

**A COUPLED SYSTEM OF INTEGRO-DIFFERENTIAL
EQUATIONS INVOLVING RIEMANN-LIOUVILLE INTEGRAL
AND CAPUTO DERIVATIVE**

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ABSTRACT. A coupled system of fractional integro-differential equations involving Riemann-Liouville integral and Caputo derivative is considered in this paper. Some existence and uniqueness results are obtained using Banach contraction principle. Other existence results are also studied using Shaefer's fixed point theorem. At the end, some illustrative examples are discussed.

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1. INTRODUCTION

In recent years, differential equations of fractional order have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as visco-elasticity, electrochemistry, control, porous media, electromagnetic, aerodynamics, etc. For more details, we refer the reader to [5, 17, 18]. Recently, there is a large number of papers dealing with such equations, see [4, 6, 10]. More recently, a basic theory for the initial boundary value problems of fractional differential equations has been discussed in [11, 18, 21]. On the other hand, existence and uniqueness of solutions to boundary value problems for fractional differential equations have attracted the attention of many authors, see for example, [2, 7, 18] and the references therein. Moreover, the study of coupled systems of fractional order is also important in various problems of applied nature [3, 8, 9, 15, 20, 22, 23]. Recently, many people have established the existence and uniqueness for solutions of some fractional systems, see [1, 14, 17, 19] and the reference therein. This paper deals with the existence and uniqueness of solutions to the following coupled system of fractional integro-differential equations:

$$\left\{ \begin{array}{l} D^\alpha x(t) = (\varphi_1(t) + \varphi_2(t)) f_1\left(t, D^{\frac{\alpha}{2}} x(t), y(t)\right) \\ \quad + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f_1\left(s, D^{\frac{\alpha}{2}} x(s), y(s)\right) ds, t \in J, \\ D^\beta y(t) = (\varphi_1(t) - \varphi_2(t)) f_2\left(t, x(t), D^{\frac{\beta}{2}} y(t)\right) \\ \quad + \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} f_2\left(s, x(s), D^{\frac{\beta}{2}} y(s)\right) ds, t \in J, \\ x(0) = x_0^*, y(0) = y_0^*, \end{array} \right. \quad (1)$$

where D^α, D^β denote the Caputo fractional derivatives, with $0 < \alpha < 1, 0 < \beta < 1$, and σ, δ are non negative real numbers, φ_1, φ_2 are two continuous functions on $J := [0, 1]$, $x_0^*, y_0^* \in \mathbb{R}^*$, $f_1, f_2 : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are two functions that will be specified later.

The paper is organized as follows: In section 2, we present some preliminaries and lemmas. In Section 3, we present our main results for existence and uniqueness of solutions for the problem (1). Some examples to illustrate our results are presented in Section 4.

2. PRELIMINARIES

In this section, we present some useful definitions and lemmas [12, 13, 16, 19]:

Definition 1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a, b]$ is defined as:*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, a \leq t \leq b \quad (2)$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2. *The fractional derivative of $f \in C^n([a, b])$ in the Caputo's sense is defined as:*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, n-1 < \alpha, n \in \mathbb{N}^*, a \leq t \leq b. \quad (3)$$

The following lemmas give some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative [12, 13]:

Lemma 1. *Let $r, s > 0, f \in L_1([a, b])$. Then $J^r J^s f(t) = J^{r+s} f(t), D^s I^s f(t) = f(t), t \in [a, b]$.*

Lemma 2. *Let $s > r > 0, f \in L_1([a, b])$. Then $D^r J^s f(t) = J^{s-r} f(t), t \in [a, b]$.*

Let us now introduce the spaces

$$X = \{x : x \in C([0, 1]), D^{\frac{\alpha}{2}} x \in C([0, 1])\},$$

and

$$Y = \{y : y \in C([0, 1]), D^{\frac{\beta}{2}} y \in C([0, 1])\},$$

endowed respectively with the norms

$$\|x\|_X = \|x\| + \|D^{\frac{\alpha}{2}} x\|; \|x\| = \sup_{t \in J} |x(t)|, \|D^{\frac{\alpha}{2}} x\| = \sup_{t \in J} \left| D^{\frac{\alpha}{2}} x(t) \right|,$$

and

$$\|y\|_Y = \|y\| + \|D^{\frac{\beta}{2}} y\|; \|y\| = \sup_{t \in J} |y(t)|, \|D^{\frac{\beta}{2}} y\| = \sup_{t \in J} \left| D^{\frac{\beta}{2}} y(t) \right|.$$

Obviously, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, are two Banach spaces. The product space $(X \times Y, \|(x, y)\|_{X \times Y})$ is also a Banach space with norm $\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$.

We also give the following lemmas [12]:

Lemma 3. *For $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad (4)$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1, n = [\alpha] + 1$.

Lemma 4. *Let $\alpha > 0$. Then*

$$J^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad (5)$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1, n = [\alpha] + 1$.

We need also the following auxiliary result:

Lemma 5. For a given $g \in C([0, 1], \mathbb{R})$, the solution of the boundary value problem

$$\begin{aligned} D^\alpha x(t) &= (\varphi_1 g + \varphi_2 g)(t) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} g(s) ds, t \in J, 0 < \alpha < 1, \sigma > 0, \\ x(0) &= x_0^*, \end{aligned} \tag{6}$$

is given by:

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ((\varphi_1 g)(s) + (\varphi_2 g)(s)) ds + \int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)} g(s) ds + x_0^*. \tag{7}$$

Proof. By lemma 5 and Lemma 6, the general solution of (6) is written as

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ((\varphi_1 g)(s) + (\varphi_2 g)(s)) ds + \int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)} g(s) ds - c_0. \tag{8}$$

Applying the boundary condition of (6), we find that

$$c_0 = -x_0^*. \tag{9}$$

Substituting the value of c_0 in (7), we obtain the solution (6).

