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NUMERICAL SOLUTION OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS USING ALPERT MULTIWAVELETS

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Abstract. A numerical technique is presented for the solution of onlinear ordinary differential equations. This method uses Alpert multiwavelet system. The orthonormality and high vanishing moment properties of this system result in efficient and accurate solutions. Finally, numerical results for some test problems with known solutions are presented and the absolute errors are compared with the errors resulting from B-spline bases and Flatlet multiwavelet.

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1. Introduction

Recently, scalar wavelets are used widely which are generated by one scaling function. But, one can imagine a situation when there is more than one scaling function. This leads to the multiwavelets. Multiwavelets are revealed to possess several advantages with respect to scalar wavelets. The reason of their success is due to the fact that, unlike scalar wavelets, multiwavelets can be constructed with several simultaneous properties, such as orthogonality, symmetry, high-order vanishing moments and the simple structure, etc. Multiwavelets are useful both in theory and in applications such as signal and image processing [9, 10, 8], numerical solution of ODE, PDE and IE [2, 1, 4, 5, 3, 6, 7]. Sparse representation of differential and integral operators due to moments of the simple functions is another property of multiwavelets [5, 11, 1]. The use of operator modelling converts differential equations to systems of algebraic equations. Alpert multiwavelet system with multiplicity r consists of a pair of r multiscaling functions and a corresponding pair of r multiwavelets.

In this paper, we use Alpert multiwavelets with multiplicity r. Also we derive an algorithm to compute the operational matrix of the integral for solving ordinary differential equations of the general form

$$y''(x) = f(x, y(x)), \qquad x \in [0, 1], \tag{1}$$

$$y'(0) = y_0, \quad y'(1) = y_1. \tag{2}$$

Here f, is a known function, y_0 and y_1 are given real numbers and y is the unknown function to be found.

The existence of solution of Equation 1 with Neumann boundary conditions is studied in [12] using the quasi-linearization method. Also wavelet method is used by many papers [13, 14]. For this purpose, different approaches such as the finite-element method, boundary element method, Galerkin and collocation methods are used. In this work, the functions are approximated by Alpert multiwavelets. Then these multiwavelets are used to obtain the coefficients of the expansions.

2. Alpert multiwavelet systems

2.1. Multiresolution analysis

For functions $\phi^m \in L^2(R)$, $m = 0, \ldots, r-1$, let a reference subspace or sample space V_0 be generated as the L^2-closure of the linear span of the integer translates of ϕ^m, namely:

$$V_0 = clos_{L^2} \langle \phi^m(. - k) : k \in Z \rangle, \qquad m = 0, \ldots, r-1,$$

and consider other subspace

$$V_j = clos_{L^2} \left\langle \phi^m_{j,k} : k \in Z \right\rangle, \qquad j \in Z, m = 0, \ldots, r-1,$$

where $\phi^m_{j,k} = \phi^m(2^j x - k)$, $j, k \in Z$, $m = 0, \ldots, r-1$.

Definition 1. *[16]: Functions $\phi^m \in L^2(R)$, is said to generate a multiresolution analysis (MRA) if they generate a nested sequence of closed subspaces V_j that satisfy*

$$\begin{cases} i) & \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, \\ ii) & clos_{L^2}\left(\bigcup_{j\in Z} V_j\right) = L^2(R), \\ iii) & \bigcap_{j \in Z} V_j = 0, \\ iv) & f(x) \in V_j \Longleftrightarrow f(x + 2^{-j}) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}, \\ v) & \{\phi^m(. - k)\}_{k\in Z}, \qquad \text{form a Riesz basis of } V_0. \end{cases} \tag{3}$$

If ϕ^m generate an MRA, then ϕ^m are called scaling functions. In case the different integer translate of ϕ^m are orthogonal and where is with respect to the standard inner product $< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$ for two functions in $L^2(R)$), denoted by $\phi^m(. - k) \perp \phi^{\tilde{m}}(. - \tilde{k})$ for $m \neq \tilde{m}, k \neq \tilde{k}$, the scaling functions are called an orthogonal scaling functions.

As the subspaces V_j are nested, there exist complementary orthogonal subspaces W_j such that

$$V_{j+1} = V_j \bigoplus W_j, \qquad j \in Z,$$

here and in the following $\bigoplus$ denotes orthogonal sums.

