Acta Universitatis Apulensis No. 40/2014
ISSN: 1582-5329 pp. 219-231
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2014.40.18

SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and study some classes of meromorphic univalent functions defined in the punctured open unit disc. These classes are defined by using convolution technique. Coefficient bounds and inclusion results are solved.

2000 *Mathematics Subject Classification*: 30C45

Keywords: Meromorphic functions, convolution , subordination, integral operator, coefficient bounds.

1. Introduction

Let M denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \tag{1}$$

which are analytic and univalent in $E^* = \{z : 0 < |z| < 1\} = E \setminus \{0\}$. We denote MS^*, MC and M_λ, as the classes of meromorphic starlike, convex and $\lambda-$convex functions respectively. These classes were extensively studied by Pommerenke [18], Clunie [3], Miller [10, 11], Rosihan et al [1] and many others. For any two meromorphic functions f and g with

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \text{ and } g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m, \quad z \in E^*,$$

the convolution $(*)$ is defined as

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m, \qquad z \in E^*.$$

Salagean [20] defined a differential operator K^n, $n \in N_0 == N \cup \{0\}$, by

$$K^n f_1(z) = \Big[\underbrace{(k * k * . . . * k)}_{n\text{-times}} * f_1\Big](z), \tag{2}$$

with $k(z) = \frac{z}{(1-z)^2}$ and $f_1(z) = z + \sum_{j=2}^{\infty} a_j z^j$, analytic in E. Using convolution, we here define an operator analogue of the operator defined in (2). Let

$$S(z) = \frac{1-2z}{z(1-z)^2} = \frac{1}{z} - \sum_{m=1}^{\infty} m z^m, \qquad z \in E^*.$$

We define the function f_n by

$$f_n(z) = \underbrace{S(z) * S(z) * ... * S(z)}_{n\text{-times}}. \tag{3}$$

Next we define the differential operator D^n, $n \in N_0$, by

$$\begin{aligned} D^n f(z) &= f_n(z) * f(z) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} (-m)^n a_m z^m, \qquad z \in E^*. \end{aligned} \tag{4}$$

Clearly $D^0 f = f$ and $D^1 f = -zf'$. It is noted that

$$z\left(D^n f(z)\right)' = -D^{n+1} f(z), \qquad z \in E^*. \tag{5}$$

Next we define an integral operator by using the same technique as Noor [15] and Noor et-al [16] used for analytic case. Let f_n^{-1} be defined as

$$f_n^{-1}(z) * f_n(z) = S(z). \tag{6}$$

Then

$$\begin{aligned} I_n f(z) &= f_n^{-1}(z) * f(z) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} (-m)^{1-n} a_m z^m, \qquad z \in E^*. \end{aligned} \tag{7}$$

Clearly $I_0 f = -zf'$ and $I_1 f = f$. The following identity holds for I_n

$$z\left(I_{n+1} f(z)\right)' = -I_n f(z). \tag{8}$$

Let f and g be two analytic functions in E. We say that f is subordinate to g, written $f(z) \prec g(z)$, if there exist a Schwarz function w, analytic in E with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, see [9]. If g is univalent in E, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$. Using linear operators some important subclasses of analytic and meromorphic functions are introduced and for the recent work on this topic, we refer, [12, 5, 22, 6, 21, 8]. Now we define the following classes of functions by using the operator defined in (4). A function $f \in M$ is said to be from the class $MT^*(n)$, if and only if,

$$-\operatorname{Re}\left\{\frac{z\left(D^n f(z)\right)'}{D^n f(z)}\right\} > 0, \quad z \in E, \qquad (n \in N_0). \tag{9}$$

Using subordination, we can write the above relation as

$$-\left\{\frac{z\left(D^n f(z)\right)'}{D^n f(z)}\right\} \prec \frac{1+z}{1-z}, \quad z \in E, \quad (n \in N_0),$$