3. MAIN RESULTS

First of all, we consider the following quantities:

$$\begin{aligned} \theta_1 &= \frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\sigma+1)}, \\ \theta_2 &= \frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\frac{\alpha}{2}+1)} + \frac{1}{\Gamma(\frac{\alpha}{2}+\sigma+1)}, \\ \theta_3 &= \frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+\delta+1)}, \\ \theta_4 &= \frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\frac{\beta}{2}+1)} + \frac{1}{\Gamma(\frac{\beta}{2}+\delta+1)}. \end{aligned} \tag{10}$$

Also, we suppose the following conditions:

(H1) : The functions $f_1, f_2 : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous.

(H2) : There exist non negative continuous functions $a_i(t), b_i(t), i = 1, 2$ such that for all $t \in J$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} |f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| &\leq a_1(t) |x_1 - x_2| + b_1(t) |y_1 - y_2|, \\ |f_2(t, x_1, y_1) - f_2(t, x_2, y_2)| &\leq a_2(t) |x_1 - x_2| + b_2(t) |y_1 - y_2|, \end{aligned} \quad (11)$$

with

$$\omega_1 = \sup_{t \in J} a_1(t), \omega_2 = \sup_{t \in J} b_1(t), \varpi_1 = \sup_{t \in J} a_2(t), \varpi_2 = \sup_{t \in J} b_2(t).$$

(H3) : There exists non negative continuous functions m_1 and m_2 , such that

$$|f_1(t, x, y)| \leq m_1(t), |f_2(t, x, y)| \leq m_2(t), \text{ for each } t \in J \text{ and all } x, y \in \mathbb{R},$$

with

$$M_1 = \sup_{t \in J} m_1(t), M_2 = \sup_{t \in J} m_2(t).$$

Our first result is obtained using Banach contraction principle. We have:

Theorem 6. *Assume that the hypothesis (H2) holds. If*

$$(\theta_1 + \theta_2)(\omega_1 + \omega_2) + (\theta_3 + \theta_4)(\varpi_1 + \varpi_2) < 1, \quad (12)$$

then the fractional integro-differential system (1) has a unique solution on J .

Proof. Define the operator $T : X \times Y \rightarrow X \times Y$ by:

$$T(x, y)(t) := (T_1(x, y)(t), T_2(x, y)(t)), t \in J, \quad (13)$$

where

$$\begin{aligned} T_1(x, y)(t) &: = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\varphi_1(s) + \varphi_2(s)) f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) ds \\ &+ \frac{1}{\Gamma(\alpha + \sigma)} \int_0^t (t-s)^{\alpha+\sigma-1} f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) ds + x_0^*, \end{aligned} \quad (14)$$

and

$$\begin{aligned} T_2(x, y)(t) &: = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (\varphi_1(s) - \varphi_2(s)) f_2\left(t, x(t), D^{\frac{\beta}{2}}y(t)\right) ds \\ &+ \frac{1}{\Gamma(\beta + \delta)} \int_0^t (t-s)^{\beta+\delta-1} f_2\left(s, x(s), D^{\frac{\beta}{2}}y(s)\right) ds + y_0^*. \end{aligned} \quad (15)$$

We shall prove that T is a contraction mapping :

Let $(x, y), (x_1, y_1) \in X \times Y$. Then, for each $t \in J$, we can write

$$|T_1(x, y)(t) - T_1(x_1, y_1)(t)| \leq \quad (16)$$

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0 \leq s \leq 1} |\varphi_1(s) + \varphi_2(s)| |f_1(s, D^{\frac{\alpha}{2}}x(s), y(s)) - f_1(s, D^{\frac{\alpha}{2}}x_1(s), y_1(s))| ds \\ & + \frac{1}{\Gamma(\alpha + \sigma)} \int_0^t (t-s)^{\alpha+\sigma-1} |f_1(s, D^{\frac{\alpha}{2}}x(s), y(s)) - f_1(s, D^{\frac{\alpha}{2}}x_1(s), y_1(s))| ds. \end{aligned}$$

Thanks to (H2), we obtain

$$|T_1(x, y)(t) - T_1(x_1, y_1)(t)| \leq \quad (17)$$

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\|\varphi_1\|_\infty + \|\varphi_2\|_\infty) ds \left(\sup_{t \in J} a_1(t) \left\| D^{\frac{\alpha}{2}}x - D^{\frac{\alpha}{2}}x_1 \right\| + \sup_{t \in J} b_1(t) \|y - y_1\| \right) \\ & + \frac{1}{\Gamma(\alpha + \sigma)} \int_0^t (t-s)^{\alpha+\sigma-1} ds \left(\sup_{t \in J} a_1(t) \left\| D^{\frac{\alpha}{2}}x - D^{\frac{\alpha}{2}}x_1 \right\| + \sup_{t \in J} b_1(t) \|y - y_1\| \right). \end{aligned}$$

Consequently,

$$|T_1(x, y)(t) - T_1(x_1, y_1)(t)| \leq \quad (18)$$

$$\left[\frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha + \sigma + 1)} \right] (\omega_1 + \omega_2) \left(\left\| D^{\frac{\alpha}{2}}x - D^{\frac{\alpha}{2}}x_1 \right\| + \|y - y_1\| \right),$$

which implies that

$$\|T_1(x, y) - T_1(x_1, y_1)\| \leq \theta_1 (\omega_1 + \omega_2) \left(\left\| D^{\frac{\alpha}{2}}x - D^{\frac{\alpha}{2}}x_1 \right\| + \|y - y_1\| \right), \quad (19)$$

and

$$\left| D^{\frac{\alpha}{2}}T_1(x, y)(t) - D^{\frac{\alpha}{2}}T_1(x_1, y_1)(t) \right| \leq \quad (20)$$

$$\begin{aligned} & \frac{1}{\Gamma(\alpha - \frac{\alpha}{2})} \int_0^t (t-s)^{\alpha - \frac{\alpha}{2} - 1} \sup_{0 \leq s \leq 1} |\varphi_1(s) + \varphi_2(s)| |f_1(s, D^{\frac{\alpha}{2}}x(s), y(s)) - f_1(s, D^{\frac{\alpha}{2}}x_1(s), y_1(s))| ds \\ & + \frac{1}{\Gamma(\alpha + \sigma - \frac{\alpha}{2})} \int_0^t (t-s)^{\alpha + \sigma - \frac{\alpha}{2} - 1} |f_1(s, D^{\frac{\alpha}{2}}x(s), y(s)) - f_1(s, D^{\frac{\alpha}{2}}x_1(s), y_1(s))| ds. \end{aligned}$$

