

## ON $h$ -TRICHOTOMY OF LINEAR DISCRETE-TIME SYSTEMS IN BANACH SPACES

IOAN-LUCIAN POPA, MIHAIL MEGAN AND TRAIAN CEAUȘU

**ABSTRACT.** The aim of this paper is to give characterizations of a general concept of trichotomy of time-varying linear systems described by difference equations with noninvertible operators in Banach spaces. This concept contains as particular cases the classical properties of (uniform and nonuniform) exponential trichotomy and polynomial trichotomy. The approach is motivated by two examples.

*2000 Mathematics Subject Classification:* 34D09, 39A06.

*Keywords:*  $h$ -trichotomy, exponential trichotomy, linear discrete-time systems.

### 1. INTRODUCTION

As a natural generalization of exponential dichotomy, exponential trichotomy is one of the most important properties of dynamical systems ( see for eg. [1], [2], [3], [4], [5], [6], [10], [11], [12], [15], [16], [18] and the references therein) playing a crucial role in the study of centers manifolds.

The concepts of trichotomy was identified for the first time in literature in 1976 by R.J. Sacker and G.R. Sell in [17]. They described trichotomy of linear differential systems by linear skew-product flows. The first studies devoted to the trichotomic behavior for differential equations was initiated by S. Elaydi and O. Hajek [8]. Later, for the case of difference equations, an interesting study concerning several trichotomy concepts was considered by S. Elaydi and K. Janglajew in [7].

The importance of exponential trichotomy in the study of dynamical systems is well-established, as we can see for example in [9] where it is proved that this property is a necessary condition for the presence of ergodic solutions of linear differential and difference equations.

The case of a particular concept of nonuniform exponential trichotomy was considered by L. Barreira and C. Valls [5] opening a new direction of study by treating the trichotomy in the context of nonuniform hiperbolicity theory. As we will see later

in this paper, there are linear systems which do not admit a uniform exponential trichotomy which motivates our study of the nonuniform exponential trichotomy.

The aim of this paper is to give some characterizations for a more general concept of nonautonomous discrete dynamical systems in Banach spaces. Thus we obtain variants for  $h$ -trichotomy (in particular for nonuniform exponential trichotomy) of some known results in the theory of uniform exponential trichotomy [14] and nonuniform exponential trichotomy [13]. Some examples are included to illustrate the connections between the uniform and nonuniform trichotomy concepts considered.

## 2. PRELIMINARIES

Let  $X$  be a Banach space and let  $I$  be the identity operator on  $X$ . The norm on  $X$  and on the space  $\mathcal{B}(X)$  of all bounded linear operators on  $X$ , will be denoted by  $\|\cdot\|$ . We also denote by  $\Delta$  the set of all pairs of natural numbers  $(m, n)$  with  $m \geq n$  and  $T = \Delta \times X$ .

Let  $(A_n)$  be a sequence in  $\mathcal{B}(X)$ . We consider the linear discrete-time system

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{N}. \tag{2}$$

The *discrete evolution operator* associated to (2) is

$$\mathcal{A}_m^n := \begin{cases} A_{m-1} \cdots A_n & m > n \\ I & m = n, \end{cases} \text{ for } (m, n) \in \Delta. \tag{1}$$

**Definition 1.** A sequence  $(P_n)$  in  $\mathcal{B}(X)$  is called a *projections sequence on  $X$*  if

$$P_n^2 = P_n, \text{ for all } n \in \mathbb{N}.$$

A *projections sequence  $(P_n)$  with the property*

$$A_n P_n = P_{n+1} A_n$$

for every  $n \in \mathbb{N}$  is called *invariant for the system (2)*.