When $n = 0$, we obtain the class of meromorphic starlike functions, which has been studied by Clunie [3] and Pommerenke [18], and for $n = 1$, we have the class of meromorphic convex functions. See [10, 11]. Further for λ real and $n \in N_0$, the class $MT^*_\lambda(n)$ consists of functions $f \in M$ satisfying, $D^n f \neq 0$, $D^{n+1} f \neq 0$ in E^* and

$$\left\{(1-\lambda)\frac{D^{n+1} f(z)}{D^n f(z)} + \lambda\frac{D^{n+2} f(z)}{D^{n+1} f(z)}\right\} \prec \frac{1+z}{1-z}, \quad z \in E.$$

For $n = 0$, we have the class of meromorphic $\lambda-$convex functions, studied in [1, 13], and for $n = 0 = \lambda$, we have the class MS^*, studied by Clunie [3] and Pommerenke [18], and for $n = 0$ and $\lambda = 1$, we obtain the class MC, investigated by Miller [10, 11].

2. Preliminary Results

We need the following results.

Lemma 1. [17] *Let p be analytic in E with $p(0) = 1$ and suppose that*

$$Re\left\{p(z) - \frac{zp'(z)}{p(z)}\right\} > 0, \quad z \in E.$$

Then we have

$$Re p(z) > 0 \quad in\ E.$$

Lemma 2. *[4] Let β and γ be complex numbers. Also let the function h be convex univalent in E with*

$$h(0) = 1 \text{ and } Re\{\beta h(z) + \gamma\} > 0, \quad z \in E.$$

Suppose that the function

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots ,$$

is analytic in E and satisfying the following differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad z \in E. \tag{10}$$

If the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = 1, \tag{11}$$

has a univalent solution q, then

$$p(z) \prec q(z) \prec h(z), \quad z \in E,$$

and q is the best dominant in (10).

Remark 1. [4] *The differential equation (11) has its formal solution given by*

$$q(z) = \frac{zF'(z)}{F(z)} = \frac{\beta + \gamma}{\beta} \left(\frac{H(z)}{F(z)}\right)^{\beta} - \frac{\gamma}{\beta},$$

where

$$F(z) = \left\{\frac{\beta + \gamma}{\beta} \int_0^z \left(\frac{H(t)}{t}\right)^{\beta} t^{\beta+\gamma-1} dt\right\}^{\frac{1}{\beta}},$$

and

$$H(z) = z.\exp\left(\int_0^z \frac{h(t) - 1}{t} dt\right).$$

Lemma 3. *[19] Let $p \in P$ for $z \in E$. Then, for $t > 0$, $\mu \neq -1$ (complex),*

$$Re\left\{p(z) + \frac{tzp'(z)}{p(z) + \mu}\right\} > 0,$$

for

$$|z| < \frac{|\mu + 1|}{\sqrt{A + \sqrt{A^2 - |\mu^2 - 1|^2}}}, \quad A = 2(t+1)^2 + |\mu|^2 - 1.$$

This bound is best possible.

3. Main Results

In this section we shall prove our main results.

Theorem 4.
$$MT^*(n+1) \subset MT^*(n), \quad \textit{for } n \in N_o.$$

Proof. Let $f \in MT^*(n+1)$, then

$$\operatorname{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} > 0, \quad z \in E.$$

Set
$$p(z) = \frac{D^{n+1}f(z)}{D^n f(z)}. \tag{12}$$

Then p is analytic in E with $p(0) = 1$. Differentiating logarithmically (12), and after manipulations, we obtain

$$\frac{zp'(z)}{p(z)} = \frac{z\left(D^{n+1}f(z)\right)'}{D^{n+1}f(z)} - \frac{z\left(D^n f(z)\right)'}{D^n f(z)}.$$

Now (5) coupled with (12), yields

$$p(z) - \frac{zp'(z)}{p(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)},$$

that is
$$\operatorname{Re}\left\{p(z) - \frac{zp'(z)}{p(z)}\right\} > 0, \quad z \in E.$$

Now by using Lemma 1, we have that

$$f(z) \in MT^*(n), \quad z \in E^*.$$

Corollary 5. *For $n = 0$, we obtain the result of Nunokawa [17] that every meromorphic convex function is meromorphic starlike function.*

From Theorem 4, one has

$$MT^*(n+1) \subset MT^*(n) ... \subset MT^*(1) \subset MT^*(0), \quad n \in N_0.$$