By (H2), we have

$$\left| D^{\frac{\alpha}{2}}T_1(x, y)(t) - D^{\frac{\alpha}{2}}T_1(x_1, y_1)(t) \right| \leq \quad (21)$$

$$\frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^t (t-s)^{\frac{\alpha}{2}-1} (\|\varphi_1\|_\infty + \|\varphi_2\|_\infty) ds \left(\sup_{t \in J} a_1(t) \left\| D^{\frac{\alpha}{2}}x - D^{\frac{\alpha}{2}}x_1 \right\| + \sup_{t \in J} b_1(t) \|y - y_1\| \right)$$

$$+ \frac{1}{\Gamma\left(\frac{\alpha}{2} + \sigma\right)} \int_0^t (t-s)^{\frac{\alpha}{2} + \sigma - 1} ds \left(\sup_{t \in J} a_1(t) \left\| D^{\frac{\alpha}{2}} x - D^{\frac{\alpha}{2}} x_1 \right\| + \sup_{t \in J} b_1(t) \|y - y_1\| \right).$$

Hence,

$$\left| D^{\frac{\alpha}{2}} T_1(x, y)(t) - D^{\frac{\alpha}{2}} T_1(x_1, y_1)(t) \right| \leq \tag{22}$$

$$\left[\frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma\left(\frac{\alpha}{2} + 1\right)} + \frac{1}{\Gamma\left(\frac{\alpha}{2} + \sigma + 1\right)} \right] (\omega_1 + \omega_2) \left(\left\| D^{\frac{\alpha}{2}} x - D^{\frac{\alpha}{2}} x_1 \right\| + \|y - y_1\| \right).$$

Therefore,

$$\left| D^{\frac{\alpha}{2}} T_1(x, y)(t) - D^{\frac{\alpha}{2}} T_1(x_1, y_1)(t) \right| \leq \theta_2 (\omega_1 + \omega_2) \left(\left\| D^{\frac{\alpha}{2}} x - D^{\frac{\alpha}{2}} x_1 \right\| + \|y - y_1\| \right). \tag{23}$$

And consequently,

$$\left\| D^{\frac{\alpha}{2}} T_1(x, y) - D^{\frac{\alpha}{2}} T_1(x_1, y_1) \right\| \leq \theta_2 (\omega_1 + \omega_2) \left(\left\| D^{\frac{\alpha}{2}} x - D^{\frac{\alpha}{2}} x_1 \right\| + \|y - y_1\| \right). \tag{24}$$

By (19) and (24), we can state that

$$\|T_1(x, y) - T_1(x_1, y_1)\|_X \leq (\theta_1 + \theta_2) (\omega_1 + \omega_2) \left(\left\| D^{\frac{\alpha}{2}} x - D^{\frac{\alpha}{2}} x_1 \right\| + \|y - y_1\| \right). \tag{25}$$

With the same arguments as before, we have

$$\|T_2(x, y) - T_2(x_1, y_1)\|_Y \leq (\theta_3 + \theta_4) (\varpi_1 + \varpi_2) \left(\|x - x_1\| + \left\| D^{\frac{\beta}{2}} y - D^{\frac{\beta}{2}} y_1 \right\| \right). \tag{26}$$

By (25) and (26), we obtain

$$\|T(x, y) - T(x_1, y_1)\|_{X \times Y} \leq [(\theta_1 + \theta_2) (\omega_1 + \omega_2) + (\theta_3 + \theta_4) (\varpi_1 + \varpi_2)] \|(x - x_1, y - y_1)\|_{X \times Y}. \tag{27}$$

Thanks to (14), we conclude that T is contraction. As a consequence of Banach fixed point theorem, we deduce that T has a fixed point which is a solution of the problem (1).

Our second result is the following theorem:

Theorem 7. *Suppose that the hypotheses (H1) and (H3) are satisfied. Then, the integro-differential system (1) has at least a solution on J .*

Proof. We shall use Schaefer's fixed point theorem to prove that T has at least a fixed point on $X \times Y$:

The continuity of f_1 and f_2 (hypothesis (H1)) implies that the operator T is continuous on $X \times Y$.

[1*] : We shall prove that T maps bounded sets into bounded sets in $X \times Y$:
 Taking $\rho > 0$, and $(x, y) \in B_\rho := \{(x, y) \in X \times Y; \|(x, y)\|_{X \times Y} \leq \rho\}$, then for each $t \in J$, we have:

$$|T_1(x, y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds \\ + \frac{1}{\Gamma(\alpha + \sigma)} \int_0^t (t-s)^{\alpha+\sigma-1} \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds + |x_0^*|. \quad (28)$$

Thanks to (H3), we can write

$$|T_1(x, y)(t)| \leq \frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty \sup_{t \in J} m_1(t)}{\Gamma(\alpha+1)} + \frac{\sup_{t \in J} m_1(t)}{\Gamma(\alpha+\sigma+1)} + |x_0^*| \quad (29) \\ \leq \sup_{t \in J} m_1(t) \left[\frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\sigma+1)} \right] + |x_0^*|.$$