**Remark 1.** The *projections sequence  $(P_n)$  is invariant for the system (2) if and only if*

$$\mathcal{A}_m^n P_n = P_m \mathcal{A}_m^n, \text{ for all } (m, n) \in \Delta.$$

**Definition 2.** Three *projections sequences  $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(X)$  are called supplementary if*

$$P_n^1 + P_n^2 + P_n^3 = I, \text{ for every } n \in \mathbb{N}; \tag{s_1}$$

$$P_n^i P_n^j = 0, \text{ for all } n \in \mathbb{N} \text{ and all } i, j \in \{1, 2, 3\} \text{ with } i \neq j; \quad (s_2)$$

$$\|P_n^i x + P_n^j x\|^2 = \|P_n^i x\|^2 + \|P_n^j x\|^2, \text{ for all } (n, x) \in \mathbb{N} \times X \quad (s_3)$$

and all  $i, j \in \{1, 2, 3\}$  with  $i \neq j$ , where  $P_n^j = P_j(n)$ .

Let  $h : \mathbb{N} \rightarrow [1, \infty)$  be a nondecreasing sequence of real numbers with  $h(n) = h_n$  for every  $n \in \mathbb{N}$ .

**Definition 3.** We say that the system  $(\mathfrak{A})$  is  $h$ -trichotomic if there exist three supplementary projections sequences  $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(X)$  which are invariant for  $(\mathfrak{A})$  and there are two positive constants  $a, b$  and a sequence of nonnegative real numbers  $(c_n)$  such that

$$h_m^a \|\mathcal{A}_m^n P_n^1 x\| \leq c_n h_n^a \|P_n^1 x\|, \quad (ht_1)$$

$$h_m^a \|P_n^2 x\| \leq c_m h_n^a \|\mathcal{A}_m^n P_n^2 x\|, \quad (ht_2)$$

$$h_n^b \|\mathcal{A}_m^n P_n^3 x\| \leq c_n h_m^b \|P_n^3 x\|, \quad (ht_3)$$

$$h_n^b \|P_n^3 x\| \leq c_m h_m^b \|\mathcal{A}_m^n P_n^3 x\|, \quad (ht_4)$$

for all  $(m, n, x) \in T$ .

In particular case when the sequence  $(c_n)$  is constant, we say that  $(\mathfrak{A})$  is *uniformly  $h$ -trichotomic*.

**Remark 2.** As particular cases of  $h$ -trichotomy we observe that

- (a) if  $h_n = e^n$ , then we recover notion of nonuniform exponential trichotomy and in particular when  $(c_n)$  is constant and  $h_n = e^n$  we obtain the classical property of uniform exponential trichotomy;
- (b) if  $h_n = n + 1$ , then we recover the notion of nonuniform polynomial trichotomy and in particular when  $(c_n)$  is constant and  $h_n = n + 1$  we obtain the property of uniform polynomial trichotomy;
- (c) if  $P_3 = 0$  the we recover the properties of  $h$ -dichotomy, nonuniform exponential dichotomy (for  $h_n = e^n$ ), uniform exponential dichotomy (for  $h_n = e^n$  and  $(c_n)$  constant), nonuniform polynomial dichotomy (when  $h_n = n + 1$ ) and uniform polynomial dichotomy (for  $h_n = n + 1$  and  $(c_n)$  constant).

An example of linear discrete-time system  $(\mathfrak{A})$  which is uniformly  $h$ -trichotomic is given below.

**Example 1.** On  $X = \mathbb{R}^3$  endowed with the norm  $\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}$ , we consider the projections sequences

$$\begin{aligned} P_n^1(x_1, x_2, x_3) &= (x_1 + (n+1)x_2, 0, 0), \\ P_n^2(x_1, x_2, x_3) &= ((n+1)x_2, x_2, 0), \\ P_n^3(x_1, x_2, x_3) &= (0, x_2, x_3). \end{aligned}$$

