. 17 (1966), 1344–1349.

[7] H. Sharp. Jr., *Cardinality of finite topologies*, J. Combin. Theory Ser. A 5 (1968), 82–86.

[8] A. Zomorodian, *Computational topology*, in: M. J. Atallah, M. Blanton, (eds.) *Algorithms and Theory of Computation* handbook 2, Chapman & Hall/CRC Press, 2009.

Ghadir Sadeghi
Department of Mathematics and Computer Sciences
Hakim Sabzevari University,
Sabzevar, IRAN
email: *g.sadeghi@hsu.ac.ir;ghadir54@gmail.com*

Ali Akbar Arefijamaal
Department of Mathematics and Computer Sciences
Hakim Sabzevari University,
Sabzevar, IRAN
email: *arefijamaal@hsu.ac.ir;arefijamaal@gmail.com*

Esmaeel Zekaee
Department of Mathematics,
Shahrood University of Technology,
Shahrood, IRAN
email: *zekaee.esmaeel@gmail.com*