This give rise to an orthogonal decomposition of $L^2(R)$, namely:

$$L^2(R) = \bigoplus_{j \in Z} W_j.$$

Definition 2. *[16]: Functions $\psi^m \in L^2(R)$ are called wavelets, if they generate the complementary orthogonal subspaces W_j of an MRA, i.e.,*

$$W_j = clos_{L^2} < \psi^m_{j,k}, k \in Z >, \qquad j \in Z, m = 0, \dots, r-1,$$

where $\psi^m_{j,k} = \psi^m(2^j x - k)$, $j, k \in Z$.

If, $\psi^m_{j,k} \perp \psi^{\tilde{m}}_{\tilde{j},\tilde{k}}$ for $j \neq \tilde{j}$, $m \neq \tilde{m}$ and $k \neq \tilde{k}$ if $< 2^{j/2}\psi^m_{j,k}, 2^{\tilde{j}/2}\psi^{\tilde{m}}_{\tilde{j},\tilde{k}} >= \delta_{j,\tilde{j}}\delta_{k,\tilde{k}}\delta_{m,\tilde{m}}$ then ψ^m are called orthonormal wavelets.

Now we define Alpert scaling functions and its corresponding multiwavelets according to above MRA.

2.2. Construction of Scaling Functions

Suppose P_r is the Legendre polynomial of order r and r is any fixed nonnegative integer number and let τ_k for $k = 0, ..., r-1$ denote the roots of P_r. The interpolating scaling functions (ISF) are given by [4, 6, 3]

$$\phi^k(t) = \begin{cases} \sqrt{\frac{2}{\omega_k}} L_k(2t-1) & t \in [0,1] \\ 0 & otherwise \end{cases}$$

Where ω_k, $k = 0, ..., r-1$ are the Gauss-Legendre quadrature weights

$$\omega_k = \frac{2}{r\acute{P}_r(\tau_k)P_{r-1}(\tau_k)}$$

and $L_k(t)$, $k = 0, ..., r-1$ are the lagrange interpolating polynomials [6]

$$L_k(t) = \prod_{i=0, i \neq k}^{r-1} \left(\frac{t - \tau_i}{\tau_k - \tau_i} \right)$$

that they have characterized by Kronecker property $L_k(\tau_i) = \delta_{ik}$ where

$$\delta_{ki} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

We can expand any polynomial g of degree less than r with the function $\phi^0, ..., \phi^{r-1}$ that they formed an orthonormal basis on $[0, 1)$

$$g(t) = \sum_{k=0}^{r-1} d_k \phi^k(t)$$

where the coefficients are given by

$$d_k = \sqrt{\frac{\omega_k}{2}} g\big(\hat{\tau_k}\big), \qquad k = 0, ..., r-1$$

and

$$\hat{\tau_k} = \frac{\tau_k + 1}{2}.$$

Let $\phi^k_{Jl}(t)$, $k = 0, ..., r-1$, $l = 0, ..., 2^J - 1$ be obtained form $\phi^k(t)$ by dilation and translation

$$\phi^k_{Jl}(t) = 2^{(J/2)} \phi^k(2^J t - l) \tag{4}$$

where J is any fixed nonnegative integer number.

Note that we have the following orthonormality relation

$$\int_0^1 \phi^k_{Jl}(t) \phi^{\acute{k}}_{J\acute{l}}(t) dt = \delta_{l\acute{l}} \delta_{k\acute{k}}$$

$$k, \acute{k} = 0, ..., r-1$$

$$l, \acute{l} = 0, ..., 2^J - 1$$

2.3. Construction of Wavelets

The two-scale relations for the r-th order Alpert multiwavelets are in the form [2]:

$$\psi^i(x) = \sum_{j=0}^{r-1} h_{i,j} \phi^j(2x) + \sum_{j=0}^{r-1} h_{i,r+j+1} \phi^j(2x - 1). \tag{5}$$

As we have $2r^2$ unknown coefficients $\{h\}$ in (5), we use the following $2r(r-1)$ vanishing moment conditions and $2r$ orthonormal conditions to determine them.