We have that  $\|P_n^1 x\| = (n+2)\|x\|$ ,  $\|P_n^2 x\| = (n+1)\|x\|$ ,  $\|P_n^3 x\| = \|x\|$ . We consider the system  $(\mathfrak{A})$  defined by the sequence  $(A_n)$  given by

$$A_n = \frac{h_n}{h_{n+1}} P_n^1 + \frac{h_{n+1}}{h_n} P_{n+1}^2 + P_n^3.$$

We have that the discrete evolution operator associated to  $(\mathfrak{A})$  is

$$\mathcal{A}_m^n = \frac{h_n}{h_m} P_n^1 + \frac{h_m}{h_n} P_m^2 + P_n^3,$$

for all  $(m, n) \in \Delta$ . We can see that

$$\begin{aligned} h_n \|\mathcal{A}_m^n P_n^1 x\| &= h_n \|P_n^1 x\| \\ h_m \|P_n^2 x\| &\leq h_m \|P_m^2 x\| = h_n \|\mathcal{A}_m^n P_n^2 x\| \\ \|\mathcal{A}_m^n P_n^3 x\| &= \|P_n^3 x\| \end{aligned}$$

for all  $(m, n, x) \in T$ , thus system  $(\mathfrak{A})$  is uniformly  $h$ -trichotomic

**Remark 3.** A uniformly  $h$ -trichotomic system  $(\mathfrak{A})$  is  $h$ -trichotomic. Now, we present an example which shows that the converse implication is not true.

**Example 2.** On  $X = \mathbb{R}^3$  we consider the linear discrete-time system  $(\mathfrak{A})$  defined by the sequence  $(A_n)$  given by

$$A_n = \frac{a_n}{a_{n+1}} P_n^1 + \frac{a_{n+1}}{a_n} P_{n+1}^2 + P_n^3 \tag{2}$$

where  $P_n^1, P_n^2, P_n^3$  are the canonical projections and  $a_n = e^{\frac{n}{1+u_n}}$  with  $u_n$  is a periodic sequence defined by  $\{1, 2, 3, 1, 2, 3, \dots\}$ .

It is easy to check that for  $h_n = e^{\alpha n}$ ,  $c_n = e^{n/4}$  and  $\alpha \in (0, \frac{1}{4})$  the system  $(\mathfrak{A})$  is  $h$ -trichotomic.

On the other side, if we suppose that the system  $(\mathfrak{A})$  admits a uniform approach, tacking into account  $(ht_1)$  from Definition 3 with  $h_n = e^{\alpha n}$ ,  $m = n + 2$ ,  $n = 3p$ ,  $u_m = 3$  and  $u_n = 1$  we have that

$$e^{2\alpha a} e^{\frac{3p-2}{4}} \leq c$$

which shows that nonuniform part cannot be removed.

3.  $h$ -TRICHOTOMY WITH TWO PROJECTIONS SEQUENCES

It is well-known that the dichotomy properties of discrete dynamical systems are defined with aid of two projections sequences. In order to give a characterization of the  $h$ -trichotomy property using two projections sequences, we introduce

**Definition 4.** *Two projections sequences  $Q_1, Q_2 : \mathbb{N} \rightarrow \mathcal{B}(X)$  are said to be orthogonal if*

$$Q_n^1 Q_n^2 = Q_n^2 Q_n^1 = 0 \tag{o_1}$$

$$\|(Q_n^1 + Q_n^2)x\|^2 = \|Q_n^1 x\|^2 + \|Q_n^2 x\|^2 \tag{o_2}$$

$$\|x - (Q_n^1 + Q_n^2)x\|^2 = \|x - Q_n^2 x\|^2 - \|Q_n^1 x\|^2 \tag{o_3}$$

$$\|x - (Q_n^1 + Q_n^2)x\|^2 = \|x - Q_n^1 x\|^2 - \|Q_n^2 x\|^2 \tag{o_4}$$

for all  $(n, x) \in \mathbb{N} \times X$ , where  $Q_n^j = Q_j(n)$ .

**Remark 4.** *If  $X$  is a Hilbert space then the projections sequences  $Q_1$  and  $Q_2$  are orthogonal if and only if  $Q_1 Q_2 = Q_2 Q_1 = 0$  (i.e. in Hilbert spaces (o<sub>1</sub>) implies (o<sub>2</sub>), (o<sub>3</sub>) and (o<sub>4</sub>)).*

Let  $P_1, P_2, P_3, Q_1, Q_2 : \mathbb{N} \rightarrow \mathcal{B}(X)$  be five projections sequences with  $P_1 = Q_1$ ,  $P_2 = Q_2$ ,  $P_3 = I - Q_1 - Q_2$ . Then  $P_1, P_2, P_3$  are supplementary if and only if  $Q_1$  and  $Q_2$  are orthogonal.