Therefore, for each $t \in J$,

$$|T_1(x, y)(t)| \leq M_1 \theta_1 + |x_0^*|. \quad (30)$$

Hence, we have

$$\|T_1(x, y)\| \leq M_1 \theta_1 + |x_0^*|. \quad (31)$$

On the other hand,

$$\left| D^{\frac{\alpha}{2}} T_1(x, y)(t) \right| \leq \quad (32) \\ \frac{1}{\Gamma\left(\alpha - \frac{\alpha}{2}\right)} \int_0^t (t-s)^{\alpha - \frac{\alpha}{2} - 1} \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds \\ + \frac{1}{\Gamma\left(\alpha + \sigma - \frac{\alpha}{2}\right)} \int_0^t (t-s)^{\alpha + \sigma - \frac{\alpha}{2} - 1} \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds + |x_0^*|. \quad (33)$$

By (H3), we obtain

$$\left| D^{\frac{\alpha}{2}} T_1(x, y)(t) \right| \leq \sup_{t \in J} m_1(t) \left[\frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma\left(\frac{\alpha}{2} + 1\right)} + \frac{1}{\Gamma\left(\frac{\alpha}{2} + \sigma + 1\right)} \right] + |x_0^*|. \quad (34)$$

Consequently,

$$\left| D^{\frac{\alpha}{2}} T_1(x, y)(t) \right| \leq M_2 \theta_2 + |x_0^*|. \quad (35)$$

Therefore,

$$\left\| D^{\frac{\alpha}{2}} T_1(x, y) \right\| \leq M_2 \theta_2 + |x_0^*|. \quad (36)$$

Combining (31) and (35), yields the following inequality

$$\|T_1(x, y)\|_X \leq M_1(\theta_1 + \theta_2) + |x_0^*|. \quad (37)$$

Similarly, it can be shown that

$$\|T_2(x, y)\|_Y \leq M_2(\theta_3 + \theta_4) + |y_0^*|. \quad (38)$$

It follows from (36) and (37) that

$$\|T(x, y)\|_{X \times Y} \leq M_1(\theta_1 + \theta_2) + M_2(\theta_3 + \theta_4) + |x_0^*| + |y_0^*|. \quad (39)$$

Consequently

$$\|T(x, y)\|_{X \times Y} < \infty. \quad (40)$$

[2*] : Now, we will prove that T is equi-continuous on J :

For $(x, y) \in B_\rho$, and $t_1, t_2 \in J$, such that $t_2 < t_1$. We have:

$$|T_1(x, y)(t_1) - T_1(x, y)(t_2)| \leq \quad (41)$$

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right) \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| \left| f_1 \left(s, D^{\frac{\alpha}{2}} x(s), y(s) \right) \right| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t_1} (t_1 - s)^{\alpha-1} \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| \left| f_1 \left(s, D^{\frac{\alpha}{2}} x(s), y(s) \right) \right| ds \\ & + \frac{1}{\Gamma(\alpha + \sigma)} \int_0^{t_1} \left((t_1 - s)^{\alpha+\sigma-1} - (t_2 - s)^{\alpha+\sigma-1} \right) \left| f_1 \left(s, D^{\frac{\alpha}{2}} x(s), y(s) \right) \right| ds \\ & + \frac{1}{\Gamma(\alpha + \sigma)} \int_{t_1}^{t_2} (t - s)^{\alpha+\sigma-1} \left| f_1 \left(s, D^{\frac{\alpha}{2}} x(s), y(s) \right) \right| ds. \end{aligned}$$

Thus,

$$\begin{aligned} |T_1(x, y)(t_1) - T_1(x, y)(t_2)| & \leq \frac{M_1(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)}{\Gamma(\alpha + 1)} [(t_1 - t_2)^\alpha + t_2^\alpha - t_1^\alpha] \\ & + \frac{M_1}{\Gamma(\alpha + \sigma + 1)} [(t_1 - t_2)^{\alpha+\sigma} + t_2^{\alpha+\sigma} - t_1^{\alpha+\sigma}], \end{aligned} \quad (42)$$

and

$$\left| D^{\frac{\alpha}{2}} T_1(x, y)(t_1) - D^{\frac{\alpha}{2}} T_1(x, y)(t_2) \right| \leq \quad (43)$$

$$\begin{aligned}
 & \frac{1}{\Gamma(\alpha - \frac{\alpha}{2})} \int_0^{t_2} \left((t_2 - s)^{\alpha - \frac{\alpha}{2} - 1} - (t_1 - s)^{\alpha - \frac{\alpha}{2} - 1} \right) \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| |f_1(s, D^{\frac{\alpha}{2}} x(s), y(s))| ds \\
 & + \frac{1}{\Gamma(\alpha - \frac{\alpha}{2})} \int_{t_2}^{t_1} (t_1 - s)^{\alpha - \frac{\alpha}{2} - 1} \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| |f_1(s, D^{\frac{\alpha}{2}} x(s), y(s))| ds \\
 & + \frac{1}{\Gamma(\alpha + \sigma - \frac{\alpha}{2})} \int_0^{t_1} \left((t_1 - s)^{\alpha + \sigma - \frac{\alpha}{2} - 1} - (t_2 - s)^{\alpha + \sigma - \frac{\alpha}{2} - 1} \right) |f_1(s, D^{\frac{\alpha}{2}} x(s), y(s))| ds \\
 & + \frac{1}{\Gamma(\alpha + \sigma - \frac{\alpha}{2})} \int_{t_1}^{t_2} (t - s)^{\alpha + \sigma - \frac{\alpha}{2} - 1} |f_1(s, D^{\frac{\alpha}{2}} x(s), y(s))| ds.
 \end{aligned}$$