Theorem 6. *Let* $n \in N_0$ *and let* $M(r) = \underset{|z|<1}{Max} \left| D^{n+1} f \right|$. *Suppose*

$$f(z) \in MT^*(n).$$

Then

$$L_r G(z) = L_r D^n f(z) = 2\pi r M(r).$$

Proof. It is know that

$$\begin{aligned} L_r G(z) &= \int_0^{2\pi} \left| -z^2 G'(z) \right| d\theta \\ &\leq \int_0^{2\pi} \left| -z^2 \left(D^n f(z) \right)' \right| d\theta \\ &= rM(r) \int_0^{2\pi} d\theta \\ &= 2\pi r M(r), \end{aligned}$$

where we have used (5). This completes the proof.

Theorem 7. *Let* $n \in N_0$ *and let* $M(r) = \underset{|z|<1}{Max} \left| D^{n+1} f \right|$. *Suppose*

$$f(z) \in MT^*(n).$$

Then

$$|a_m| = O(1) m^{-(1+n)}, \quad (m \geq 2).$$

This result is sharp.

Proof. For

$$G(z) = D^n f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} A_m z^m,$$

with $z = re^{i\theta}$, $0 < r < 1$ and $A_m = (-m)^n a_m$, we have, using Theorem 6,

$$\begin{aligned} |mA_m| &= \frac{1}{2\pi r^{m+1}} L_r G(z) \\ &\leq \frac{1}{2\pi r^{m+1}} 2\pi r M(r), \end{aligned}$$

from which, we have

$$|A_m| \leq \frac{M(r)}{r^m} m^{-1}.$$

We take $r = 1 - \frac{1}{m}$ and $A_m = (-m)^n a_m$, to have

$$|a_m| = O(1)m^{-(1+n)},$$

which is the required result. The function $zf'_n(z)$, shows that the bounds are sharp, where $f_n(z)$ is defined in (3).

Corollary 8. *For $n = 0$, we have $f \in MT^*(0) = MS^*$. Then for $m \geq 2$*

$$|a_m| = O(1)m^{-1}.$$

This result is same to that of Clunie [3].

Corollary 9. *For $n = 1$, we have $f \in MT^*(1) = MC$. Then for $m \geq 2$*

$$|a_m| = O(1)m^{-2},$$

which is same to that obtained by Noonan in [14], for the case $k = 2$.

Next, we derive an integral representation of functions belonging to the class $MT^*(n)$.

Theorem 10. *Let $f \in MT^*(n)$. Then*

$$D^n f(z) = z^{-1}.exp \int_0^z \frac{2w(t)}{t(w(t)-1)} dt, \tag{13}$$

where w is analytic in E with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in E.$$

Proof. For $f \in MT^*(n)$, then

$$\frac{-z(D^n f(z))'}{D^n f(z)} = \frac{1 + w(z)}{1 - w(z)},$$

where is w analytic in E with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in E.$$

From which, we have

$$\frac{(D^n f(z)))'}{D^n f(z)} + \frac{1}{z} = \frac{2w(z)}{z(w(z)-1)},$$

which upon integration yields

$$\ln(zD^n f(z)) = \int_0^z \frac{2w(t)}{t(w(t)-1)} dt. \tag{14}$$

The assertion (13) can easily be obtained from (14).

Theorem 11. *A function* $f \in MT^*_\lambda(n)$, $n \in N_0$, *if and only if, there is a function* $g \in MT^*(n)$ *such that*

$$D^n g(z) = \frac{1}{z}\left[zD^n f(z)\right]^{1-\lambda}\left[-z^2\left(D^n f(z)\right)'\right]^{\lambda}, \tag{15}$$

for all $z \in E^*$.

Proof. Differentiation of (15), coupled with 5, yields

$$\frac{D^{n+1}g(z)}{D^n g(z)} = \left\{(1-\lambda)\frac{D^{n+1}f(z)}{D^n f(z)} + \lambda\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\}.$$

If the right hand side belongs to P, the class of Caratheodory functions, so does the left hand side and conversely.

Theorem 12. *Let* $n \in N_0$ *and* $\lambda < \lambda_1 < 0$. *Then*

$MT^*_\lambda(n) \subset MT^*_{\lambda_1}(n)$.