1. Vanishing moments

$$\int_0^1 \psi^i(x)x^j = 0, \qquad \text{for } i = 0, 1, ..., r-1 \; j = 0, 1, ..., i+r-1. \tag{6}$$

2. Orthonormality

$$\int_0^1 \psi^i(x)\psi^j(x) = \delta_{i,j}, \qquad for \; i, j = 0, 1, ..., r-1. \tag{7}$$

2.4. Two scale relations

The representation of two scale relations is proposed for scaling functions and wavelets as

$$\phi^k(x) = \sum_{j=0}^{r-1} g^0_{k+1,j+1}\phi^j(2x) + g^1_{k+1,j+1}\phi^j(2x-1),$$

$$\psi^k(x) = \sum_{j=0}^{r-1} h^0_{k+1,j+1}\phi^j(2x) + h^1_{k+1,j+1}\phi^j(2x-1).$$

By using the function $\phi^k(x)$ and $\psi^k(x)$ for $k = 0, \ldots, r-1$, we construct the filter coefficients $g^l_{i,j}$ and $h^l_{i,j}$, $l = 0, 1$. In these representation of two scale relation, four matrices $(r \times r)$ is used to show the filter coefficients $g^l_{i,j}$ and $h^l_{i,j}$, $l = 0, 1$ as

$$G^0 = \begin{bmatrix} g^0_{11} & \cdots & g^0_{1r} \\ \vdots & & \vdots \\ g^0_{r1} & \cdots & g^0_{rr} \end{bmatrix}, G^1 = \begin{bmatrix} g^1_{11} & \cdots & g^1_{1r} \\ \vdots & & \vdots \\ g^1_{r1} & \cdots & g^1_{rr} \end{bmatrix},$$

$$H^0 = \begin{bmatrix} h^0_{11} & \cdots & h^0_{1r} \\ \vdots & & \vdots \\ h^0_{r1} & \cdots & h^0_{rr} \end{bmatrix}, H^1 = \begin{bmatrix} h^1_{11} & \cdots & h^1_{1r} \\ \vdots & & \vdots \\ h^1_{r1} & \cdots & h^1_{rr} \end{bmatrix},$$

The matrices G^0 and G^1 consist of the filter coefficients of two scale relation for scaling functions and their components are given by following equations

$$g^0_{k,\acute{k}} = \sqrt{w_{\acute{k}}}\phi^k(\frac{\hat{\tau}_{\acute{k}}}{2}), \tag{8}$$

$$g^1_{k,\acute{k}} = \sqrt{w_{\acute{k}}}\phi^k(\frac{\hat{\tau}_{\acute{k}}+1}{2}). \tag{9}$$

These equations are obtained by using the interpolating property of scaling functions.

In general, the two scale relation for the neighbour scales J and J+1 is given by the following matrix form

$$\Phi^r_J(x) = G_J \Phi^r_{J+1}(x), \tag{10}$$

where G_J define the transform matrix between two neighbour scales for scaling functions and is getting by

$$G_J = \begin{bmatrix} G & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G \end{bmatrix}_{r2^J, r2^{J+1}}, \tag{11}$$

where $\Phi^r_J(x)$ consist of $r2^J$ bases for V^r_J and $G = [G^0 G^1]$. We note that the filter coefficients of two scale relation for wavelets is constructed in subsection 2.3.

Hence the wavelet transform matrix [15, 5] between Ψ^r_J and Φ^r_J is obtained as

$$\Psi^r_J = T_J \Phi^r_J, \tag{12}$$

where T_J is a $(r2^J, r2^J)$ matrix which are obtained by the following scheme. Suppose that $H = [H^0 H^1]$ and

$$H_J = \begin{bmatrix} H & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H \end{bmatrix}_{r2^J, r2^{J+1}}, \tag{13}$$

By using these matrices, we get

$$T_J = \begin{bmatrix} \frac{1}{2^J}(G_0 \times G_1 \times \ldots \times G_{J-1}) \\ \frac{1}{2^J}(H_0 \times G_1 \times \ldots \times G_{J-1}) \\ \frac{1}{2^{J-1}}(H_1 \times G_2 \times \ldots \times G_{J-1}) \\ \vdots \\ \frac{1}{2^2}(H_{J-2} \times G_{J-1}) \\ \frac{1}{2}H_{J-1} \end{bmatrix}. \tag{14}$$