Using (H3), we obtain:

$$\begin{aligned}
 \left| D^{\frac{\alpha}{2}} T_1(x, y)(t_1) - D^{\frac{\alpha}{2}} T_1(x, y)(t_2) \right| & \leq \frac{M_1(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)}{\Gamma(\frac{\alpha}{2} + 1)} \left[(t_1 - t_2)^{\frac{\alpha}{2}} + t_2^{\frac{\alpha}{2}} - t_1^{\frac{\alpha}{2}} \right] \quad (44) \\
 & + \frac{1}{\Gamma(\frac{\alpha}{2} + \sigma + 1)} \left[(t_1 - t_2)^{\frac{\alpha}{2} + \sigma} + t_2^{\frac{\alpha}{2} + \sigma} - t_1^{\frac{\alpha}{2} + \sigma} \right].
 \end{aligned}$$

By (41) and (43), we can write

$$\begin{aligned}
 \|T_1(x, y)(t_1) - T_1(x, y)(t_2)\|_X & \leq \frac{M_1(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)}{\Gamma(\alpha + 1)} \left[(t_1 - t_2)^\alpha + t_2^\alpha - t_1^\alpha \right] \quad (45) \\
 & + \frac{M_1}{\Gamma(\alpha + \sigma + 1)} \left[(t_1 - t_2)^{\alpha + \sigma} + t_2^{\alpha + \sigma} - t_1^{\alpha + \sigma} \right] \\
 & + \frac{M_1(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)}{\Gamma(\frac{\alpha}{2} + 1)} \left[(t_1 - t_2)^{\frac{\alpha}{2}} + t_2^{\frac{\alpha}{2}} - t_1^{\frac{\alpha}{2}} \right] \\
 & + \frac{M_1}{\Gamma(\frac{\alpha}{2} + \sigma + 1)} \left[(t_1 - t_2)^{\frac{\alpha}{2} + \sigma} + t_2^{\frac{\alpha}{2} + \sigma} - t_1^{\frac{\alpha}{2} + \sigma} \right].
 \end{aligned}$$

With the same arguments as before, we get

$$\begin{aligned}
 \|T_2(x, y)(t_1) - T_2(x, y)(t_2)\|_Y & \leq \frac{M_2(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)}{\Gamma(\beta + 1)} \left[(t_1 - t_2)^\beta + t_2^\beta - t_1^\beta \right] \quad (46) \\
 & + \frac{M_2}{\Gamma(\beta + \delta + 1)} \left[(t_1 - t_2)^{\beta + \delta} + t_2^{\beta + \delta} - t_1^{\beta + \delta} \right] \\
 & + \frac{M_2(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)}{\Gamma(\frac{\beta}{2} + 1)} \left[(t_1 - t_2)^{\frac{\beta}{2}} + t_2^{\frac{\beta}{2}} - t_1^{\frac{\beta}{2}} \right] \\
 & + \frac{M_2}{\Gamma(\frac{\beta}{2} + \delta + 1)} \left[(t_1 - t_2)^{\frac{\beta}{2} + \delta} + t_2^{\frac{\beta}{2} + \delta} - t_1^{\frac{\beta}{2} + \delta} \right].
 \end{aligned}$$

Thanks to (44) and (45), we can state that $\|T(x, y)(t_1) - T(x, y)(t_2)\|_{X \times Y} \rightarrow 0$ as $t_1 \rightarrow t_2$. Combining [1*] and [2*] and using Arzela-Ascoli theorem, we conclude that T is completely continuous operator.

[3*] : Finally, we shall show that

$$\Omega = \{(x, y) \in X \times Y, (x, y) = \mu T(x, y), 0 < \mu < 1\}, \quad (47)$$

is a bounded set:

Let $(x, y) \in \Omega$, then $(x, y) = \mu T(x, y)$, for some $0 < \mu < 1$. Thus, for each $t \in J$, we have:

$$x(t) = \mu T_1(x, y)(t), y(t) = \mu T_2(x, y)(t). \quad (48)$$

Then

$$\begin{aligned} \frac{1}{\mu} |x(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds \\ &\quad + \frac{1}{\Gamma(\alpha+\sigma)} \int_0^t (t-s)^{\alpha+\sigma-1} \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds + |x_0^*|. \end{aligned} \quad (49)$$

Thanks to (H3), we can write

$$\begin{aligned} \frac{1}{\mu} |x(t)| &\leq \frac{(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty) \sup_{t \in J} m_1(t)}{\Gamma(\alpha+1)} + \frac{\sup_{t \in J} m_1(t)}{\Gamma(\alpha+\sigma+1)} + |x_0^*|. \\ &\leq \frac{M_1(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)}{\Gamma(\alpha+1)} + \frac{M_1}{\Gamma(\alpha+\sigma+1)} + |x_0^*|. \end{aligned} \quad (50)$$

Therefore,

$$\frac{1}{\mu} |x(t)| \leq M_1 \left[\frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\sigma+1)} \right] + |x_0^*|. \quad (51)$$

Hence,

$$|x(t)| \leq \mu M_1 \theta_1 + \mu |x_0^*|. \quad (52)$$

Similarly, we can get,

$$\begin{aligned} \frac{1}{\mu} |D^{\frac{\alpha}{2}}x(t)| &\leq \frac{1}{\Gamma(\alpha-\frac{\alpha}{2})} \int_0^t (t-s)^{\alpha-\frac{\alpha}{2}-1} \sup_{0 \leq s \leq 1} |(\varphi_1(s) + \varphi_2(s))| \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds \\ &\quad + \frac{1}{\Gamma(\alpha+\sigma-\frac{\alpha}{2})} \int_0^t (t-s)^{\alpha+\sigma-\frac{\alpha}{2}-1} \left| f_1\left(s, D^{\frac{\alpha}{2}}x(s), y(s)\right) \right| ds + |x_0^*|. \end{aligned} \quad (53)$$