Proof. Let $f \in MT^*_\lambda(n)$. Then

$$\begin{aligned}
&\left\{(1-\lambda_1)\frac{D^{n+1}f(z)}{D^n f(z)} + \lambda_1\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\}\\
=\;&\left\{(1-\frac{\lambda_1}{\lambda})\frac{D^{n+1}f(z)}{D^n f(z)} + \frac{\lambda_1}{\lambda}\left[(1-\lambda)\frac{D^{n+1}f(z)}{D^n f(z)} + \lambda\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right]\right\}\\
=\;&(1-\frac{\lambda_1}{\lambda})G_1(z) + \frac{\lambda_1}{\lambda}G_2(z), \quad G_1(z),\ G_z(z) \in P, \quad z \in E,\\
=\;&G(z), \quad G(z) \in P, \quad z \in E.
\end{aligned}$$

Since P is a convex set. Therefore $f \in MT^*_{\lambda_1}(n)$. This completes the proof.

Theorem 13. *Let* $n \in N_0$ *and* $Re\left\{\frac{1}{\lambda}\left[\frac{1+z}{1-z}\right]\right\} < 0$. *Then* $f \in MT^*_\lambda(n)$, *we have* $f \in MT^*(n)$. *Further*

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z) \prec \frac{1+z}{1-z}, \quad z \in E, \tag{16}$$

where

$$\left\{\begin{array}{ll}
q(z) = \frac{zF'(z)}{F(z)} = \left[\frac{H(z)}{F(z)}\right]^{\frac{-1}{\lambda}}, & \textit{with}\\
F(z) = \left\{\frac{-1}{\lambda}\int_0^z \left[\frac{H(t)}{t}\right]^{\frac{-1}{\lambda}} t^{-(1+\frac{1}{\lambda})}dt\right\}^{-\lambda}, & \textit{and}\\
H(z) = \frac{z}{(1-z)^2}. &
\end{array}\right. \tag{17}$$

Proof. Let $f \in MT^*_\lambda(n)$, where $n \in N_0$. Set

$$\phi(z) = z\left[zD^n f(z)\right]^{-1},$$

and

$$r_1 = \sup\{r : \phi(z) \neq 0, \quad 0 < |z| < 1\}.$$

Then ϕ is single valued in $0 < |z| < r_1$ and using (5), it follows that the function p_1 defined by

$$p_1(z) = \frac{z\phi'(z)}{\phi(z)} = \frac{D^{n+1} f(z)}{D^n f(z)}, \tag{18}$$

is analytic in $|z| < r_1$ and $p_1(0) = 1$. Now differentiating (18) and with the use of (5), we have

$$p_1(z) - \frac{zp_1'(z)}{p_1(z)} = \frac{D^{n+2} f(z)}{D^{n+1} f(z)}.$$

This implies

$$p_1(z) + \frac{zp_1'(z)}{\frac{-1}{\lambda}p_1(z)} = \left\{(1-\lambda)\frac{D^{n+1} f(z)}{D^n f(z)} + \lambda\frac{D^{n+2} f(z)}{D^{n+1} f(z)}\right\} \prec \frac{1+z}{1-z}. \tag{19}$$

Now from the hypothesis of the theorem and using Lemma 2 with $\beta = \frac{-1}{\lambda}$ and $\gamma = 0$, we have

$$p_1(z) \prec q(z) \prec \frac{1+z}{1-z},$$

where q is given by (17). From (19) and the hypothesis of the theorem it can be seen that $\mathrm{Re}p_1 > 0$ in $|z| < r_1$. Now (18), shows that ϕ is starlike univalent in $|z| < r_1$. Thus it is not possible that ϕ vanishes in $|z| < r_1$, if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore p_1 is analytic in E. Thus from (18) and (19), we have the required result.