2.5. Function Approximation

It can be verified that $V_j \oplus W_j = V_{j+1}$, thus we can write $V_j = V_0 \oplus (\oplus_{i=0}^{j-1} W_i)$ and we have two kind of basis sets for $J \in N$

$$\Phi^r_J(x) = \left[\phi^0_{J,0}(x), ..., \phi^{r-1}_{J,0}(x), | \cdots, \phi^0_{J,(2^J-1)}(x), ..., \phi^{r-1}_{J,(2^J-1)}(x)\right]^T, \tag{15}$$

$$\Psi^r_J(x) = \Big[\phi^0_{0,0}(x), ..., \phi^{r-1}_{0,0}(x), |\psi^0_{0,0}(x), \ldots, \psi^{r-1}_{0,0}(x)|, \tag{16}$$

$$\ldots|\psi^0_{J-1,0}(x),\ldots,\psi^{r-1}_{J-1,0}(x)|,\ldots,\psi^0_{J-1,2^{J-1}-1}(x),\ldots,\psi^{r-1}_{J-1,2^{J-1}-1}(x)\Big]^T.$$

Now any function $f(x)$ on $[0,1]$ can be approximated using scaling functions as

$$f(x)\approx P^r_J f=\sum_{k=0}^{r-1}\sum_{l=0}^{2^J-1} c^k_{J,l}\phi^k_{J,l}(x)=C^T\Phi^r_J(x), \tag{17}$$

and the corresponding wavelet functions as

$$f(x)\approx P^r_J f=\sum_{k=0}^{r-1}\left\{c^k_{0,0}\phi^k_{0,0}(x)+\sum_{j=0}^{J-1}\sum_{l=0}^{2^j-1} d^k_{j,l}\psi^k_{j,l}(x)\right\}=D^T\Psi^r_J(x), \tag{18}$$

where

$$c^k_{J,l}=\int_0^1 f(x)\phi^k_{J,l}(x)dx=\int_{h_l}^{h_{l+1}} f(t)\phi^k_{Jl}(t)dt, \tag{19}$$

and

$$h_l=\frac{l}{2^J},\qquad l=0,...,2^J-1.$$

These coefficients may be computed using Gauss-Legendre quadrature [6, 3].

$$c^k_{Jl}=2^{-J/2}\sqrt{\frac{\omega_k}{2}}f(2^{-J}(\hat{\tau}_k+l)),\ k=0,...,r-1,\ l=0,...,2^J-1. \tag{20}$$

Lemma 1. Suppose that the function $f:[0,1]\to R$ is r times continuously differentiable. Then $P^r_J f$ approximates f with mean error bounded as follow [2]:

$$\|P^r_J f-f\|\le 2^{-Jr}\frac{2}{4^r r!}\sup_{x\in[0,1]}|f^{(r)}(x)|,$$

By using Eq. (12), the elements of matrix D in Eq. (18) are obtained as

$$D^T=C^T T_J^{-1}. \tag{21}$$

Where D and C are $(m\times 1)$ vectors with $m=r2^J$ given by

$$D=\Big[c^0_{0,0},...,c^r_{0,0}|d^0_{0,0},...,d^r_{0,0}|...|d^0_{J-1,0},...,d^r_{J-1,0}|,...,d^0_{J-1,2^{J-1}-1},...d^r_{J-1,2^{J-1}-1}\Big]^T, \tag{22}$$

$$C=\Big[c^0_{J,0},...,c^r_{J,0}|...|c^0_{J,2^J-1},...,c^r_{J,2^J-1}\Big]^T. \tag{23}$$

2.6. The Operational Matrix of Integration

The integral of vectors $\Psi_J^r(x)$ and $\Phi_J^r(x)$ can be expressed as

$$\int_0^x \Psi_J^r(t)dt \approx I_\psi \Psi_J^r(x), \tag{24}$$

$$\int_0^x \Phi_J^r(t)dt \approx I_\phi \Phi_J^r(x), \tag{25}$$

where I_ϕ and I_ψ are $(N \times N)$ operational matrices of integration for Alpert scaling functions and multiwavelets respectively. The matrix I_ψ can be obtained by the following process.