By (H3), we have

$$\frac{1}{\mu} \left| D^{\frac{\alpha}{2}}x(t) \right| \leq \sup_{t \in J} m_1(t) \left[\frac{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty}{\Gamma(\frac{\alpha}{2}+1)} + \frac{1}{\Gamma(\frac{\alpha}{2}+\sigma+1)} \right] + |x_0^*|. \quad (54)$$

Therefore,

$$\frac{1}{\mu} \left| D^{\frac{\alpha}{2}}x(t) \right| \leq M_1 \theta_2 + |x_0^*|. \quad (55)$$

Thus,

$$\left| D^{\frac{\alpha}{2}} x(t) \right| \leq \mu M_1 \theta_2 + \mu |x_0^*|. \quad (56)$$

From (51) and (55), we get

$$\|x\|_X \leq \mu M_1 (\theta_1 + \theta_2) + \mu |x_0^*|. \quad (57)$$

Analogously, we can obtain

$$\|y\|_Y \leq \mu M_2 (\theta_3 + \theta_4) + \mu |y_0^*|. \quad (58)$$

It follows from (56) and (57) that

$$\|(x, y)\|_{X \times Y} \leq \mu [M_1 (\theta_1 + \theta_2) + M_2 (\theta_3 + \theta_4) + |x_0^*| + |y_0^*|]. \quad (59)$$

Hence,

$$\|T(x, y)\|_{X \times Y} < \infty. \quad (60)$$

This shows that the set Ω is bounded.

Thanks to [1*], [2*] and [3*], we deduce that T has at least one fixed point; which is a solution of the problem (1).

Corollary 8. *Suppose there exist two constants $k_1 > 0$ and $k_2 > 0$ such that for all $t \in J, (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,*

$$\begin{aligned} |f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| &\leq k_1 (|x_2 - x_1| + |y_2 - y_1|), \\ |f_2(t, x_1, y_1) - f_2(t, x_2, y_2)| &\leq k_2 (|x_2 - x_1| + |y_2 - y_1|), \end{aligned}$$

and let

$$k_1 (\theta_1 + \theta_2) + k_2 (\theta_3 + \theta_4) < 1. \quad (61)$$

Then, the integro-differential system (1) has a unique solution on J .

Corollary 9. *Suppose that there exist two positive constants L_1, L_2 , such that $|f_1(t, x, y)| \leq L_1, |f_2(t, x, y)| \leq L_2; t \in J, x, y \in \mathbb{R}$. Then, the problem (1) has at least a solution on J .*

4. EXAMPLES

To illustrate our main results, we treat the following examples.

Example 1. *Let us consider the following system:*

$$\left\{ \begin{array}{l} D^{\frac{1}{2}}x(t) = \left(\frac{\ln(1+t)}{17(1+t^2)} + \frac{e^{-\pi t}}{20\sqrt{1+\pi t^2}} \right) \left(\frac{\sqrt{\pi} \cos(\pi t) (|D^{\frac{1}{4}}x(t)| + |y(t)|)}{16(t+2)^2 (2 + |D^{\frac{1}{4}}x(t)| + |y(t)|)} \right) \\ \quad + \int_0^1 \frac{(t-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \left(\frac{\sqrt{\pi} \cos(\pi s) (|D^{\frac{1}{4}}x(s)| + |y(s)|)}{16(s+2)^2 (2 + |D^{\frac{1}{4}}x(s)| + |y(s)|)} \right) ds, t \in [0, 1], \\ D^{\frac{1}{3}}y(t) = \left(\frac{\ln(1+t)}{17(1+t^2)} - \frac{e^{-\pi t}}{20\sqrt{1+\pi t^2}} \right) \left(\frac{|x(t)| + |D^{\frac{1}{6}}y(t)|}{(25\sqrt{\pi} + e^{-\pi t})(1 + |x(t)| + |y(t)|)} \right) \\ \quad + \int_0^1 \frac{(t-s)^{\frac{1}{5}}}{\Gamma(\frac{6}{5})} \left(\frac{|x(s)| + |D^{\frac{1}{6}}y(s)|}{(25\sqrt{\pi} + e^{-\pi s})(1 + |x(s)| + |y(s)|)} \right) ds, t \in [0, 1], \\ x(0) = 2, y(0) = 3. \end{array} \right. \quad (62)$$

For this example, we have

$$f_1(t, x, y) = \frac{\sqrt{\pi} \cos(\pi t) (|x| + |y|)}{16(t+2)^2 (2 + |x| + |y|)}, t \in [0, 1], x, y \in \mathbb{R},$$

$$f_2(t, x, y) = \frac{|x| + |y|}{(25\sqrt{\pi} + e^{-\pi t})(1 + |x| + |y|)}, t \in [0, 1], x, y \in \mathbb{R}$$

and

$$\varphi_1(t) = \frac{\ln(1+t)}{17(1+t^2)}, \varphi_2(t) = \frac{e^{-\pi t}}{20\sqrt{1+\pi t^2}}.$$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, t \in [0, 1]$, we can write:

$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| \leq \frac{\sqrt{\pi} \cos(\pi t)}{16(t+2)^2} |x_1 - x_2| + \frac{\sqrt{\pi} \cos(\pi t)}{16(t+2)^2} |y_1 - y_2|,$$

$$|f_2(t, x, y) - f_2(t, x_1, y_1)| \leq \frac{1}{25\sqrt{\pi} + e^{-\pi t}} |x_1 - x_2| + \frac{1}{25\sqrt{\pi} + e^{-\pi t}} |y_1 - y_2|.$$

So, we can take

$$a_1(t) = b_1(t) = \frac{\sqrt{\pi} \cos(\pi t)}{16(t+2)^2}, a_2(t) = b_2(t) = \frac{1}{25\sqrt{\pi} + e^{-\pi t}}.$$