*Proof. Necessity.* If  $P_1, P_2, P_3$  are supplementary then from (s<sub>2</sub>) and (s<sub>3</sub>) we obtain

(o<sub>1</sub>)

$$Q_n^1 Q_n^2 = P_n^1 P_n^2 = 0 = P_n^2 P_n^1 = Q_n^2 Q_n^1$$

(o<sub>2</sub>)

$$\begin{aligned} \|(Q_n^1 + Q_n^2)x\|^2 &= \|(P_n^1 + P_n^2)x\|^2 = \|P_n^1 x\|^2 + \|P_n^2 x\|^2 \\ &= \|Q_n^1 x\|^2 + \|Q_n^2 x\|^2 \end{aligned}$$

(o<sub>3</sub>)

$$\begin{aligned} \|x - (Q_n^1 + Q_n^2)x\|^2 &= \|P_n^3 x\|^2 = \|(P_n^1 + P_n^3)x\|^2 - \|P_n^1 x\|^2 \\ &= \|x - Q_n^2 x\|^2 - \|Q_n^1 x\|^2 \end{aligned}$$

(o<sub>4</sub>)

$$\begin{aligned} \|x - (Q_n^1 + Q_n^2)x\|^2 &= \|P_n^3 x\|^2 = \|(P_n^2 + P_n^3)x\|^2 - \|P_n^2 x\|^2 \\ &= \|x - Q_n^1 x\|^2 - \|Q_n^2 x\|^2 \end{aligned}$$

and hence  $Q_1$  and  $Q_2$  are orthogonal.

*Sufficiency.* If  $Q_1$  and  $Q_2$  are orthogonal then

(s<sub>1</sub>)

$$P_n^1 + P_n^2 + P_n^3 = I$$

(s<sub>2</sub>)

$$\begin{aligned} P_n^1 P_n^2 &= Q_n^1 Q_n^2 = Q_n^2 Q_n^1 = 0 = P_n^2 P_n^1 \\ P_n^1 P_n^3 &= Q_n^1 (I - Q_n^1 - Q_n^2) = 0 = (I - Q_n^1 - Q_n^2) Q_n^1 = P_n^3 P_n^1 \\ P_n^2 P_n^3 &= Q_n^2 (I - Q_n^1 - Q_n^2) = 0 = (I - Q_n^1 - Q_n^2) Q_n^2 = P_n^3 P_n^2 \end{aligned}$$

(s<sub>3</sub>)

$$\begin{aligned} \|(P_n^1 + P_n^2)x\|^2 &= \|Q_n^1 x + Q_n^2 x\|^2 = \|Q_n^1 x\|^2 + \|Q_n^2 x\|^2 \\ &= \|P_n^1 x\|^2 + \|P_n^2 x\|^2 \end{aligned}$$

$$\begin{aligned} \|(P_n^1 + P_n^3)x\|^2 &= \|x - Q_n^2 x\|^2 = \|Q_n^1 x\|^2 + \|x - (Q_n^1 + Q_n^2)x\|^2 \\ &= \|P_n^1 x\|^2 + \|P_n^3 x\|^2 \end{aligned}$$

$$\begin{aligned} \|(P_n^2 + P_n^3)x\|^2 &= \|x - Q_n^1 x\|^2 = \|Q_n^2 x\|^2 + \|x - (Q_n^1 + Q_n^2)x\|^2 \\ &= \|P_n^2 x\|^2 + \|P_n^3 x\|^2 \end{aligned}$$

and hence  $P_1, P_2, P_3$  are supplementary.