Using Eq. (25) we have

$$\int_0^x \Phi_J^r(t)dt \approx \sum_{k'=0}^{r-1} \sum_{l'=0}^{2^J-1} [I_\phi]_{lr+(k+1),l'r+(k'+1)} \phi_{Jl'}^{k'}(x), \tag{26}$$

$$k = 0, \cdots, r-1, \;\; l = 0, \cdots, 2^J - 1.$$

Now we use Eq. (20) to obtain

$$\begin{aligned} [I_\phi]_{lr+(k+1),l'r+(k'+1)} &= 2^{\frac{-J}{2}} \sqrt{\frac{\omega_{k'}}{2}} \int_0^{2^{-J}(\hat{\tau}_{k'}+l')} \phi_{Jl}^k(t)dt \\ &= 2^{\frac{-J}{2}} \sqrt{\frac{\omega_{k'}}{2}} \int_{\frac{l}{2^J}}^{2^{-J}(\hat{\tau}_{k'}+l')} \phi_{Jl}^k(t)dt. \end{aligned} \tag{27}$$

To find the entries of matrix I_ϕ we assume the following three cases.

Case 1: $l' < l$

The support of ϕ_{Jl}^k is $[\frac{l}{2^J}, \frac{l+1}{2^J}]$ and $2^{-J}(\hat{\tau}_{k'} + l') < \frac{l}{2^J}$ Thus we get

$$[I_\phi]_{lr+(k+1),l'r+(k'+1)} = 0. \tag{28}$$

Case 2: $l' = l$

Changing the variable $2^J t - l = \hat{\tau}_k x$ we have

$$[I_\phi]_{lr+(k+1),l'r+(k'+1)} = 2^{\frac{-J}{2}} \sqrt{\frac{\omega_{k'}}{2}} \int_0^{\hat{\tau}_k} \phi_{Jl}^k(t)dt.$$

These coefficients may be computed using the Gauss-Legendre quadrature as

$$[I_\phi]_{lr+(k+1),l'r+(k'+1)} = 2^{-J} \sqrt{\frac{\omega_{k'}}{2}} \hat{\tau}_{k'} \sum_{i=0}^{r-1} \frac{\omega_{k'}}{2} \phi^k(\hat{\tau}_{k'} \hat{\tau}_i). \tag{29}$$

Case 3: $l' > l$

Again the support of ϕ^k_{Jl} is $[\frac{l}{2^J}, \frac{l+1}{2^J}]$ and $2^{-J}(\hat{\tau}_{k'} + l') > \frac{l'}{2^J} > l + \frac{l+1}{2^J}$ Thus we obtain

$$
\begin{aligned}
[I_\phi]_{lr+(k+1),l'r+(k'+1)} &= 2^{\frac{-J}{2}}\sqrt{\frac{\omega_{k'}}{2}}\int_{\frac{l}{2^J}}^{\frac{l+1}{2^J}} \phi^k_{Jl}(t)dt \\
&= 2^{\frac{-J}{2}}\sqrt{\frac{\omega_{k'}}{2}}\int_0^1 \phi^k(t)dt = 2^{-J}\sqrt{\frac{\omega_k}{2}}\sqrt{\frac{\omega_{k'}}{2}}. \qquad (30)
\end{aligned}
$$

Now we use these three cases to obtain the operational matrix of integration as

$$
I_\phi = 2^{-J}\begin{bmatrix} M & P & \cdots & \cdots & P & P \\ & M & P & \cdots & P & P \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & M & P \\ & & & & & M \end{bmatrix},
$$

where M and P are $r \times r$ matrices which can be obtained by the following equations:

$$
[M]_{k+1,k'+1} = \sqrt{\frac{\omega_{k'}}{2}}\hat{\tau}_{k'}\sum_{i=0}^{r-1}\frac{\omega_{k'}}{2}\phi^k(\hat{\tau}_{k'}\hat{\tau}_i), \quad k, k' = 0, 1, \ldots, r-1,
$$

$$
[P]_{k+1,k'+1} = \sqrt{\frac{\omega_k}{2}}\sqrt{\frac{\omega_{k'}}{2}}, \quad k, k' = 0, 1, \ldots, r-1.
$$

Using Eqs. (12), (24) and (25) we get

$$
\int_0^x \Psi^r_J(t)dt = T_J\int_0^x \Phi^r_J(t)dt = T_J I_\phi \Phi^r_J(x) = T_J I_\phi T_J^{-1}\Psi^r_J(x), \qquad (31)
$$

comparing Eqs. (24) and (31) we get

$$
I_\psi = T_J I_\phi T_J^{-1}. \qquad (32)
$$

3. Description of Numerical Method

In this section, we solve nonlinear ordinary differential equation of the form in (1) with conditions (2), by using Alpert multiwavelets.

