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SOME PROPERTIES OF GENERALIZED HADAMARD PRODUCTS FOR ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Using the generalized modified Hadamard product, we obtain some interesting characterization theorems for classes of uniformly starlike and uniformly convex functions.

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1. Introduction

Let $\mathcal{T}(n)$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z - \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geq 0; n \in \mathbb{N} \backslash \{1\} = \{2, 3, ...\}). \tag{1.1}$$

For $0 \leq \alpha < 1$, let $\mathcal{ST}(\alpha)$ and $\mathcal{CT}(\alpha)$ be the subclasses of T which are starlike functions of order α and convex functions of order α, respectively, (see Silverman [8]).

A function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{ST}(\alpha, \beta)$ of uniformly starlik functions of order α and type β in $\mathbb{U}$ if the following inequality holds

$$Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (0 \leq \alpha < 1;\ \beta \geq 0; z \in \mathbb{U}). \tag{1.2}$$

Also, a function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{CT}(\alpha, \beta)$ of uniformly convex functions of order α and type β in $\mathbb{U}$ if the following inequality holds

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta \left|\frac{zf''(z)}{f'(z)}\right| \quad (0 \leq \alpha < 1;\ \beta \geq 0; z \in \mathbb{U}). \tag{1.3}$$

The classes $\mathcal{ST}(\alpha,\beta)$ and $\mathcal{CT}(\alpha,\beta)$ were introduced and studied by Bharati et al. [2], Goodman [5, 6] and Kanas and Srivastava [7]. From (1.2) and (1.3), we note that

$$f \in \mathcal{CT}(\alpha,\beta) \Leftrightarrow zf' \in \mathcal{ST}(\alpha,\beta), \tag{1.4}$$

and

$$\mathcal{ST}(\alpha,0) = \mathcal{ST}(\alpha), \quad \mathcal{CT}(\alpha,0) = \mathcal{CT}(\alpha) \quad (0 \le \alpha < 1).$$

For $p_i \ge 1$ and $\sum_{i=1}^{m}\left(\frac{1}{p_i}\right) \ge 1$, the Hölder inequality is defined by (see [1]):

$$\sum_{i=2}^{\infty}\left(\prod_{j=1}^{m} a_{i,j}\right) \le \prod_{j=1}^{m}\left(\sum_{i=2}^{\infty} a_{i,j}^{p_i}\right)^{\frac{1}{p_i}}. \tag{1.5}$$

Let $f_j \in \mathcal{T}(n)\ (j=1,2)$ be given by

$$f_j(z) = z - \sum_{k=n}^{\infty} a_{k,j} z^k \quad (n \ge 2; j = 1,2), \tag{1.6}$$

then the modified Hadamard product (or convolution) $(f_1 * f_2)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \quad (n \ge 2). \tag{1.7}$$

For any real numbers p and q, the generalized modified Hadamard product $(f_1\ \Delta f_2)(p,q;z)$ defined by (see Choi and Yong [3]):

$$(f_1\ \Delta f_2)(p,q;z) = z - \sum_{k=n}^{\infty} (a_{k,1})^p (a_{k,2})^q z^k \quad (n \ge 2). \tag{1.8}$$

In the special case, if we take $p = q = 1$, then

$$(f_1\ \Delta f_2)(1,1;z) = (f_1 * f_2)(z) \quad (z \in \mathbb{U}). \tag{1.9}$$

In the present paper, we make use of the generalized Hadamard product to obtain some interesting characterization theorems involving the classes $\mathcal{ST}(\alpha,\beta)$ and $\mathcal{CT}(\alpha,\beta)$.

2. Main results

In order to prove our results for functions belonging to class $\mathcal{ST}(\alpha,\beta)$ and $\mathcal{CT}(\alpha,\beta)$, we shall need the following lemmas given by Bharati et al. [2].