A characterization of the  $h$ -trichotomy property in terms of two orthogonal projections sequences is presented in

**Theorem 1.** *The system  $(\mathfrak{A})$  is  $h$ -trichotomic if and only if there are two orthogonal projections sequences  $Q_1, Q_2 : \mathbb{N} \rightarrow \mathcal{B}(X)$  which are invariant for  $(\mathfrak{A})$  and there are some positive constants  $a, b$  and a sequence of nonnegative real numbers  $(c_n)$  such that*

$$h_m^a \|\mathcal{A}_m^n Q_n^1 x\| \leq c_n h_n^a \|Q_n^1 x\| \tag{ht'_1}$$

$$h_m^a \|Q_n^2 x\| \leq c_m h_n^a \|\mathcal{A}_m^n Q_n^2 x\| \tag{ht'_2}$$

$$h_n^b \|\mathcal{A}_m^n (I - Q_n^2)x\| \leq c_n h_m^b \|(I - Q_n^2)x\| \tag{ht'_3}$$

$$h_n^b \|(I - Q_n^1)x\| \leq c_m h_m^b \|\mathcal{A}_m^n (I - Q_n^1)x\| \tag{ht'_4}$$

for all  $(m, n, x) \in T$ .

*Proof.* We have that  $(ht'_1) \Leftrightarrow (ht_1)$  and  $(ht'_2) \Leftrightarrow (ht_2)$ .

*Necessity.* Suppose that  $(\mathfrak{A})$  is  $h$ -trichotomic. Let  $Q_1, Q_2$  be the orthogonal projections sequences (given by Proposition 3) associated to the supplementary projections  $P_1, P_2, P_3$  given by Definition 2. To prove  $(ht'_3)$  we observe that  $(ht_1), (ht_3)$  and  $(s_3)$  imply

$$\begin{aligned}
 h_n^{2b} \|\mathcal{A}_m^n(x - Q_n^2 x)\|^2 &= h_n^{2b} \|\mathcal{A}_m^n(P_n^1 + P_n^3)x\|^2 = h_n^{2b} \|P_m^1 \mathcal{A}_m^n x + P_m^3 \mathcal{A}_m^n x\|^2 \\
 &= h_n^{2b} (\|P_m^1 \mathcal{A}_m^n x\|^2 + \|P_m^3 \mathcal{A}_m^n x\|^2) \\
 &= h_n^{2b} (\|\mathcal{A}_m^n P_n^1 x\|^2 + \|\mathcal{A}_m^n P_n^3 x\|^2) \\
 &\leq c_n^2 h_n^{2b} \left( \frac{h_n^{2a}}{h_m^{2a}} \|P_n^1 x\|^2 + \frac{h_m^{2b}}{h_n^{2b}} \|P_n^3 x\|^2 \right) \\
 &\leq c_n^2 (h_n^{2b} \|P_n^1 x\|^2 + h_m^{2b} \|P_n^3 x\|^2) \\
 &\leq c_n^2 h_m^{2b} (\|P_n^1 x\|^2 + \|P_n^3 x\|^2) \\
 &= c_n^2 h_m^{2b} \|(P_n^1 + P_n^3)x\|^2 = c_n^2 h_m^{2b} \|x - Q_n^2 x\|^2,
 \end{aligned}$$

for all  $(m, n, x) \in T$ , which proves  $(ht'_3)$ .

Similarly, by  $(ht_2), (ht_4)$  and  $(s_3)$  we obtain

$$\begin{aligned}
 h_n^{2b} \|x - Q_n^1 x\|^2 &= h_n^{2b} \|P_n^2 x + P_n^3 x\|^2 = h_n^{2b} (\|P_n^2 x\|^2 + \|P_n^3 x\|^2) \\
 &\leq h_n^{2b} \left( c_m^2 \frac{h_n^{2a}}{h_m^{2a}} \|\mathcal{A}_m^n P_n^2 x\|^2 + c_m^2 \frac{h_m^{2b}}{h_n^{2b}} \|\mathcal{A}_m^n P_n^3 x\|^2 \right) \\
 &\leq c_m^2 h_m^{2b} (\|\mathcal{A}_m^n P_n^2 x\|^2 + \|\mathcal{A}_m^n P_n^3 x\|^2) \\
 &= c_m^2 h_m^{2b} (\|P_m^2 \mathcal{A}_m^n x\|^2 + \|P_m^3 \mathcal{A}_m^n x\|^2) \\
 &= c_m^2 h_m^{2b} \|(P_m^2 + P_m^3) \mathcal{A}_m^n x\|^2 = c_m^2 h_m^{2b} \|\mathcal{A}_m^n(x - Q_n^1 x)\|^2,
 \end{aligned}$$

for all  $(m, n, x) \in T$ . Thus,  $(ht'_4)$  is proved.