For this purpose, Let me to suppose that

$$
y''(x) = Y^T\Psi_J(x), \qquad (33)
$$

by integrating from both sides of Eq.(33) and by using (24) we get

$$y'(x) - y'(0) = Y^T \int_0^x \Psi_J(t)dt = Y^T I_\psi \Psi_J(x). \tag{34}$$

By using first condition of (2), we obtain

$$y'(x) = Y^T I_\Psi \Psi_J(x) + y_0. \tag{35}$$

Again by integrating from both sides of Eq. (35), we get

$$y(x) - y(0) = Y^T I_\psi^2 \Psi_J(x) + y_0 x, \tag{36}$$

suppose that

$$y(0) = \alpha,$$

thus we get

$$y(t) = Y^T I_\psi^2 \Psi_J(x) + y_0 x + \alpha. \tag{37}$$

Now by using Eq. (37), we let

$$z(x) = f(x, y(x)). \tag{38}$$

We expand $y_0 x$, α and $z(x)$ by using interpolating scaling functions as

$$y_0 x = A^T \Phi_J(x), \quad \alpha = B^T \Phi_J(x), \quad z(x) = Z^T \Phi_J(x). \tag{39}$$

We use again Eq. (12) to convert the vectors A, B and Z to the wavelet space. Thus we have

$$y_0 x = A^T T_J^{-1} \Psi_J(x), \quad \alpha = B^T T_J^{-1} \Psi_J(x), \quad z(x) = Z^T T_J^{-1} \Psi_J(x). \tag{40}$$

Applying (33),(37) and (40) in (1), we get

$$Y^T \Psi_J(x) = Z^T T_J^{-1} \Psi_J(x). \tag{41}$$

Multiplying (41) By $\Psi_J^T(t)$ and integrating from 0 to 1, we have

$$Y^T - Z^T T_J^{-1} = 0. \tag{42}$$

Now we have N algebraic equations with $N + 1$ unknowns for vector Y and α. But one of the conditions in Eq. (2) remained without using. We use Eq. (35) and second condition of (2) to obtain the $N + 1$th equation. Thus we can solve this system of equations and we obtain unknown members.

4. Test problems

In this section we give some computational results of numerical experiments with methods based on preceding section, to support our theoretical discussion. To show the efficiency of the present method for our problems in comparison with the exact solution, we report absolute values of errors of the solution at a selection of chosen points. From the tables, we can observe the convergence of numerical solutions as J and r are increased. Furthermore, the main advantages of the method are its simplicity and small computations costs which result from the sparsity of the associated matrices and also small number of the coefficients of wavelet representations.

Example 1. *Consider the following equation [17, 11]*

$$y''(x) = (4x^2 - 2)y(x),$$
$$y'(0) = 0, \quad y'(1) = -\frac{2}{e}. \tag{43}$$

The analytical solution is given in [17, 11] as

$$u(x,t) = e^{-x^2},$$

Table 1 *consist absolute values of errors of example* 1 *for* $n = 1, 2$. *Also we show that the methods represented in this paper AWGM* (*Alpert Multiwavelet Galerkin method*) *is the better than the method used in [17, 11]. Also the error function for* $r = 4$, $J = 2$ *is shown in Figure* 1.

Table 1. Absolute values of error for Example 1.