Theorem 14. *Let $n \in N_0$ and $\lambda < 0$. Then $f \in MT^*(n)$, we have $f \in MT^*_\lambda(n)$ for $|z| < r_0$,*

$$r_0 < \frac{1}{\sqrt{A + \sqrt{A^2 - 1}}}, \quad A = 2(1-\lambda)^2 - 1. \tag{20}$$

Proof. Let

$$p(z) = \frac{D^{n+1} f(z)}{D^n f(z)},$$

then p is analytic in E^* with $p(0) = 1$. Now proceeding as in previous theorem, we have

$$\mathrm{Re}\left\{(1-\lambda)\frac{D^{n+1} f(z)}{D^n f(z)} + \lambda\frac{D^{n+2} f(z)}{D^{n+1} f(z)}\right\} = \mathrm{Re}\left\{p(z) - \frac{\lambda zp'(z)}{p(z)}\right\}. \tag{21}$$

Using Lemma 3, with $t = -\lambda > 0$, $\mu = 0$, it follows that

$$\operatorname{Re}\left\{p(z) - \frac{\lambda z p'(z)}{p(z)}\right\} > 0, \qquad |z| < r_0,$$

where r_0 is given by (20). Consequently from (21), it follows that

$$(1-\lambda)\frac{D^{n+1}f(z)}{D^n f(z)} + \lambda \frac{D^{n+2}f(z)}{D^{n+1}f(z)} \in P, \quad \text{for } |z| < r_0.$$

This completes the proof.

Let $\operatorname{Re} c > -1$, $f \in M$. Bajpai [2] defined the following integral operator $F : M \to M$ as

$$\begin{aligned} F(z) &= \frac{c}{z^{c+1}} \int_0^z t^c f(z)\, dt \qquad (22) \\ &= \varphi(z) * f(z), \end{aligned}$$

where

$$\varphi(z) = \frac{1}{z} + \sum_0^\infty \frac{c}{c+n+1}.$$

We prove the following.

Theorem 15. *Let F be defined by* (22) *with $f \in MT^*(n)$, $c > -1$. Then $F \in MT^*(n)$.*

Proof. From(22), one can easily derive the formula

$$z\left(D^n F(z)\right)' = cD^n f(z) - (1+c)\, D^n F(z). \qquad (23)$$

Let

$$p_1(z) = \frac{D^{n+1}F(z)}{D^n F(z)}, \qquad (24)$$

where p_1 is analytic in E^* with $p_1(0) = 1$. From (22) and (23), we have

$$\begin{aligned} cD^{1+n} f(z) &= (1+c)\, D^{1+n}F(z) + z\left(D^{1+n}F(z)\right)' \\ &= (1+c)\left[p_1(z)\, D^n F(z)\right] + z\left(p_1(z)\, D^n F(z)\right)' \\ &= \left[(1+c)\, p_1(z) + z p_1'(z) - p_1^2(z)\right] D^n F(z). \qquad (25) \end{aligned}$$

Similarly, we have

$$cD^n f(z) = \left[(1+c) - p_1(z)\right] D^n F(z). \qquad (26)$$

Now from (25) and (26) , we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} = p_1(z) + \frac{zp_1'(z)}{1+c-p_1(z)}, \tag{27}$$

we take

$$p_1(z) = \frac{1-w(z)}{1+w(z)},$$

then (27), can be wrirren as

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1-w(z)}{1+w(z)} - \frac{2zw'(z)}{(1+w(z))(c+(2+c)w(z))}. \tag{28}$$

We claim that $|w| < 1$ for $z \in E$. Otherwise there exists a point z_0 in E such that $\max\limits_{|z|\le z_0} |w(z)| = |w(z_0)| = 1$. Then from a well known result due to Jack [7], there is a real number $\delta \ge 1$ such that

$$z_0 w'(z_0) = \delta w(z). \tag{29}$$

From (28) and (29), we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1-w(z)}{1+w(z)} - \frac{2\delta w(z)}{(1+w(z))(c+(2+c)w(z))}.$$

Therefore

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} \le \frac{-1}{2(1+c)} < 0,$$

a contradiction. Hence $|w| < 1$ for $z \in E$. Thus we have $F(z) \in MT^*(n)$.

Acknowledgements. The authors express deep gratitude to Dr. S. M. Junaid Zaidi, Rector, CIIT, for his support and providing excellent research facilities. This research is carried out under the HEC project grant No. NRPU No. 20-1966/R&D/11-2553.

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