*Sufficiency.* If we denote by  $P_1 = Q_1, P_2 = Q_2$  and  $P_3 = I - Q_1 - Q_2 = (I - Q_2)(I - Q_1) = (I - Q_1)(I - Q_2)$  then by Proposition 3 it follows that  $P_1, P_2, P_3$  are supplementary projections sequences and  $P_1, P_2, P_3$  are invariant for  $(\mathfrak{A})$ . It remains to prove the implications  $(ht'_3) \Rightarrow (ht_3)$  and  $(ht'_4) \Rightarrow (ht_4)$ . Indeed, from  $(ht'_3)$  we obtain

$$\begin{aligned}
 h_n^b \|\mathcal{A}_m^n P_n^3 x\| &= h_n^b \|\mathcal{A}_m^n (I - Q_n^2)(I - Q_n^1)x\| \leq c_n h_m^b \|(I - Q_n^2)(I - Q_n^1)x\| \\
 &= c_n h_m^b \|P_n^3 x\|
 \end{aligned}$$

and similarly by  $(ht'_4)$  it results

$$h_n^b \|P_n^3 x\| = h_n^b \|(I - Q_n^1)(I - Q_n^2)x\| \leq c_m h_m^b \|\mathcal{A}_m^n P_n^3 x\|,$$

for all  $(m, n, x) \in T$ . Finally, we obtain that  $(\mathfrak{A})$  is  $h$ -trichotomic.

4.  $h$ -TRICHOTOMY WITH FOUR PROJECTIONS SEQUENCES

In order to obtain a characterizations of the  $h$ -trichotomy property in terms of four projections sequences we introduce

**Definition 5.** *Four projections sequences  $R_1, R_2, R_3, R_4 : \mathbb{N} \rightarrow \mathcal{B}(X)$  are said to be complementary if*

$$R_n^1 + R_n^4 = R_n^2 + R_n^3 = I \quad (c_1)$$

$$R_n^1 R_n^2 = R_n^2 R_n^1 = 0 \quad (c_2)$$

$$R_n^3 R_n^4 = R_n^4 R_n^3 \quad (c_3)$$

$$\|(R_n^1 + R_n^2)x\|^2 = \|R_n^1 x\|^2 + \|R_n^2 x\|^2 \quad (c_4)$$

$$\|(R_n^1 + R_n^3 R_n^4)x\|^2 = \|R_n^1 x\|^2 + \|R_n^3 R_n^4 x\|^2 \quad (c_5)$$

$$\|(R_n^2 + R_n^3 R_n^4)x\|^2 = \|R_n^2 x\|^2 + \|R_n^3 R_n^4 x\|^2 \quad (c_6)$$

for all  $(n, x) \in \mathbb{N} \times X$ , where  $R_n^j = R_j(n)$  for all  $n \in \mathbb{N}$  and  $j \in \{1, 2, 3\}$ .

**Remark 5.** *If  $X$  is a Hilbert space then the conditions  $(c_1)$ ,  $(c_2)$  and  $(c_3)$  imply the equalities  $(c_4)$ ,  $(c_5)$  and  $(c_6)$ .*

**Remark 6.** *If  $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(X)$  are three supplementary projections sequences then the projections sequences  $R_1, R_2, R_3, R_4 : \mathbb{N} \rightarrow \mathcal{B}(X)$  defined by  $R_1 = P_1$ ,  $R_2 = P_2$ ,  $R_3 = P_1 + P_3$  and  $R_4 = P_2 + P_3$  are complementary.*