It follows then that

$$\omega_1 = \sup_{t \in [0,1]} a_1(t) = \sup_{t \in [0,1]} b_1(t) = \frac{\sqrt{\pi}}{64} = \omega_2,$$

$$\varpi_1 = \sup_{t \in [0,1]} a_2(t) = \sup_{t \in [0,1]} b_2(t) = \frac{1}{25\sqrt{\pi} + 1} = \varpi_2,$$

$$\|\varphi_1\|_{\infty} = 0,040773, \|\varphi_2\|_{\infty} = 0,024569,$$

$$\theta_1 = 0,57373, \theta_2 = 0,69383, \theta_3 = 0,99713, \theta_4 = 0,89293.$$

It is clear that

$$(\theta_1 + \theta_2)(\omega_1 + \omega_2) + (\theta_3 + \theta_4)(\varpi_1 + \varpi_2) = 0,0701191 + 0,0834461 = 0,1535652 < 1.$$

Thanks to Theorem 8, the system (61) has a unique solution on $[0, 1]$.

Example 2. As a second illustrative example, let us take

$$\left\{ \begin{array}{l} D^{\frac{6}{11}}x(t) = \left(\frac{e^{-t^2}}{16(\sqrt{\pi} + e^{-\pi t})} + \frac{\sqrt{\pi} \sinh t}{\sqrt{12\pi + t^2}} \right) \left(\frac{\cos^2(D^{\frac{3}{11}}x(t))}{16+t+t^2} + \frac{e^{-\pi t^2}|y(t)|}{(32\sqrt{\pi} + e^{-\pi t})(1+|y(t)|)} + \frac{1}{2} \right) \\ \quad + \int_0^t \frac{(t-s)^{\sqrt{13}-1}}{\Gamma(\sqrt{13})} \left(\frac{\cos^2(D^{\frac{3}{11}}x(s))}{16+s+s^2} + \frac{e^{-\pi s^2}|y(s)|}{(32\sqrt{\pi} + e^{-\pi s})(1+|y(s)|)} + \frac{1}{2} \right) ds = 0, t \in [0, 1], \\ D^{\frac{4}{7}}y(t) = \left(\frac{e^{-t^2}}{16(\sqrt{\pi} + e^{-\pi t})} - \frac{\sqrt{\pi} \sinh t}{\sqrt{12\pi + t^2}} \right) \left(\frac{|x(t)|}{(15 + \sqrt{\pi}e^t)(2+|x(t)|)} + 1 + \frac{\sin(2\pi D^{\frac{2}{7}}y(t))}{16\pi(t+2)^2} \right) \\ \quad + \int_0^t \frac{(t-s)^{\frac{2}{5}}}{\Gamma(\frac{7}{5})} \left(\frac{|x(s)|}{(15 + \sqrt{\pi}e^s)(2+|x(s)|)} + 1 + \frac{\sin(2\pi D^{\frac{2}{7}}y(s))}{16\pi(s+2)^2} \right) ds = 0, t \in [0, 1], \\ x(0) = \sqrt{5}, y(0) = \sqrt{3}. \end{array} \right. \quad (63)$$

We have

$$f_1(t, x, y) = \frac{\cos^2(D^{\frac{3}{11}}x(t))}{16+t+t^2} + \frac{e^{-\pi t^2}|y(t)|}{(32\sqrt{\pi} + e^{-\pi t})(1+|y(t)|)} + \frac{1}{2}, t \in [0, 1], x, y \in \mathbb{R},$$

$$f_2(t, x, y) = \frac{|x(t)|}{(15 + \sqrt{\pi}e^t)(2+|x(t)|)} + 1 + \frac{\sin(2\pi D^{\frac{2}{7}}y(t))}{16\pi(t+2)^2}, t \in [0, 1], x, y \in \mathbb{R},$$

and

$$\varphi_1(t) = \frac{e^{-t^2}}{16(\sqrt{\pi} + e^{-\pi t})}, \varphi_2(t) = \frac{\sqrt{\pi} \sinh t}{\sqrt{12\pi + t^2}},$$

For $(x, y), (x_1, y_1) \in \mathbb{R}^2, t \in [0, 1]$, we have

$$|f_1(t, x, y) - f_1(t, x_1, y_1)| \leq \frac{1}{16+t+t^2}|x-x_1| + \frac{e^{-\pi t^2}}{32\sqrt{\pi} + e^{-\pi t}}|y-y_1|,$$

$$|f_2(t, x, y) - f_2(t, x_1, y_1)| \leq \frac{1}{15 + \sqrt{\pi}e^t}|x-x_1| + \frac{1}{8\pi(t+2)^2}|y-y_1|.$$

Hence, we take

$$a_1(t) = \frac{1}{16+t+t^2}, b_1(t) = \frac{e^{-\pi t^2}}{32\sqrt{\pi} + e^{-\pi t}},$$