Lemma 1. [2]. *Let the function f be defined by* (1.1), *then f is in the class $\mathcal{ST}(\alpha,\beta)$ if*

$$\sum_{k=n}^{\infty}\left[(1+\beta)k-(\alpha+\beta)\right]a_k \le 1-\alpha \quad (n\ge 2). \tag{2.1}$$

Lemma 2. [2]. *Let the function f be defined by* (1.1), *then f is in the class $\mathcal{CT}(\alpha,\beta)$ if*

$$\sum_{k=n}^{\infty}k\left[(1+\beta)k-(\alpha+\beta)\right]a_k \le 1-\alpha \quad (n\ge 2). \tag{2.2}$$

Theorem 3. *If the functions f_j $(j=1,2)$ defined by* (1.6) *are in the class $\mathcal{ST}(\alpha_j,\beta)$ $(j=1,2)$. Then*

$$(f_1\ \Delta f_2)\left(\frac{1}{p},\frac{p-1}{p};z\right)\in\mathcal{ST}(\delta,\beta), \tag{2.3}$$

where $p>1$ and δ is given by

$$\delta=\min_{k\ge n}\left\{1-\frac{(k-1)(1+\beta)}{\left[\frac{(1+\beta)k-(\alpha_1+k)}{1-\alpha_1}\right]^{\frac{1}{p}}\left[\frac{(1+\beta)k-(\alpha_2+k)}{1-\alpha_2}\right]^{\frac{p-1}{p}}-1}\right\}.$$

Proof. Since $f_j\in\mathcal{ST}(\alpha_j,\beta)$, then by using Lemma 1, we have

$$\sum_{k=n}^{\infty}\frac{(1+\beta)k-(\alpha_j+\beta)}{1-\alpha_j}a_{k,j}\le 1 \quad (j=1,2;n\ge 2). \tag{2.4}$$

Moreover

$$\left[\sum_{k=n}^{\infty}\frac{(1+\beta)k-(\alpha_1+\beta)}{1-\alpha_1}a_{k,1}\right]^{\frac{1}{p}}\le 1, \tag{2.5}$$

and

$$\left[\sum_{k=n}^{\infty}\frac{(1+\beta)k-(\alpha_2+\beta)}{1-\alpha_2}a_{k,2}\right]^{\frac{p-1}{p}}\le 1. \tag{2.6}$$

Applying the Hölder inequality (1.5) to (2.5) and (2.6), we obtain

$$\sum_{k=n}^{\infty}\left[\frac{(1+\beta)k-(\alpha_1+\beta)}{1-\alpha_1}\right]^{\frac{1}{p}}\left[\frac{(1+\beta)k-(\alpha_2+\beta)}{1-\alpha_2}\right]^{\frac{p-1}{p}}(a_{k,1})^{\frac{1}{p}}(a_{k,2})^{\frac{p-1}{p}}\le 1. \tag{2.7}$$

Since

$$(f_1 \ \Delta f_2) \left(\frac{1}{p}, \frac{p-1}{p}; z\right) = z - \sum_{k=n}^{\infty} (a_{k,1})^{\frac{1}{p}} (a_{k,2})^{\frac{p-1}{p}} z^k \quad (n \geq 2), \tag{2.8}$$

we see that

$$\sum_{k=n}^{\infty} \left[\frac{(1+\beta) k - (\delta+\beta)}{1-\delta}\right] (a_{k,1})^{\frac{1}{p}} (a_{k,2})^{\frac{p-1}{p}} \leq 1 \quad (n \geq 2), \tag{2.9}$$

with

$$\delta \leq \min_{k \geq n} \left\{ 1 - \frac{(k-1)(1+\beta)}{\left[\frac{(1+\beta)k-(\alpha_1+\beta)}{1-\alpha_1}\right]^{\frac{1}{p}} \left[\frac{(1+\beta)k-(\alpha_2+\beta)}{1-\alpha_2}\right]^{\frac{p-1}{p}} - 1} \right\}.$$

Thus, by using Lemma 1, the proof of Theorem 3 is completed.