t	AWGM r=4, J=3	AWGM $r=5, J=2$	AWGM $r=5, J=3$	[11] $r=5, J=2$	[11] $r=6, J=3$
0.0	1.39×10^{-6}	4.08×10^{-6}	1.13×10^{-7}	2.4×10^{-4}	2.0×10^{-7}
0.1	1.01×10^{-6}	3.74×10^{-6}	1.05×10^{-7}	2.4×10^{-4}	4.2×10^{-10}
0.2	1.74×10^{-8}	3.53×10^{-6}	9.51×10^{-8}	2.3×10^{-4}	2.0×10^{-7}
0.3	4.58×10^{-8}	3.15×10^{-6}	1.06×10^{-7}	2.2×10^{-4}	3.9×10^{-7}
0.4	4.31×10^{-7}	2.77×10^{-6}	8.49×10^{-8}	2.0×10^{-4}	5.8×10^{-7}
0.5	2.71×10^{-7}	2.75×10^{-6}	9.36×10^{-8}	1.8×10^{-4}	3.5×10^{-7}
0.6	1.61×10^{-7}	2.78×10^{-6}	7.83×10^{-8}	1.7×10^{-4}	7.3×10^{-8}
0.7	3.58×10^{-7}	2.36×10^{-6}	5.76×10^{-8}	1.7×10^{-4}	7.4×10^{-8}
0.8	3.83×10^{-7}	1.89×10^{-6}	5.92×10^{-8}	1.6×10^{-4}	7.8×10^{-8}
0.9	2.97×10^{-7}	1.54×10^{-6}	4.69×10^{-8}	1.5×10^{-4}	7.8×10^{-8}
1.0	1.19×10^{-6}	1.12×10^{-6}	3.72×10^{-8}	1.5×10^{-4}	7.9×10^{-8}

Example 2. *The nonlinear problem, [13]*

$$y''(x) = -2y(x)^3,$$

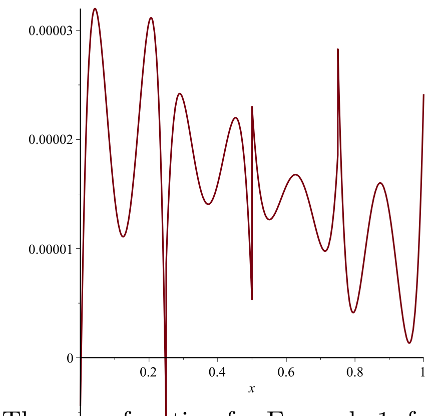

Figure 1: The error function for Example 1, for $r = 5,\ J = 2$.

$$y'(0) = -1, \quad y'(1) = -\frac{1}{4}. \tag{44}$$

has the exact solution $y(x) = 1/(x+1)$. *The absolute values of error in some points are shown in Table* 2.

Table 2. Absolute values of error for Example 2.

t	AWGM r=4, J=2	AWGM $r = 5, J = 2$	[11] $r = 6, J = 2$	[13] B-spline wavelet
0.0	3.32×10^{-5}	1.91×10^{-6}	1.9×10^{-6}	5.6×10^{-6}
0.1	9.80×10^{-6}	4.93×10^{-7}	1.9×10^{-6}	2.6×10^{-5}
0.2	1.17×10^{-5}	3.00×10^{-7}	2.0×10^{-6}	1.7×10^{-5}
0.3	2.84×10^{-6}	1.49×10^{-7}	2.2×10^{-6}	1.6×10^{-5}
0.4	4.25×10^{-6}	2.74×10^{-7}	2.6×10^{-6}	1.4×10^{-5}
0.5	6.37×10^{-6}	1.71×10^{-7}	4.0×10^{-6}	1.2×10^{-5}
0.6	3.88×10^{-6}	5.38×10^{-8}	4.3×10^{-6}	1.0×10^{-5}
0.7	9.57×10^{-7}	9.70×10^{-8}	3.9×10^{-6}	7.2×10^{-6}
0.8	2.54×10^{-6}	1.68×10^{-7}	3.7×10^{-6}	5.3×10^{-6}
0.9	4.56×10^{-6}	2.09×10^{-7}	3.5×10^{-6}	5.5×10^{-6}
1.0	7.10×10^{-6}	2.98×10^{-7}	3.4×10^{-6}	1.6×10^{-6}

Example 3. *Consider the following nonlinear problem, [11]*

$$y''(x) = -e^{-2y(x)},$$

$$y'(0) = 1, \quad y'(1) = \frac{1}{2}. \tag{45}$$

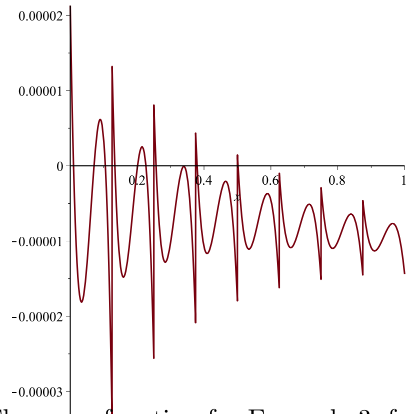

Figure 2: The error function for Example 3, for $r = 3,\ J = 3$.