$$a_2(t) = \frac{1}{15 + \sqrt{\pi}e^t}, b_2(t) = \frac{1}{8\pi(t+2)^2}.$$

It follows then that

$$\begin{aligned}\omega_1 &= \sup_{t \in [0,1]} a_1(t) = \frac{1}{16}, \omega_2 = \sup_{t \in [0,1]} b_1(t) = \frac{1}{32\sqrt{\pi} + 1}, \\ \varpi_1 &= \sup_{t \in [0,1]} a_2(t) = \frac{1}{15 + \sqrt{\pi}}, \varpi_2 = \sup_{t \in [0,1]} b_2(t) = \frac{1}{32\pi}, \\ \|\varphi_1\|_\infty &= \frac{1}{16(\sqrt{\pi} + 1)}, \|\varphi_2\|_\infty = \frac{1,1752\sqrt{\pi}}{\sqrt{12\pi + 1}}, \\ \theta_1 &= 0, \theta_2 = 0, \theta_3 = 1, \theta_4 = 1, \theta_5 = 1, \theta_6 = 1, \theta_7 = 1, \theta_8 = 1, \theta_9 = 1, \theta_{10} = 1, \theta_{11} = 1, \theta_{12} = 1, \theta_{13} = 1, \theta_{14} = 1, \theta_{15} = 1, \theta_{16} = 1, \theta_{17} = 1, \theta_{18} = 1, \theta_{19} = 1, \theta_{20} = 1, \theta_{21} = 1, \theta_{22} = 1, \theta_{23} = 1, \theta_{24} = 1, \theta_{25} = 1, \theta_{26} = 1, \theta_{27} = 1, \theta_{28} = 1, \theta_{29} = 1, \theta_{30} = 1, \theta_{31} = 1, \theta_{32} = 1, \theta_{33} = 1, \theta_{34} = 1, \theta_{35} = 1, \theta_{36} = 1, \theta_{37} = 1, \theta_{38} = 1, \theta_{39} = 1, \theta_{40} = 1, \theta_{41} = 1, \theta_{42} = 1, \theta_{43} = 1, \theta_{44} = 1, \theta_{45} = 1, \theta_{46} = 1, \theta_{47} = 1, \theta_{48} = 1, \theta_{49} = 1, \theta_{50} = 1, \theta_{51} = 1, \theta_{52} = 1, \theta_{53} = 1, \theta_{54} = 1, \theta_{55} = 1, \theta_{56} = 1, \theta_{57} = 1, \theta_{58} = 1, \theta_{59} = 1, \theta_{60} = 1, \theta_{61} = 1, \theta_{62} = 1, \theta_{63} = 1, \theta_{64} = 1, \theta_{65} = 1, \theta_{66} = 1, \theta_{67} = 1, \theta_{68} = 1, \theta_{69} = 1, \theta_{70} = 1, \theta_{71} = 1, \theta_{72} = 1, \theta_{73} = 1, \theta_{74} = 1, \theta_{75} = 1, \theta_{76} = 1, \theta_{77} = 1, \theta_{78} = 1, \theta_{79} = 1, \theta_{80} = 1, \theta_{81} = 1, \theta_{82} = 1, \theta_{83} = 1, \theta_{84} = 1, \theta_{85} = 1, \theta_{86} = 1, \theta_{87} = 1, \theta_{88} = 1, \theta_{89} = 1, \theta_{90} = 1, \theta_{91} = 1, \theta_{92} = 1, \theta_{93} = 1, \theta_{94} = 1, \theta_{95} = 1, \theta_{96} = 1, \theta_{97} = 1, \theta_{98} = 1, \theta_{99} = 1, \theta_{100} = 1.\end{aligned}$$

Therefore,

$$(\theta_1 + \theta_2)(\omega_1 + \omega_2) + (\theta_3 + \theta_4)(\varpi_1 + \varpi_2) = 0,1164702 + 0,1770020 = 0,2934722.$$

Thanks to Theorem 8, we can state that the fractional system (62) has a unique solution on $[0, 1]$.

Example 3. *Our third example is the following:*

$$\left\{ \begin{aligned} D^{\frac{10}{13}} x(t) &= \left(\frac{\cosh(\pi t + 1)}{18(t^2 + 2)^2} + \frac{e^{-t^2}}{\pi e^t + 15} \right) \frac{\sinh(e^{t^2})}{21\sqrt{\pi e^{t^2} + 7}} \sin \left(D^{\frac{5}{13}} x(t) + y(t) \right) + \\ &+ \int_0^t \frac{(t-s)^{\sqrt{21}-1}}{\Gamma(\sqrt{21})} \frac{\sinh(e^{s^2})}{21\sqrt{\pi e^{s^2} + 7}} \sin \left(D^{\frac{5}{13}} x(s) + y(s) \right) ds, t \in [0, 1], \\ D^{\frac{8}{15}} y(t) &= \left(\frac{\cosh(\pi t + 1)}{18(t^2 + 2)^2} - \frac{e^{-t^2}}{\pi e^t + 15} \right) \frac{e^{-\pi t}}{t^2 + 20\pi} \cos \left(x(t) + D^{\frac{4}{15}} y(t) \right) \\ &+ \int_0^t \frac{(t-s)^{\sqrt{17}-1}}{\Gamma(\sqrt{17})} \frac{e^{-\pi s}}{s^2 + 20\pi} \cos \left(x(s) + D^{\frac{4}{15}} y(s) \right) ds, t \in [0, 1], \\ x(0) &= \frac{3}{2}, y(0) = \sqrt{5} \end{aligned} \right. \quad (64)$$

We have

$$\begin{aligned} f_1(t, x, y) &= \frac{\sinh(e^{t^2})}{21\sqrt{\pi e^{t^2} + 7}} \sin(x + y), t \in [0, 1], x, y \in \mathbb{R}, \\ f_2(t, x, y) &= \frac{e^{-\pi t}}{t^2 + 20\pi} \cos(x + y), t \in [0, 1], x, y \in \mathbb{R}. \end{aligned}$$

For $x, y \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$|f_1(t, x, y)| \leq \frac{\sinh(e^{t^2})}{21\sqrt{\pi e^{t^2} + 7}}, |f_2(t, x, y)| \leq \frac{e^{-\pi t}}{t^2 + 20\pi}.$$

Hence,

$$m_1(t) = \frac{\sinh(e^{t^2})}{21\sqrt{\pi e^{t^2} + 7}}, m_2(t) = \frac{e^{-\pi t}}{t^2 + 20\pi},$$

and then,

$$M_1 = \sup_{t \in [0,1]} m_1(t) = 0,01757, M_2 = \sup_{t \in [0,1]} m_2(t) = 0,01592.$$

By Theorem 9, the system (63) has at least one solution on $[0, 1]$.

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