Putting $\alpha_j = \alpha \, (j = 1, 2)$ in Theorem 3, we obtain the following corollary.

Corollary 4. *If the functions $f_j \, (j = 1, 2)$ defined by (1.6) are in the class $\mathcal{ST}(\alpha, \beta)$. Then*

$$(f_1 \ \Delta f_2) \left(\frac{1}{p}, \frac{p-1}{p}; z\right) \in \mathcal{ST}(\alpha, \beta) \quad (p > 1). \tag{2.10}$$

Theorem 5. *If the functions $f_j \, (j = 1, 2)$ defined by (1.6) are in the class $\mathcal{CT}(\alpha_j, \beta)$ for $j = 1, 2$, then*

$$(f_1 \ \Delta f_2) \left(\frac{1}{p}, \frac{p-1}{p}; z\right) \in \mathcal{CT}(\delta, \beta), \tag{2.11}$$

where $p > 1$ and δ is given by

$$\delta = \min_{k \geq n} \left[1 - \frac{(k-1)(1+\beta)}{\left(\frac{(1+\beta)k-(\alpha_1+\beta)}{1-\alpha_1}\right)^{\frac{1}{p}} \left(\frac{(1+\beta)k-(\alpha_2+\beta)}{1-\alpha_2}\right)^{\frac{p-1}{p}} - 1} \right].$$

Proof. Since $f_j \in \mathcal{CT}(\alpha_j, \beta)$ by using Lemma 2, we have

$$\sum_{k=n}^{\infty} \frac{k \left[(1+\beta) k - (\alpha_j + \beta)\right]}{1-\alpha_j} a_{k,j} \leq 1 \quad (j = 1, 2; n \geq 2). \tag{2.12}$$

Thus the proof of Theorem 5 is much akin to that of Theorem 3 by using lemma 2 instead of lemma 1.

Putting $\alpha_j = \alpha \ (j = 1, 2)$ in Theorem 5, we obtain the following corollary.

Corollary 6. *If the functions $f_j\,(j = 1, 2)$ defined by (1.6) are in the class $\mathcal{CT}\,(\alpha, \beta)$. Then*

$$(\ f_1\ \Delta f_2)\left(\frac{1}{p}, \frac{p-1}{p}; z\right) \in \mathcal{CT}\,(\alpha, \beta) \qquad (p > 1)\,. \tag{2.13}$$

Theorem 7. *Let the functions $f_j\,(j = 1, 2, ..., m)$ defined by (1.6) be in the class $\mathcal{ST}\,(\alpha_j, \beta)\,(j = 1, 2, ..., m)$, and let $F_m\,(z)$ be defined by*

$$F_m\,(z) = z - \sum_{k=n}^{\infty}\left[\sum_{j=1}^{m}(a_{k,j})^p\right] z^k \quad (z \in \mathbb{U}; n \geq p \geq 2; n \geq 2)\,. \tag{2.14}$$

Then

$$F_m\,(z) \in \mathcal{ST}\,(\delta, \beta) \qquad (m \geq 2)\,, \tag{2.15}$$

where

$$\delta = 1 - \frac{m\,(n-1)\,(1+\beta)}{\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^p - m},$$

where

$$\alpha = \min_{1\leq j\leq m}\{\alpha_j\}$$

and

$$\left[\frac{(1+\beta)\,n-(\alpha+\beta)}{1-\alpha}\right]^p \geq m\,[(1+\beta)\,n - \beta]\,.$$