**Remark 7.** *If  $R_1, R_2, R_3, R_4 : \mathbb{N} \rightarrow \mathcal{B}(X)$  are four complementary projections sequences then the projections sequences  $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(X)$  defined by  $P_1 = R_1$ ,  $P_2 = R_2$  and  $P_3 = R_3 R_4$  are supplementary.*

**Theorem 2.** *The linear discrete-time system  $(\mathfrak{A})$  is  $h$ -trichotomic if and only if there are four complementary projections sequences  $R_1, R_2, R_3, R_4 : \mathbb{N} \rightarrow \mathcal{B}(X)$  which are invariant for  $(\mathfrak{A})$  and there exist some positive constants  $a, b$  and a sequence of nonnegative real numbers  $(c_n)$  such that*

$$h_m^a \|\mathcal{A}_m^n R_n^1 x\| \leq c_n h_n^a \|R_n^1 x\| \quad (ht_1'')$$

$$h_m^a \|R_n^2 x\| \leq c_m h_n^a \|\mathcal{A}_m^n R_n^2 x\| \quad (ht_2'')$$

$$h_n^b \|\mathcal{A}_m^n R_n^3 x\| \leq c_n h_m^b \|R_n^3 x\| \quad (ht_3'')$$

$$h_n^b \|R_n^4 x\| \leq c_m h_m^b \|\mathcal{A}_m^n R_n^4 x\| \quad (ht_4'')$$

for all  $(m, n, x) \in T$ .



*Proof. Necessity.* If  $(\mathfrak{A})$  is  $h$ -trichotomic,  $P_1, P_2, P_3$  three supplementary projections sequences given by Definition 2 and  $R_1, R_2, R_3, R_4$  defined as in Remark 6, then  $(ht_1'') \Leftrightarrow (ht_1)$  and  $(ht_2'') \Leftrightarrow (ht_2)$ . Thus, similarly as in the proof of the implication  $(ht_3) \Rightarrow (ht_3')$  we obtain

$$\begin{aligned} h_n^{2b} \|\mathcal{A}_m^n R_n^3 x\|^2 &= h_n^{2b} \|\mathcal{A}_m^n (P_n^1 + P_n^3) x\|^2 \leq c_n^2 h_m^{2b} (\|P_n^1 x\|^2 + \|P_n^3 x\|^2) \\ &= c_n^2 h_m^{2b} \|(P_n^1 + P_n^3) x\|^2 = c_n^2 h_m^{2b} \|R_n^3 x\|^2 \end{aligned}$$

for all  $(m, n, x) \in T$ . Thus the property  $(ht_3'')$  is proved. The proof of  $(ht_4'')$  is similar.

*Sufficiency.* Let  $P_1, P_2, P_3 : \mathbb{N} \rightarrow \mathcal{B}(X)$  given by Remark 7. Then  $P_1, P_2, P_3$  are supplementary projections sequences which are invariant for  $(\mathfrak{A})$  and satisfy the inequalities  $(ht_1)$  and  $(ht_2)$ . To prove  $(ht_3)$  we observe that from  $(ht_3'')$  we obtain

$$h_n^b \|\mathcal{A}_m^n P_n^3 x\| = h_n^b \|\mathcal{A}_m^n R_n^3 R_n^4 x\| \leq c_n h_m^b \|R_n^3 R_n^4 x\| = c_n h_m^b \|P_n^3 x\|$$

for all  $(m, n, x) \in T$ . Similarly, we prove that  $(ht_4'') \Rightarrow (ht_4)$ . Finally, we conclude that  $(\mathfrak{A})$  is  $h$ -trichotomic.