The exact solution is $y(x) = \ln(x+1)$*. The absolute values of error in some points are shown in Table* 3*. Figure* 2*, shows the error function for* $r = 3,\ J = 3$*.*

Table 3. Absolute values of error for Example 3*.*

t	*AWGM* $r{=}4,\ J{=}2$	*AWGM* $r=4, J=3$	*AWGM* $r=5, J=2$	*[11]* $r=4, J=3$	*[11]* $r=5, J=2$
0.0	1.09×10^{-5}	7.97×10^{-7}	3.62×10^{-7}	8.8×10^{-5}	6.6×10^{-6}
0.1	4.38×10^{-6}	1.73×10^{-7}	4.87×10^{-8}	8.8×10^{-5}	7.7×10^{-6}
0.2	1.53×10^{-6}	2.09×10^{-7}	2.34×10^{-12}	8.9×10^{-5}	8.9×10^{-6}
0.3	6.65×10^{-7}	1.90×10^{-7}	1.03×10^{-7}	9.3×10^{-5}	1.0×10^{-5}
0.4	3.16×10^{-6}	4.12×10^{-8}	1.38×10^{-7}	9.6×10^{-5}	1.1×10^{-5}
0.5	3.86×10^{-6}	1.13×10^{-7}	1.11×10^{-7}	1.1×10^{-5}	1.9×10^{-5}
0.6	3.14×10^{-6}	7.89×10^{-8}	7.74×10^{-8}	1.2×10^{-4}	1.9×10^{-5}
0.7	1.98×10^{-6}	1.73×10^{-7}	9.24×10^{-8}	1.1×10^{-4}	1.8×10^{-5}
0.8	2.56×10^{-6}	1.79×10^{-7}	1.16×10^{-7}	1.1×10^{-4}	1.7×10^{-5}
0.9	3.47×10^{-6}	1.45×10^{-7}	1.30×10^{-7}	1.1×10^{-4}	1.7×10^{-5}
1.0	4.60×10^{-6}	2.45×10^{-7}	1.62×10^{-7}	1.1×10^{-4}	1.6×10^{-5}

Example 4. *The problem, [11]*

$$y''(x) = 2 - 4y(x),$$
$$y'(0) = 0, \qquad y'(1) = \sin(2). \tag{46}$$

has the exact solution $y(x) = \sin(2x)$*. The absolute values of error in some points are shown in Table* 4*.*

Table 4. Absolute values of error for Example 4.

	AWGM	*AWGM*	*AWGM*	*[11]*	*[11]*
t	*r=4, J=2*	$r = 4, J = 3$	$r = 5, J = 2$	$r = 4, J = 3$	$r = 7, J = 1$
0.0	2.10×10^{-6}	6.63×10^{-8}	2.28×10^{-7}	2.8×10^{-5}	4.7×10^{-7}
0.2	2.58×10^{-5}	8.75×10^{-7}	1.06×10^{-7}	2.6×10^{-5}	4.3×10^{-7}
0.4	1.10×10^{-5}	1.15×10^{-6}	3.51×10^{-8}	2.0×10^{-5}	3.2×10^{-7}
0.6	5.61×10^{-6}	6.49×10^{-7}	1.88×10^{-7}	4.5×10^{-5}	1.5×10^{-7}
0.8	2.04×10^{-6}	3.04×10^{-8}	7.91×10^{-8}	4.8×10^{-5}	1.9×10^{-8}
1.0	5.04×10^{-6}	1.59×10^{-7}	5.49×10^{-7}	4.2×10^{-5}	1.6×10^{-7}

Acknowledgements. In this paper we presented the numerical schemes for solving the nonlinear differential equations. This technique is based on the Alpert multiwavelets and Galerkin method. The numerical results given in the previous section demonstrate the accuracy of these schemes. The obtained results showed that this technique can solve the problem effectively. We believe that this method may be applied to more complicated problems. This will hopefully be taken up in our future studies. Also the numerical test problems illustrate that this type of multiwavelets is better than other types.

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