Proof. Since $f_j \in \mathcal{ST}\,(\alpha_j, \beta)$ by using Lemma 1, we observe that

$$\sum_{k=n}^{\infty}\frac{(1+\beta)\,k-(\alpha_j+\beta)}{1-\alpha_j}a_{k,j} \leq 1 \quad (j = 1, 2, ..., m\ ; n \geq 2)\,, \tag{2.16}$$

and

$$\sum_{k=n}^{\infty}\left[\frac{(1+\beta)\,k-(\alpha_j+\beta)}{1-\alpha_j}\right]^p (a_{k,j})^p \leq \left[\sum_{k=n}^{\infty}\frac{(1+\beta)\,k-(\alpha_j+\beta)}{1-\alpha_j}a_{k,j}\right]^p \leq 1. \tag{2.17}$$

It follow from (2.17) that

$$\sum_{k=n}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left[\frac{(1+\beta)\,k-(\alpha_j+\beta)}{1-\alpha_j}\right]^p (a_{k,j})^p\right\} \leq 1. \tag{2.18}$$

Putting

$$\alpha = \min_{1 \le j \le m} \{\alpha_j\}, \tag{2.19}$$

and by virtue of Lemma 1, we find that

$$\sum_{k=n}^{\infty} \frac{(1+\beta)k-(\delta+\beta)}{1-\delta} \left[\sum_{j=1}^{m} (a_{k,j})^p\right]$$

$$\le \sum_{k=n}^{\infty} \left\{ \frac{1}{m} \left[\frac{(1+\beta)k-(\alpha+\beta)}{1-\alpha}\right]^p \left[\sum_{j=1}^{m} (a_{k,j})^p\right] \right\}$$

$$\le \sum_{k=n}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \left[\frac{(1+\beta)k-(\alpha_j+\beta)}{1-\alpha_j}\right]^p (a_{k,j})^p \right\} \le 1 \tag{2.20}$$

If

$$\delta \le 1 - \frac{m(1+\beta)(k-1)}{\left[\frac{(1+\beta)k-(\alpha+\beta)}{1-\alpha}\right]^p - m} \quad (k \ge n).$$

Now let

$$g(k) = 1 - \frac{m(1+\beta)(k-1)}{\left[\frac{(1+\beta)k-(\alpha+\beta)}{1-\alpha}\right]^p - m}. \tag{2.21}$$

Then $g'(k) \ge 0$ if $p \ge 2$. Hence

$$\delta = 1 - \frac{m(1+\beta)(n-1)}{\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^p - m}. \tag{2.22}$$

By

$$\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^p \ge m\left[(1+\beta)n-\beta\right],$$

we see that $0 \le \delta < 1$.Thus the proof of Theorem 7 is completed.

Theorem 8. *Let the functions* $f_j\,(j = 1, 2, ..., m)$ *defined by* (1.6) *be in the class* $\mathcal{CT}(\alpha_j, \beta)$ *for* $j = 1, 2, ..., m$, *and let* $F_m(z)$ *be defined by*

$$F_m(z) = z - \sum_{k=n}^{\infty} \left(\sum_{j=1}^{m} (a_{k,j})^p\right) z^k \quad (z \in \mathbb{U}; n \ge p \ge 2).$$

Then

$$F_m(z) \in \mathcal{CT}(\delta, \beta) \quad (m \ge 2), \tag{2.23}$$

where δ is given by

$$\delta = \left\{ 1 - \frac{m(1+\beta)(n-1)}{n^{p-1}\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^p - m} \right\}, \quad \alpha = \min_{1\le j\le m}\{\alpha_j\},$$

and

$$n^{p-1}\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^p \ge m\left[(1+\beta)n-\beta\right].$$

Proof. Since $f_j \in \mathcal{CT}(\alpha_j, \beta)$ by using Lemma 2, we observe that

$$\sum_{k=n}^{\infty} k\frac{(1+\beta)k-(\alpha_j+\beta)}{1-\alpha_j}a_{k,j} \le 1. \tag{2.24}$$

Thus the proof of Theorem 8 uses Lemma 2, in precisely the same manner as the above proof of Theorem 7 uses Lemma 1, The details may be omitted.

Remark 1. *Putting $\beta = 0$ in our results we obtain the results of Choi and Kim* [3] .

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