#### REFERENCES

- [1] A.I. Alonso, J. Hong, R. Obaya, *Exponential dichotomy and trichotomy for difference equations*, Comp. Math. Appl., 38(1999), 41-49.
- [2] M.-G. Babuția, M. Megan, I.-L. Popa, *On  $(h, k)$ -dichotomies for nonautonomous linear difference equations in Banach spaces*, International Journal of Differential Equations, Volume 2013, Article ID 761680, 7 pages.
- [3] M.-G. Babuția, T. Ceașu, *Connections between some well-known concepts of uniform exponential dichotomies for discrete-time systems in Banach spaces*, accepted for publication in Proceedings of the International Symposium "Research and Education in Innovation Era", 5-th Edition, Arad Publishing House, 2014.
- [4] M.-G. Babuția, M.I. Kovacs, M. Lăpădat, M. Megan, *Discrete  $(h, k)$ -dichotomy and remarks on the boundedness of the projections*, Journal of Operators, Vol. 2014, Article ID 196345.
- [5] L. Barreira, C. Valls, *Robustness of nonuniform exponential trichotomies in Banach spaces*, J. Math. Anal. Appl. 351(2009), 373-381.
- [6] A. Ducrot, P. Magal, O. Seydi, *A finite-time condition for exponential trichotomy in infinite dynamical systems*, Canad. J. Math. <http://dx.doi.org/10.4153/CJM-2014-023-3>
- [7] S. Elaydi, K. Janglajew, *Dichotomy and trichotomy of difference equations*, J. Difference Equ. Appl. 3 (1998), no. 5-6, 417-448.

- [8] S. Elaydi, O. Hajek, *Exponential dichotomy and trichotomy of nonlinear differential equations*, Differential Integral Equations 3(6) 1990, 1201-1224.
- [9] J. Hong, R. Obaya, A. Sanz, *Existence of a class of ergodic solutions implies exponential trichotomy*, Appl. Math. Lett. 12(1999), 43-45.
- [10] Z. Li, X. Song, X. Yang, *On Nonuniform Polynomial Trichotomy of Linear Discrete-Time Systems in Banach Spaces*, Journal of Applied Mathematics, Volume 2014 (2014), Article ID 807265, 6 pages.
- [11] N. Lupa, M. Megan, *Generalized Exponential Trichotomies for Abstract Evolution Operators on the Real Line*, Journal of Function Spaces and Applications Volume 2013 (2013), Article ID 409049, 8 pages.
- [12] J. López-Fenner, M. Pinto, *(h,k)-Trichotomies and Asymptotics of Nonautonomous Difference Systems*, Computers Math. Applic., Vol. 33, No. 10, 105-124, 1997.
- [13] M. Megan, L. Buliga, *Nonuniform exponential trichotomy for difference equations in Banach spaces*, Hot Topics in Operator Theory, Theta Ser. Adv. Math. 9(2008), 145-150.
- [14] M. Megan, C. Stoica, *Equivalent definitions for uniform exponential trichotomy of evolution operators in Banach spaces*, Hot Topics in Operator Theory, Theta Ser. Adv. Math. 9 (2008), 151-158.
- [15] G. Papaschinopoulos, *On exponential trichotomy of linear difference equations*, Appl. Anal. 40(1991), 89-109.
- [16] A.L. Sasu, B. Sasu, *Exponential trichotomy for variational difference equations*, J. Difference Equ. Appl., (15) 2009, 693-718.
- [17] R.S. Sacker, G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential equations III*, J. Differential Equations 3(6) 1990, 1201-1224.
- [18] X. Song, T. Yue, D. Li, *Nonuniform Exponential Trichotomy for Linear Discrete-Time Systems in Banach Spaces*, Journal of Function Spaces and Applications, vol. 2013, Article ID 645250, 6 pages, 2013

Ioan-Lucian Popa  
Department of Mathematics, Faculty of Exact Sciences and Engineering,  
University "1<sup>st</sup> December 1918" of Alba Iulia,  
510009-Alba Iulia, Romania  
email: [lucian.popa@uab.ro](mailto:lucian.popa@uab.ro)

Mihail Megan  
Academy of Romanian Scientists

Independenței 54, 050094 Bucharest, Romania  
email: *megan@math.uvt.ro*

Traian Ceașu  
Department of Mathematics, Faculty of Mathematics and Computer Science,  
West University of Timișoara,  
300223-Timișoara, Romania  
email: *ceausu@math.uvt.ro*